Solutions Manual For

CONTINUUM ELECTROMECHANICS

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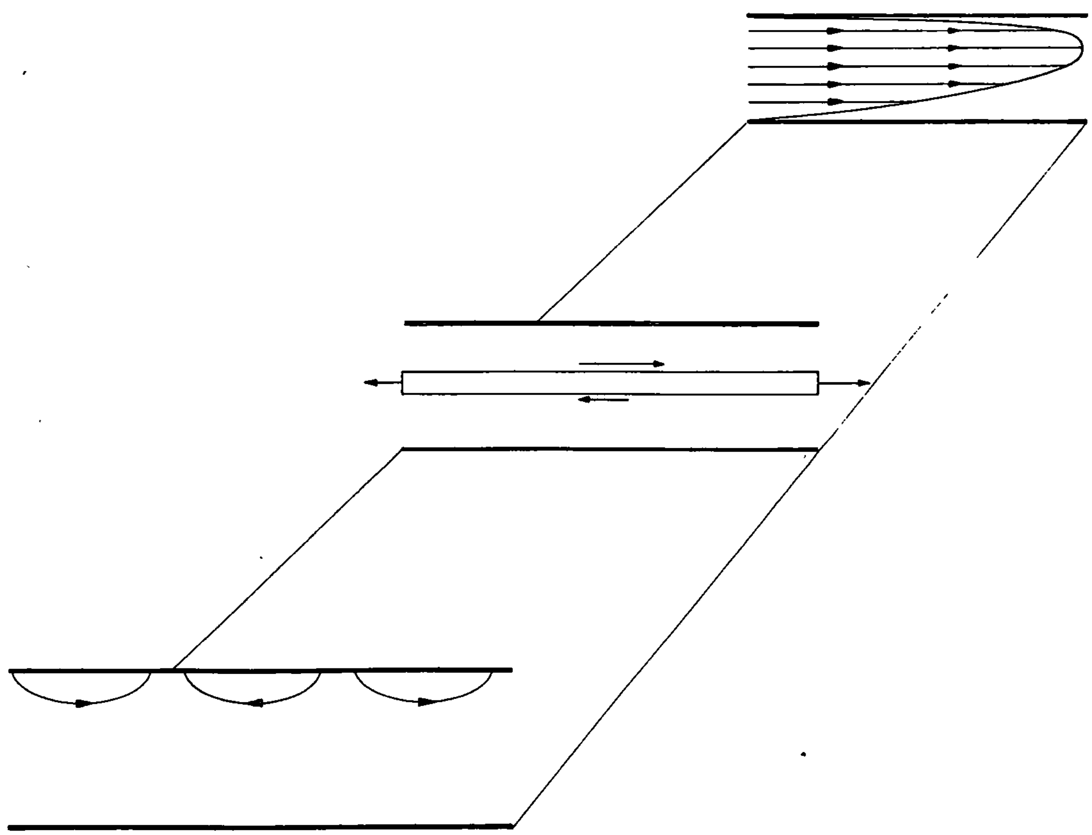

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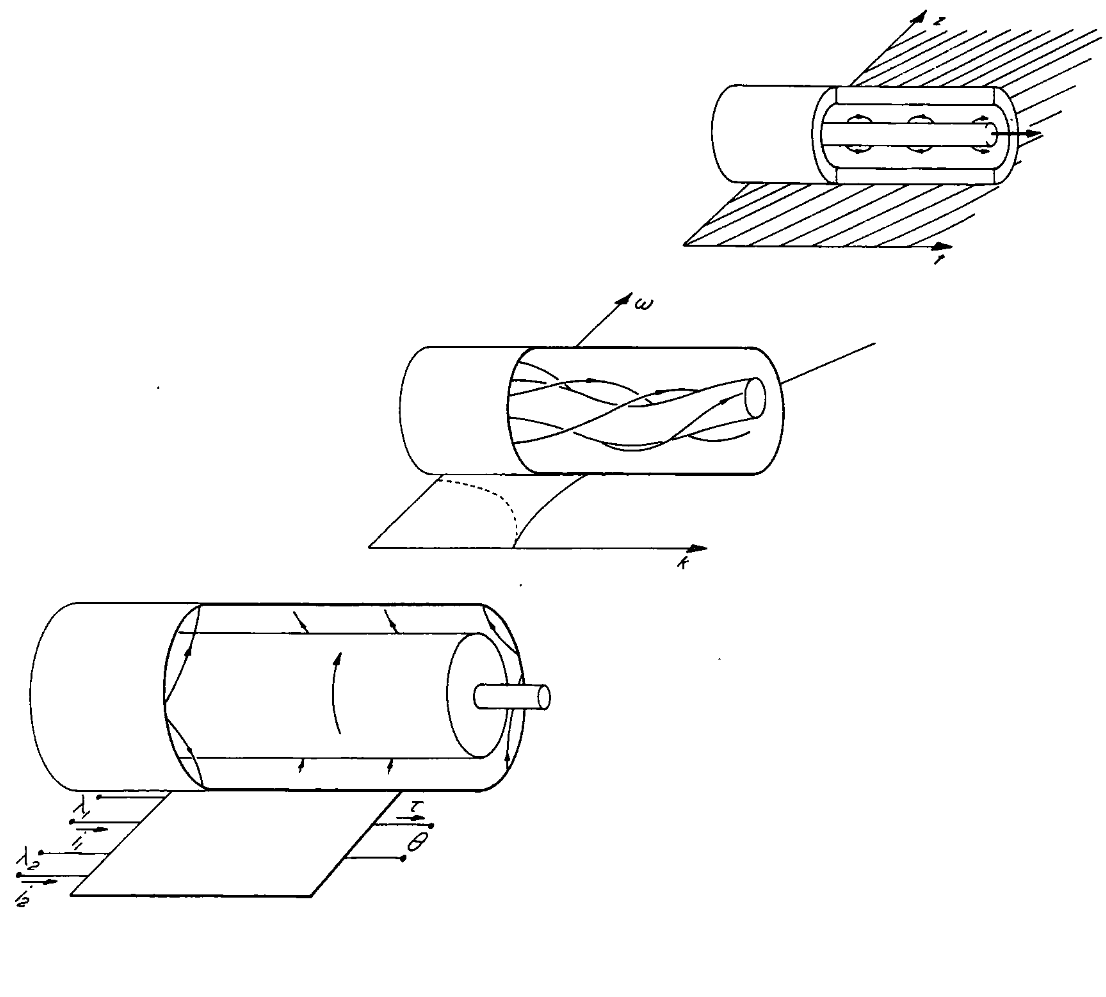
1

Introduction to Continuum Electromechanics



CONTINUUM ELECTROMECHANICS Used as a Text

Much of Chap. 2 is a summary of relevant background material and care should be taken not to become mired down in the preliminaries. The discussion of electromagnetic quasistatics in the first part of Chap. 2 is a "dry" starting point and will mean more as later examples are worked out. After a brief reading of Secs. 2.1-2.12, the subject can begin with Chap. 3. Then, before taking on Secs. 3.7 and 3.8, Secs. 2.13 and 2.14 respectively should be studied. Similarly, before starting Chap. 4, it is appropriate to take up Secs. 2.15-2.17, and when needed, Sec. 2.18. The material of Chap. 2 is intended to be a reference in all of the chapters that follow.

Chapters 4-6 evolve by first exploiting complex amplitude representations, then Fourier amplitudes, and by the end of Chap. 5, Fourier transforms. The quasi-one-dimensional models of Chap. 4 and method of characteristics of Chap. 5 also represent developing viewpoints for describing continuum systems. In the first semester, the author has found it possible to provide a taste of the "full-blown" continuum electromechanics problems by covering just enough the fluid mechanics in Chap. 7 to make it possible to cover interesting and practical examples from Chap. 8. This is done by first covering Secs. 7.1-7.9 and then Secs. 8.1-8.4 and 8.9-8.13.

The second semester, is begun with a return to Chap. 7, now bringing in the effects of fluid viscosity (and through the homework, of solid elasticity). As with Chap. 2, Chap. 7 is designed to be materials collected for reference in one chapter but best taught in conjunction with chapters where the material is used. Thus, after Secs. 7.13-7.18 are covered, the electromechanics theme is continued with Secs. 8.6, 8.7 and 8.16.

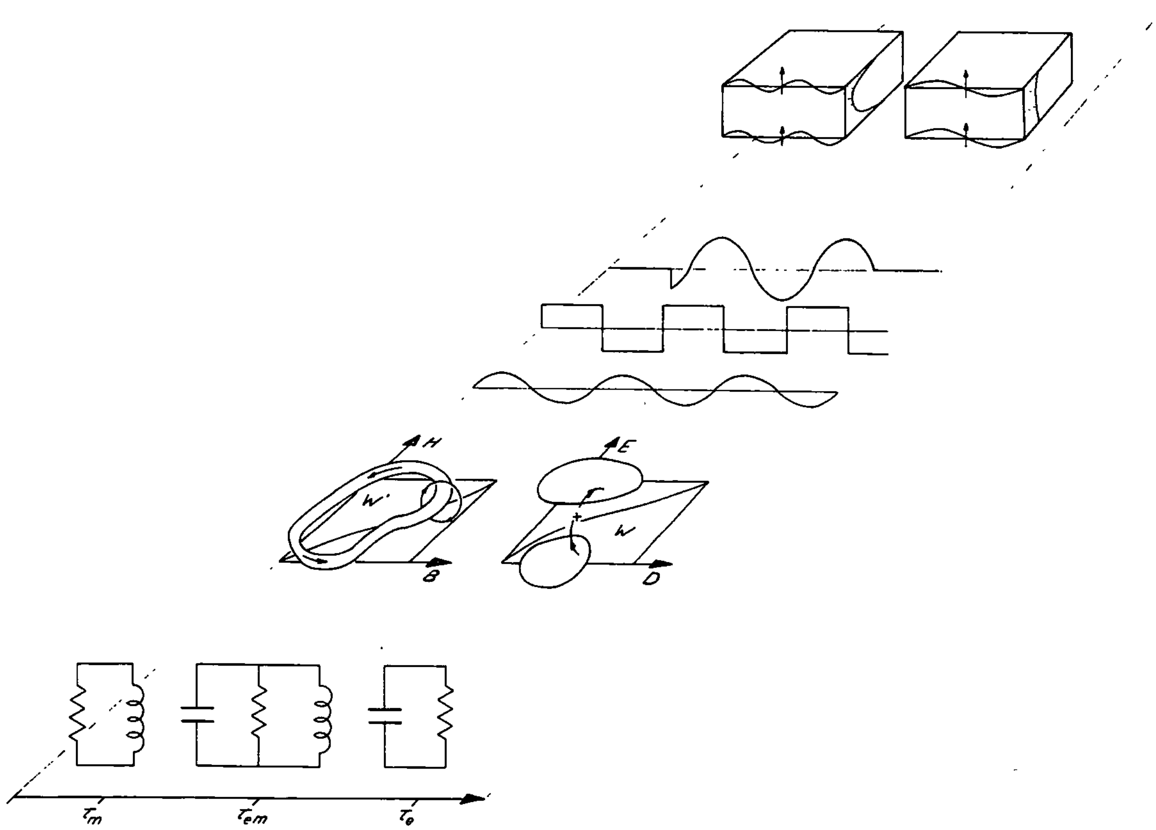
Coverage in the second semester has depended more on the interests of the class. But, if the material in Sec. 9.5 on compressible flows is covered, the relevant sections of Chap. 7 are then brought in. Similarly, in Chap. 10, where low Reynolds number flows are considered, the material from Sec. 7.20 is best brought in.

With the intent of making the material more likely to "stick", the author has found it good pedagogy to provide a staged and multiple exposure to new concepts. For example, the Fourier transform description of spatial transients is first brought in at the end of Chap. 5 (in the first semester) and then expanded to describe space-time dynamics in Chap. 11 (at the end of the second semester). Similarly, the method of characteristics for "first-order" systems is introduced in Chap. 5, and then expanded in Chap. 11 to wave-like dynamics. The magnetic diffusion (linear) boundary layers of Chap. 6 appear in the first semester and provide background for the viscous diffusion (nonlinear) boundary layers of Chap. 9, taken up in the second semester.

This Solutions Manual gives some hint of the vast variety of physical situations that can be described by combinations of results summarized throughout the text. Thus, it is that even though the author tends to discourage a dependence on the text in lower level subjects (the first step in establishing confidence in field theory often comes from memorizing Maxwell's equations), here emphasis is placed on deriving results and making them a ready reference. Quizzes, like the homework, should encourage reference to the text.

2

Electrodynamic Laws, Approximations and Relations



Prob. 2.3.1 a) In the free space region between the plates, $\bar{J}_v=\bar{P}=\bar{M}=0$ and Maxwell's equations, normalized in accordance with Eqs. 2.3.4b are

$$\nabla \times \bar{E} = -\frac{\partial \bar{H}}{\partial t} \tag{1}$$

$$\nabla \times \bar{H} = \beta \frac{\partial \bar{E}}{\partial t} \tag{2}$$

$$\nabla \cdot \bar{E} = 0 \tag{3}$$

$$\nabla \cdot \bar{H} = 0 \tag{4}$$

For fields of the form given, these reduce to just two equations.

$$\frac{\partial E_x}{\partial z} = -\frac{\partial H_y}{\partial t} \tag{5}$$

$$\frac{\partial H_y}{\partial z} = -\beta \frac{\partial E_x}{\partial t} \tag{6}$$

Here, the characteristic time is taken as $1/\omega$ so that time dependences exp $j\omega t$ take the form

$$E_x = Re\, \hat{E}_x(z)\, e^{jt} \; ; \; H_y = Re\, \hat{H}_y(z)\, e^{jt} \tag{7}$$

For the time-rate expansion, the dependent variables are expanded in $\beta = \omega^2 \mu \epsilon \ell^2$

$$\hat{E}_x = \sum_{n=0}^{\infty} \hat{E}_{xn} \beta^n \; ; \; H_y = Re\, \hat{H}_{yn} \beta^n \tag{8}$$

so that Eqs. 5 and 6 become

$$\frac{\partial}{\partial z}\left[\sum_{n=0}^{\infty} \hat{E}_{xn} \beta^n\right] = -j\left[\sum_{n=0}^{\infty} \hat{H}_{yn} \beta^n\right] \tag{9}$$

$$\frac{\partial}{\partial z}\left[\sum_{n=0}^{\infty} \hat{H}_{yn} \beta^n\right] = -j\beta\left[\sum_{n=0}^{\infty} \hat{E}_{xn} \beta^n\right] \tag{10}$$

Equating like powers of β results in a hierarchy of expressions

$$\frac{\partial \hat{E}_{xn}}{\partial z} = -j\, \hat{H}_{yn} \tag{11}$$

$$\frac{\partial \hat{H}_{yn}}{\partial z} = -j\, \hat{E}_{x(n-1)} \tag{12}$$

Boundary conditions on the upper and lower plates are satisfied identically. (No tangential $\bar{E}$ and no normal $\bar{B}$ at the surface of a perfect conductor.) At $z=0$ where there is also a perfectly conducting plate, $E_x=0$. At $z=-\ell$, Ampere's law requires that $i/w=H_y$ (boundary condition, 2.10.21). (Because $w \gg s$, the magnetic field intensity outside the region between the plates is negligible compared to that inside.) With the characteristic magnetic field taken as I_o/w, where $i(t) = \underline{i}(t)\; I_o$, it follows that the normalized boundary conditions are

$$\hat{E}_x(0) = 0 \; ; \; \hat{H}_y(-1) = 1 \tag{13}$$

Prob. 2.3.1 (cont)

The zero order Eq. 12 requires that

$$\frac{\partial \hat{H}_{yo}}{\partial z} = 0 \tag{13}$$

and reflects the nature of the magnetic field distribution in the static limit $\beta \rightarrow 0$. The boundary condition on H_y, Eq. 13, evaluates the integration constant.

$$\hat{H}_{yo} = 1 \tag{14}$$

The electric field induced through Faraday's law follows by using this result in the zero order statement of Eq. 11. Because what is on the right is independent of z, it can be integrated to give

$$\hat{E}_{xo} = -jz \tag{15}$$

Here, the integration constant is zero because of the boundary condition on E_x, Eq. 13. These zero order fields are now used to find the first order fields The n=1 version of Eq. 12 with the right hand side evaluated using Eq. 15 can be integrated. Because the zero order fields already satisfy the boundary conditions, it is clear that all higher order terms must vanish at the appropriate boundary, E_{xn} at z=0 and H_{yn} at z=1. Thus, the integration constant is evaluated and

$$\hat{H}_{y1} = -\frac{1}{2}(z^2-1) \tag{16}$$

This expression is inserted into Eq. 11 with n=1, integrated and the constant evaluated to give

$$\hat{E}_{x1} = j\frac{1}{2}\left(\frac{1}{3}z^3 - z\right) \tag{17}$$

If the process is repeated, it follows that

$$\hat{H}_{y2} = \frac{1}{4}\left(\frac{1}{6}z^4 - z^2 + \frac{5}{6}\right) \tag{18}$$

$$\hat{E}_{x2} = -j\frac{1}{4}\left(\frac{1}{30}z^5 - \frac{1}{3}z^3 + \frac{5}{6}z\right) \tag{19}$$

so that, with the coefficients defined by Eqs. 15-19, solutions to order are

$$\hat{E}_x = E_{xo} + E_{x1}\beta + E_{x2}\beta^2;\ \hat{H}_y = \hat{H}_{yo} + \hat{H}_{y1}\beta + \hat{H}_{y2}\beta^2 \tag{20}$$

Prob. 2.3.1(cont.)

Note that the surface charge on the lower electrode, as well as the surface current density there, are related to the fields between the electrodes by

$$\sigma_f = E_x \quad ; \; K_z = H_y \tag{21}$$

The respective quantities on the upper electrode are the negatives of these quantities. (Gauss' law and Ampere's law). With Eqs. 7 used to recover the time dependence, what have been found to second order in β are the normalized fields

$$E_x = z\left[1-\tfrac{1}{2}\left(\tfrac{1}{3}z^2-1\right)\beta+\tfrac{1}{4}\left(\tfrac{1}{30}z^4-\tfrac{1}{3}z^2+\tfrac{5}{6}\right)\beta^2\right]\sin t = \sigma_f \tag{22}$$

$$H_y = \left[1-\tfrac{1}{2}(z^2-1)\beta+\tfrac{1}{4}\left(\tfrac{1}{6}z^4-z^2+\tfrac{5}{6}\right)\beta^2\right]\cos t = K_z \tag{23}$$

The dimensioned forms follow by identifying

$$E_x = \frac{\mu_o \omega l I_o}{w} \tag{24}$$

e) Now, consider the exact solutions. Eqs. 7 substituted into Eqs. 5 and 6 give

$$\frac{d^2\hat{H}_y}{dz^2} + \beta\hat{H}_y = 0 \tag{25}$$

$$\hat{E}_x = \frac{j}{\beta}\frac{d\hat{H}_y}{dz} \tag{26}$$

Solutions that satisfy these expressions as well as Eqs. 13 are

$$\hat{H}_y = \cos(\sqrt{\beta}\,z)/\cos\sqrt{\beta} \tag{27}$$

$$\hat{E}_x = \frac{-j}{\sqrt{\beta}}\sin(\sqrt{\beta}\,z)/\cos\sqrt{\beta} \tag{28}$$

These can be expanded to second order in β as follows.

$$\hat{H}_y \cong \frac{1-\frac{1}{2}\beta z^2+\frac{1}{4!}\beta^2 z^4+\cdots}{1-\frac{1}{2}\beta+\frac{1}{4!}\beta^2+\cdots} \tag{29}$$

$$\cong \left(1-\tfrac{1}{2}\beta z^2+\tfrac{1}{4!}\beta^2 z^4\right)\left(1-\left(-\tfrac{1}{2}\beta+\tfrac{1}{4!}\beta^2\right)+\left(-\tfrac{1}{2}\beta+\tfrac{1}{4!}\beta^2\right)^2\right)$$

$$= \left(1-\tfrac{1}{2}\beta z^2+\tfrac{1}{24}\beta^2 z^4\right)\left(1+\tfrac{1}{2}\beta+\tfrac{5}{24}\beta^2-\tfrac{1}{24}\beta^3+\tfrac{1}{576}\beta^4\right)$$

$$\cong 1-\tfrac{1}{2}(z^2-1)\beta+\tfrac{1}{4}\left(\tfrac{1}{6}z^4-z^2+\tfrac{5}{6}\right)\beta^2$$

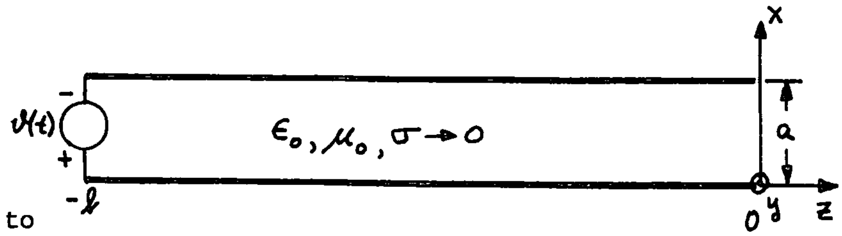
Prob. 2.3.1(cont.)

$$\hat{E}_x \cong \frac{-j}{\sqrt{\beta}} \frac{(\sqrt{\beta}\, z) - \frac{1}{3!}(\sqrt{\beta}\, z)^3 + \frac{1}{5!}(\sqrt{\beta}\, z)^5 - \cdots}{1 - \frac{1}{2}\beta + \frac{1}{4!}\beta^2 - \cdots} \tag{30}$$

$$\cong -jz\left[1 - \frac{1}{2}\left(\frac{1}{3}z^2 - 1\right)\beta + \frac{1}{4}\left(\frac{1}{30}z^4 - \frac{1}{3}z^2 + \frac{5}{6}\right)\beta^2\right]$$

These expressions thus prove to be the same expansions as found from the time-rate expansion.

Prob. 2.3.2 Assume



$$\bar{E} = \bar{i}_x E_x(z,t)$$
$$\bar{H} = \bar{i}_y H_y(z,t)$$

and Maxwell's equations reduce to

$$\frac{\partial E_x}{\partial z} = -\frac{\partial \mu_o H_y}{\partial t} \quad ; \quad -\frac{\partial H_y}{\partial z} = \frac{\partial \epsilon_o E_x}{\partial t} \tag{1}$$

In normalized form (Eqs. 2.3.5a-2.3.10a) these are

$$\frac{\partial E_x}{\partial z} = -\beta \frac{\partial H_y}{\partial t} \quad ; \quad -\frac{\partial H_y}{\partial z} = \frac{\partial E_x}{\partial t} \tag{2}$$

Let

$$E_x = E_{xo} + \beta E_{x1} + \beta^2 E_{x2} + \cdots$$
$$H_y = E_{yo} + \beta H_{y1} + \beta^2 H_{y2} + \cdots \tag{3}$$

Then, Eqs. 2 become

$$\frac{\partial E_{xo}}{\partial z} + \beta\left[\frac{\partial E_{x1}}{\partial z} + \frac{\partial H_{yo}}{\partial t}\right] + \beta^2\left[\frac{\partial E_{x2}}{\partial z} + \frac{\partial H_{y1}}{\partial t}\right] + \cdots = 0$$
$$\left[\frac{\partial H_{yo}}{\partial z} + \frac{\partial E_{xo}}{\partial t}\right] + \beta\left[\frac{\partial H_{y1}}{\partial z} + \frac{\partial E_{x1}}{\partial t}\right] + \beta^2\left[\frac{\partial H_{y2}}{\partial z} + \frac{\partial E_{x2}}{\partial t}\right] + \cdots = 0 \tag{4}$$

Zero order terms in β require

$$\frac{\partial E_{xo}}{\partial z} = 0 \Rightarrow E_{xo} = E_{xo}(t) = \frac{v(t)}{a}\frac{1}{\mathcal{E}} \tag{5}$$

$$\frac{\partial H_{yo}}{\partial z} = -\frac{\partial E_{xo}}{\partial t} = -\frac{1}{a\mathcal{E}}\frac{dv}{dt} \Rightarrow H_{yo} = -\frac{1}{a\mathcal{E}}\frac{dv}{dt} z \tag{6}$$

Boundary conditions have been introduced to insure $E_x(-\ell,t) = v/a$ and, because $K_z(0,t) = 0$, $H_y(0,t) = 0$

Now consider first order terms.

$$\frac{\partial E_{x1}}{\partial z} = -\frac{\partial H_{yo}}{\partial t} = \frac{1}{a\mathcal{E}}\frac{d^2 v}{dt^2} z \Rightarrow E_{x1} = \frac{1}{a\mathcal{E}}\frac{d^2 v}{dt^2}\frac{1}{2}(z^2 - \ell^2)$$
$$\frac{\partial H_{y1}}{\partial z} = -\frac{\partial E_{x1}}{\partial t} = -\frac{1}{a\mathcal{E}}\frac{d^3 v}{dt^3}\frac{1}{2}(z^2 - \ell^2) \Rightarrow H_{y1} = \frac{-1}{a\mathcal{E}}\frac{d^3 v}{dt^3}\frac{1}{2}\left(\frac{z^3}{3} - z\right) \tag{7}$$

Prob. 2.3.2 (cont.)

The integration functions in these last two functions are determined by the boundary conditions which, because the first terms satisfy the boundary conditions, must satisfy homogeneous boundary conditions; $\underline{E}_{x1}(\underline{z}=-1)=0, \underline{H}_{y1}(0)=0$. In normalized form, we have

$$E_x = \frac{v(t)}{a\mathcal{E}} + \beta\frac{1}{a\mathcal{E}}\frac{d^2v}{dt^2}\frac{1}{2}(z^2-1)+\cdots$$

$$H_y = -\frac{1}{a\mathcal{E}}\frac{dv}{dt}z - \frac{\beta}{a\mathcal{E}}\frac{d^3v}{dt^3}\frac{1}{2}\left(\frac{z^3}{3}-z\right)+\cdots \tag{8}$$

In unnormalized form

$$E_x = \frac{v(t)}{a} + \frac{\mu_0\epsilon_0}{a}\frac{d^2v}{dt^2}\frac{1}{2}(z^2-\ell^2)+\cdots$$

$$H_y = -\frac{\epsilon_0}{a}\frac{dv}{dt}z - \frac{\mu_0\epsilon_0^2}{a}\frac{d^3v}{dt^3}\frac{1}{2}\left(\frac{z^3}{3}-z\ell^2\right)+\cdots \tag{9}$$

Compare these series to the exact solutions, which by inspection are

$$E_x = \frac{v_0}{a}\frac{\cos\frac{\omega}{c}z}{\cos\frac{\omega}{c}\ell}\cos\omega t \cong \frac{v_0}{a}\cos\omega t\left[1-\frac{1}{2}\frac{\omega^2}{c^2}(z^2-\ell^2)+\cdots\right]$$

$$H_y = \frac{v_0}{a\mu_0 c}\frac{\sin\left(\frac{\omega}{c}z\right)}{\cos\left(\frac{\omega}{c}\ell\right)}\sin\omega t \simeq \frac{v_0}{a}\sqrt{\frac{\epsilon_0}{\mu_0}}\left[\frac{\omega z}{c}+\frac{1}{2}\left(\frac{\omega}{c}\right)^3\left(\ell^2 z-\frac{1}{3}z^3\right)+\cdots\right]$$

Thus, the formal expansion gives the same result as a series expansion of the exact solution. Note that what is being expanded is

$$\left(\frac{\omega}{c}z\right)^2 = \left[\frac{\sqrt{\mu_0\epsilon_0}\,\ell}{1/\omega}\left(\frac{z}{\ell}\right)\right]^2 \equiv \beta\left(\frac{z}{\ell}\right)^2$$

The quasi-static equations are Eqs. 5 and 6 in unnormalized form, which respectively represent the one-dimensional forms of $\nabla\times\bar{E}=0$ and conservation

Prob. 2.3.2 (cont.)

of charge ($H_y \leftrightarrow K_z$ in lower electrode), give the zero order solutions. Conservation of charge on electrode gives linearly increasing K_z which is the same as H_y.

Prob. 2.3.3 In the volume of the Ohmic conductor, Eqs. 2.2.1-2.2.5, with $\bar{P}=\bar{M}=\bar{v}=0$) become

$$\nabla \times \bar{E} = -\mu_o \frac{\partial \bar{H}}{\partial t} \tag{1}$$

$$\nabla \times \bar{H} = \epsilon_o \frac{\partial \bar{E}}{\partial t} + \sigma \bar{E} \tag{2}$$

$$\nabla \cdot \epsilon_o \bar{E} = \rho_f \tag{3}$$

$$\nabla \cdot \mu_o \bar{H} = 0 \tag{4}$$

Fields are now assumed that are transverse to their spatial dependence, z, that satisfy the boundary conditions on the electrodes at x=0 and x=a (no tangential $\bar{E}$ or normal $\bar{H}$) and that have the same temporal dependence as the excitation.

$$\bar{E} = \bar{E}(z,t) = \bar{i}_x \,\mathrm{Re}[\hat{E}_x(z) \exp j\omega t] \tag{5}$$

$$\bar{H} = \bar{H}(z,t) = \bar{i}_y \,\mathrm{Re}[\hat{H}_y(z) \exp j\omega t] \tag{6}$$

It follows that $\rho_f = 0$ and that all components of Eqs. 1 and 2 are identically satisfied except the y component of Eq. 1 and the x component of Eq. 2, which require that

$$\frac{d\hat{E}_x}{dz} = -j\omega\mu_o \hat{H}_y \tag{7}$$

$$-\frac{d\hat{H}_y}{dz} = (\sigma + j\omega\epsilon_o)\hat{E}_x \tag{8}$$

Transverse fields are solenoidal, so Eqs. 3 and 4 are identically satisfied with $\rho_f = 0$. (See Sec. 5.10 for a discussion of why $\rho_f = 0$ in the volume of a uniform conductor. Note that the arguments given there can be applied to a conductor at rest without requiring that the system be EQS.)

Elimination of E_x between Eqs. 8 and 7 shows that

$$\frac{d^2\hat{H}_y}{dz^2} + k^2 \hat{H}_y = 0 \; ; \; k^2 \equiv \omega^2\mu_o\epsilon_o - j\omega\mu_o\sigma \tag{9}$$

and in terms of $\hat{H}_y$, $\hat{E}_x$ follows from Eq. 8.

$$\hat{E}_x = \frac{-1}{\sigma + j\omega\epsilon_o} \frac{d\hat{H}_y}{dz} \tag{10}$$

b) Solutions to Eq. 9 take the form

$$\hat{H}_y = H_+ e^{-jkz} + H_- e^{jkz} \tag{11}$$

Prob. 2.3.3(cont.)

In terms of these same coefficients, H_+ and H_-, it follows from Eq. 10 that

$$\hat{E}_x = \frac{-jk}{\sigma + j\omega\epsilon_o}\left[H_+ e^{-jkz} - H_- e^{jkz}\right] \tag{12}$$

Because the electrodes are very long in the y direction compared to the spacing a, and because fringing fields are ignored at z=0, the magnetic field outside the region between the perfectly conducting electrodes is essentially zero. It follows from the boundary condition required by Ampere's law at the respective ends (Eq. 21 of Table 2.10.1) that

$$H_y(0,t) = 0 \Rightarrow \hat{H}_y(0) = 0 \tag{13}$$

$$H_y(-\ell,t) = \mathrm{Re}\,\hat{K}\exp j\omega t \Rightarrow \hat{H}_y(-\ell) = \hat{K} \tag{14}$$

Thus, the two coefficients in Eq. 11 are evaluated and the expressions of Eqs. 11 and 12 become those given in the problem statement.

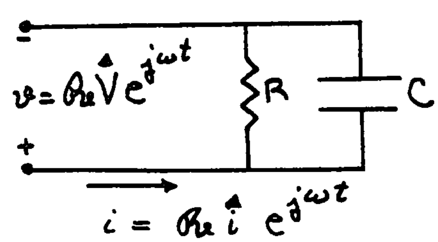
c) Note that

$$k\ell = \sqrt{\omega^2\mu_o\epsilon_o\ell^2 - j\omega\mu_o\sigma\ell^2} = \sqrt{(\omega\tau_{em})^2 - j(\omega\tau_m)^2} \tag{15}$$

so, $|k\ell| \ll 1$ provided that $\omega\tau_{em} \ll 1$ and $\omega\tau_m \ll 1$. To obtain the limiting form of E_x, the exponentials are expanded to first order in $k\ell$. In itself, the approximation does not imply an ordering of the characteristic times. However, if the frequency dependence of E_x expressed by the limiting form is to have any significance, then it is clear that the ordering must be $\tau_m < \tau_{em} < \tau_e$ as illustrated by Fig. 2.3.1 for the EQS approximation.

With the voltage and current defined as $v = E_x(-\ell,t)a$, $i = Kd$, it follows from the limiting form of E_x that

$$\hat{V} = \frac{\hat{I}}{\frac{\sigma\ell d}{a} + j\omega\left(\frac{\epsilon_o\ell d}{a}\right)} \tag{16}$$



This is of the same form as the relation

$$\hat{V} = \frac{\hat{I}}{\frac{1}{R} + j\omega C} \tag{17}$$

found for the circuit shown. Thus, as expected, $C = \epsilon_o\ell d/a$ and $R = a/\sigma\ell d$.

In the MQS approximation, where $\omega\tau_m$ is arbitrary, it is helpful to write Eq. 15 in the form

$$k\ell = \sqrt{-j\omega\tau_m(1+j\omega\tau_e)} \qquad (18)$$

The second terms is negligible (the displacement current is small compared to the conduction current) if $\omega\tau_e \ll 1$, in which case

$$k \approx (-1+j)/\delta_m \; ; \; \delta_m \equiv \sqrt{\frac{2}{\omega\mu_o\sigma}} \qquad (19)$$

Then, the magnetic field distribution assumes the limiting form

$$H_y = Re\left\{\frac{\hat{K}}{e^{-j\frac{\ell}{\delta_m}}e^{-\frac{\ell}{\delta_m}} - e^{j\frac{\ell}{\delta_m}}e^{\frac{\ell}{\delta_m}}}\left(e^{\frac{z}{\delta_m}}e^{j(\omega t+\frac{z}{\delta_m})} - e^{-\frac{z}{\delta_m}}e^{j(\omega t-\frac{z}{\delta_m})}\right)\right\} \qquad (20)$$

That is, Eddy currents induced in the conductor tend to shield out the magnetic field, which tends to be confined to the neighborhood of the current source. The skin depth, δ_m, serves notice that the phenomena accounting for the superimposed decaying waves represented by Eq. 20 is magnetic diffusion. With the exclusion of the displacement current, the dynamics no longer have the attributes of an electromagnetic wave.

It is easy to see that this MQS approximation is valid only if $\omega\tau_e \ll 1$, but how does this imply that $\omega\tau_{em} \ll 1$? Here, the implicite relation between τ_m, τ_e and τ_{em} comes into play. What is considered negligible in Eq. 18 by making $\omega\tau_e \ll 1$ is neglected in the same expression written in terms of τ_{em} and τ_m as Eq. 15 by making $\tau_{em} \ll \tau_m$. Thus, the ordering of characteristic times is $\tau_e < \tau_{em} < \tau_m$, as summarized by the MQS sketch of Fig. 2.3.1.

d) The electroquasistatic equations, Eqs. 2.3.23a-2.3.25a, require that

$$\frac{\partial E_x}{\partial z} = 0 \qquad (21)$$

so that E_x is independent of z (uniform) and

$$\frac{d\hat{H}_y}{dz} = -(\sigma + j\omega\epsilon_o)\hat{E}_x \qquad (22)$$

It follows that this last expression can be integrated on z with the constant of integration taken as zero because of boundary condition, Eq. 13. That H_y also satisfy Eq. 14 then results in

Prob. 2.3.3(cont.)

$$\hat{E}_x = (\hat{K}/\sigma\ell)/(1 + j\omega\tau_e) \tag{23}$$

which is the same as the EQS limit of the exact solution, Eq. 16.

e) In the MQS limit, where Eqs. 2.3.23a-2.3.25a apply, equations combine to show that H_y satisfies the diffusion equation.

$$\frac{1}{\mu_o\sigma}\frac{\partial^2 H_y}{\partial z^2} = \frac{\partial H_y}{\partial t} \Rightarrow \frac{d^2\hat{H}_y}{dz^2} = -j\omega\mu_o\sigma\hat{H}_y \tag{24}$$

Formal solution of this expression is the same as carried out in general, and results in Eq. 20.

Why is it that in the EQS limit the electric field is uniform, but that in the MQS limit the magnetic field is not? In the EQS limit, the fundamental field source is ρ_f while for the magnetic field it is $\bar{J}_f$. For this particular problem, where the volume is filled by a uniformly conducting material, there is no accumulation of free charge density, and hence no shielding of $\bar{E}$ from the volume. By contrast, the volume currents can shield the magnetic field from the volume by "skin effect"....the result of having a continuum of inductances and resistances. To have a case study exemplifying how the accumulation of ρ_f (at an interface) can shield out an electric field, consider this same configuration but with the region $0 < x < a$ half filled with conductor ($0 < x < b$) and half free space ($b < x < a$).

Prob. 2.3.4 The conduction constitutive law can be used to eliminate $\bar{E}$ in the law of induction. Then, Eqs. 23b-26b determine $\bar{H}$, $\bar{M}$ and hence $\bar{J}_f$. That the curl of $\bar{E}$ is then specified is clear from the law of induction, Eq. 25b, because all quantities on the right are known from the MQS solution. The divergence of $\bar{E}$ follows by solving the constitutive law for $\bar{E}$ and taking its divergence.

$$\nabla\cdot\bar{E} = \nabla\cdot\left(\frac{\bar{J}_f}{\sigma}\right) - \nabla\times(\bar{v}\times\mu_o\bar{H}) \tag{1}$$

All quantities on the right in this expression have also been found by solving the MQS equations. Thus, both the curl and divergence of $\bar{E}$ are known and $\bar{E}$ is uniquely specified. Given a constitutive law for $\bar{P}$, Gauss Law, Eq. 27b, can be used to evaluate ρ_f.

Prob. 2.4.1 For the given displacement vector in Lagrangian coordinates, the velocity follows from Eq. 2.6.1 as

$$\bar{v} = \frac{\partial \bar{\xi}}{\partial t} = -r_o \Omega \sin(\Omega t + \theta_o)\bar{i}_x + \Omega r_o \cos(\Omega t + \theta_o)\bar{i}_y \qquad (1)$$

In turn, the acceleration follows from Eq. 2.6.2.

$$\bar{a} = \frac{\partial \bar{v}}{\partial t} = -r_o \Omega^2 \left[\cos(\Omega t + \theta_o)\bar{i}_x + \sin(\Omega t + \theta_o)\bar{i}_y\right] \qquad (2)$$

But, in view of Eq. 1, this can also be written in the more familiar form

$$\bar{a} = -\Omega^2 \bar{\xi} \qquad (3)$$

Prob. 2.4.2 From Eq. 2.4.4, it follows that in Eulerian coordinates the acceleration is

$$\bar{a} = \left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y}\right)\bar{i}_x + \left(v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y}\right)\bar{i}_y = -\Omega^2 x \bar{i}_x - \Omega^2 y \bar{i}_y \qquad (4)$$

Using coordinates defined in the problem, this is converted to cylindrical form.

$$\bar{a} = -\Omega^2 r\left[\cos\theta(\cos\theta\,\bar{i}_r - \sin\theta\,\bar{i}_\theta) + \sin\theta(\sin\theta\,\bar{i}_r + \cos\theta\,\bar{i}_\theta)\right] \qquad (5)$$

Because $\cos^2\theta + \sin^2\theta = 1$, it follows that

$$\bar{a} = -\Omega^2 r\,\bar{i}_r \qquad (6)$$

which is equivalent to Eq. 3 of Prob. 2.4.1.

Prob. 2.5.1 By definition, the convective derivative is the time rate of change for an observer moving with the velocity $\bar{v}$, which in this case is $U\bar{i}_x$. Hence,

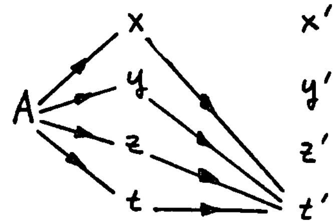
$$\frac{D\Phi}{Dt} = \frac{\partial \Phi'}{\partial t'}$$

and evaluation gives

$$j(\omega - kU)\hat{\Phi} = j\omega'\hat{\Phi}'$$

Because the amplitudes are known to be equal at the same position and time it follows that $\omega - kU = \omega'$. Here, ω is the doppler shifted frequency. The special case where the frequency in the moving frame is zero makes evident why the shift in frequency. In that case $\omega' = 0$ and the moving observer sees a static distribution of Φ that varies sinusoidally with position. The fixed observer sees this distribution moving by with the velocity $U = \omega/k$ and hence observes the frequency kU.

Prob. 2.5.2 To take the derivative with respect to primed variables, say t'; observe in $\bar{A}(x,y,z,t)$, that each variable can in general depend on that variable (say t').



(1)

Thus

$$\frac{\partial A_i}{\partial t'} = \frac{\partial A_i}{\partial t}\frac{\partial t}{\partial t'} + \frac{\partial A_i}{\partial x}\frac{\partial x}{\partial t'} + \frac{\partial A_i}{\partial y}\frac{\partial y}{\partial t'} + \frac{\partial A_i}{\partial z}\frac{\partial z}{\partial t'} \qquad (2)$$

From Eq. 1,

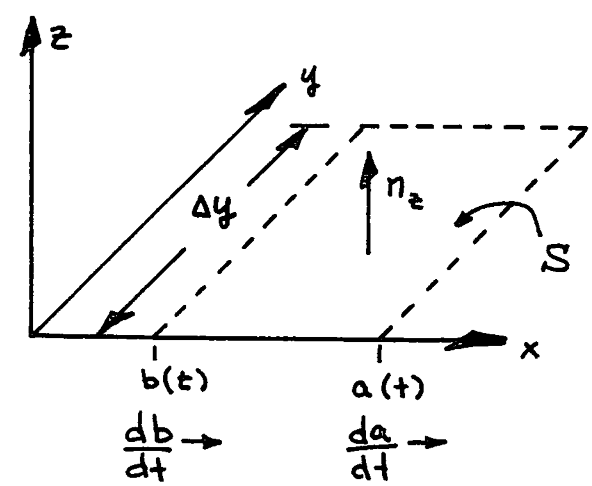
$$\begin{aligned} x &= x' + U_x t' \\ y &= y' + U_y t' \\ z &= z' + U_z t' \\ t &= t' \end{aligned} \quad \Rightarrow \quad \begin{aligned} \frac{\partial t}{\partial t'} &= 1 \\ \frac{\partial x}{\partial t'} &= U_x \\ \frac{\partial y}{\partial t'} &= U_y \\ \frac{\partial z}{\partial z'} &= U_z \end{aligned} \qquad (3)$$

so

$$\frac{\partial A_i}{\partial t'} = \frac{\partial A_i}{\partial t}(1) + \frac{\partial A_i}{\partial x}U_x + \frac{\partial A_i}{\partial y}U_y + \frac{\partial A_i}{\partial z}U_z = \frac{\partial A_i}{\partial t} + \bar{U}\cdot\nabla A_i \qquad (4)$$

Here, if $\bar{A}$ is a vector then A_i is one of its cartesian components. If $A_i \to \psi$, the scalar form is obtained.

Prob. 2.6.1 For use in Eq. 2.6.4, take as A the given one dimensional function with the surface of integration that shown in the figure. The edges at x=a and x=b have the velocities in the x direction indicated. Thus, Eq. 2.6.4 becomes



$$\Delta y \frac{d}{dt}\int f(x,t)\,dx =$$

$$\Delta y\left[\int \frac{\partial f}{\partial t}\,dx + \int \frac{\partial f}{\partial z} v_z\,dx\right] + \Delta y\left[f(a)\frac{da}{dt} - f(b)\frac{db}{dt}\right] \qquad (1)$$

The second term on the right is zero because A has no divergence. Thus, Δy can be divided out to obtain the given one-dimensional form of Leibnitz' rule.

Prob. 2.6.2 a) By Gauss' theorem,

$$\int_V \nabla\cdot\bar{A}\,dV = \oint_S \bar{A}\cdot\bar{i}_n\,da \tag{1}$$

where on S_1, $\bar{i}_n=\bar{n}$, on S_2, $\bar{i}_n=-\bar{n}$ and on the sides $\bar{i}_n$ has the direction of $-\bar{v}\times d\bar{l}$. Also, $\bar{i}_n da$ integrated between S_1 and S_2 is approximated by $-\bar{v}\Delta t\times d\bar{l}$. Thus, it follows that if all integrals are taken at the same instant in time,

$$\int_V \nabla\cdot\bar{A}\,dV = \int_{S_2}\bar{A}(t)\cdot\bar{n}\,da - \int_{S_1}\bar{A}(t)\cdot\bar{n}\,da - \oint_{C_1}\bar{A}\cdot\bar{v}\Delta t\times d\bar{l} \tag{2}$$

b) At any location,

$$\bar{A}(t+\Delta t) = \bar{A}(t) + \frac{\partial\bar{A}}{\partial t}\Delta t + \cdots \tag{3}$$

Thus, the integral over S_2 when it actually has that location gives

$$\int_{S_2}\bar{A}(t+\Delta t)\cdot\bar{n}\,da = \int_{S_2}\bar{A}(t)\cdot\bar{n}\,da + \int_{S_2}\frac{\partial\bar{A}}{\partial t}\Delta t\cdot\bar{n}\,da + \cdots \tag{4}$$

Because S_2 differs from S_1 by terms of higher order than Δt , the second integral can be evaluated to first order in Δt on S_1.

$$\int_{S_2}\bar{A}(t+\Delta t)\cdot\bar{n}\,da = \int_{S_2}\bar{A}(t)\cdot\bar{n}\,da + \int_{S_1}\frac{\partial\bar{A}}{\partial t}\Delta t\cdot\bar{n}\,da \tag{5}$$

c) For the elemental volume pictured, the height is $\Delta t\,\bar{v}\cdot\bar{n}$, while the area of the base is da , so to first order in Δt , the volume integral reduces to

$$\int_V \nabla\cdot\bar{A}\,dV \cong \int_{S_1}\nabla\cdot\bar{A}\,\Delta t\,\bar{v}\cdot\bar{n}\,da \tag{6}$$

d) What is desired is

$$\frac{d}{dt}\int_S \bar{A}\cdot\bar{n}\,da = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{S_2}\bar{A}(t+\Delta t)\cdot\bar{n}\,da - \int_{S_1}\bar{A}(t)\cdot\bar{n}\,da\right] \tag{7}$$

Substitution from Eq. 5 into this expression gives

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{S_2}\bar{A}(t)\cdot\bar{n}\,da + \int_{S_1}\frac{\partial\bar{A}}{\partial t}(t)\,\Delta t\cdot\bar{n}\,da - \int_{S_1}\bar{A}(t)\cdot\bar{n}\,da\right] \tag{8}$$

The first and last terms on the right can be replaced using Eq. 2

Prob. 2.6.2(cont.)

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_V \nabla\cdot\bar{A}\,dV + \oint_C \bar{A}\cdot\bar{v}\Delta t\times d\bar{l} + \int_{S_1}\frac{\partial\bar{A}}{\partial t}\Delta t\cdot\bar{n}\,da\right] \tag{9}$$

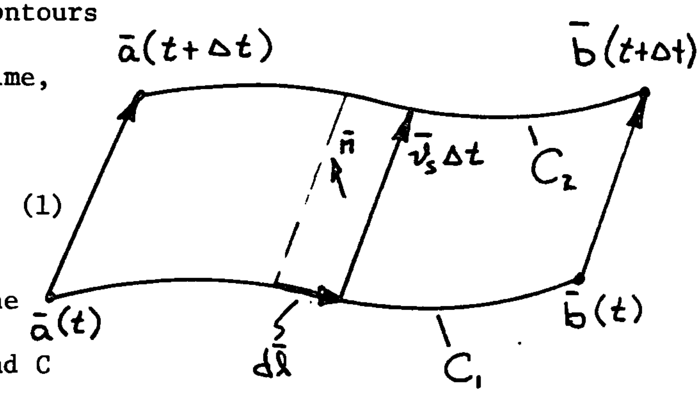
Finally, given that $\bar{A}\cdot\bar{v}\Delta t\times d\bar{l} = \bar{A}\times\bar{v}\Delta t\cdot d\bar{l}$, Eq. 6 is substituted into this expression to obtain

$$\frac{d}{dt}\int_S \bar{A}\cdot\bar{n}\,da = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{S_1}\nabla\cdot\bar{A}\,\Delta t\,\bar{v}\cdot\bar{n}\,da + \int_{S_1}\frac{\partial\bar{A}}{\partial t}\Delta t\cdot\bar{n}\,da + \Delta t\int_C \bar{A}\times\bar{v}\cdot d\bar{l}\right] \tag{10}$$

With Δt divided out, this is the desired Leibnitz rule generalized to three dimensions.

Prob. 2.6.3 Given the geometry of contours C_1 and C_2, if $\bar{A}$ is evaluated at one time, t, Stoke's theorem applies

$$\int_S \nabla\times\bar{A}\cdot\bar{n}\,da = \oint_C \bar{A}\cdot d\bar{l} \tag{1}$$



Here, S is the surface swept out by the open contour during the interval Δt and C is composed of C_1, C_2 and the side segments represented to first order in Δt by $\bar{v}_s(\bar{b}(t),t)\Delta t$ and $\bar{v}_s(\bar{a}(t),t)\Delta t$. Note that for Δt small, $\bar{n} = d\bar{l}\times\bar{v}_s\Delta t$ with $\bar{v}_s$ evaluated at time t. Thus, to linear terms in Δt , Eq. 1 becomes

$$\int_{\bar{a}(t)}^{\bar{b}(t)} \nabla\times\bar{A}\Big|_{C_1,t}\cdot d\bar{l}\times\bar{v}_s\Big|_{C_1,t}\Delta t = \int_{\bar{a}}^{\bar{b}}\bar{A}\Big|_t\cdot d\bar{l} + \bar{A}\cdot\bar{v}_s\Big|_{\bar{b}(t),t}\Delta t - \int_{\bar{a}(t+\Delta t)}^{\bar{b}(t+\Delta t)}\bar{A}\Big|_t\cdot d\bar{l} - \bar{A}\cdot\bar{v}_s\Big|_{a(t),t}\Delta t \tag{2}$$

Note that, again to linear terms in Δt,

$$\int_{\bar{a}(t+\Delta t)}^{\bar{b}(t+\Delta t)} \bar{A}(t+\Delta t)\cdot d\bar{l} \cong \int_{\bar{a}(t+\Delta t)}^{\bar{b}(t+\Delta t)} \bar{A}\Big|_t\cdot d\bar{l} + \int_{\bar{a}(t+\Delta t)}^{\bar{b}(t+\Delta t)}\frac{\partial\bar{A}}{\partial t}\Big|_t\Delta t\cdot d\bar{l} \tag{3}$$

Prob. 2.6.3 (cont.)

The first term on the right in this expression is substituted for the third one on the right in Eq. 2, which then becomes

$$\int_{\bar{a}(t)}^{\bar{b}(t)} \nabla\times\bar{A}\Big|_t \cdot d\bar{\ell}\times\bar{v}_s\Big|_{C_1,t}\Delta t = \int_{\bar{a}(t)}^{\bar{b}(t)} \bar{A}\Big|_{C_1,t}\cdot d\bar{\ell} + \bar{A}\cdot\bar{v}_s\Big|_{b(t),t}\Delta t \tag{4}$$

$$- \int_{\bar{a}(t+\Delta t)}^{\bar{b}(t+\Delta t)} \bar{A}(t+\Delta t)\cdot d\bar{\ell} + \int_{\bar{a}(t+\Delta t)}^{\bar{b}(t+\Delta t)} \frac{\partial\bar{A}}{\partial t}\Big|_{C_1,t}\Delta t\cdot d\bar{\ell} - \bar{A}\cdot\bar{v}_s\Big|_{a(t),t}\Delta t$$

The first and third terms on the right comprise what is required to evaluate the derivative. Note that because the integrand of the fourth term is already first order in Δt, the end points can be evaluated when t=t.

$$\frac{d}{dt}\int_{\bar{a}(t)}^{\bar{b}(t)} \bar{A}\cdot d\bar{\ell} = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\int_{\bar{a}(t)}^{\bar{b}(t)}\frac{\partial\bar{A}}{\partial t}\Delta t\cdot d\bar{\ell}\right.$$

$$\left. + \bar{A}\cdot\bar{v}_s\Big|_{b(t),t}\Delta t - \bar{A}\cdot\bar{v}_s\Big|_{a(t),t}\Delta t + \int_{\bar{a}(t)}^{\bar{b}(t)}\nabla\times\bar{A}\cdot\bar{v}_s\times d\bar{\ell}\,\Delta t\right\} \tag{5}$$

The sign of the last term has been reversed because the order of the cross product is reversed. The Δt cancels out on the right-hand side and the expression is the desired generalized Leibnitz rule for a time-varying contour integration.

Prob. 2.8.1 a) In the steady state and in the absence of a conduction current, $\bar{J}_f$, Ampere's law requires that

$$\nabla\times\bar{H} = \nabla\times(\bar{P}\times\bar{v}) \tag{1}$$

so one solution follows by setting the arguments equal.

$$\bar{H} = \bar{P}\times\bar{v} = -\left(\frac{P_o a}{\pi}\right)U\sin\left(\frac{\pi x}{a}\right)\bar{i}_z \tag{2}$$

Because the boundary conditions, $H_z(x=\pm a)=0$ are also satisfied, this is the required solution. For different boundary conditions, a "homogeneous" solution would have to be added.

Prob. 2.8.2 (cont.)

b) The polarization current density follows by direct evaluation.

$$\bar{J}_p = \nabla\times(\bar{P}\times\bar{v}) = \rho_o U \cos(\pi x/a)\bar{i}_y \tag{3}$$

Thus, Ampere's law reads

$$\nabla\times\bar{H} = -\frac{\partial H_z}{\partial x}\bar{i}_y = \rho_o U \cos(\pi x/a)\bar{i}_y \tag{4}$$

where it has been assumed that $\partial(\ )/\partial y$ and $\partial(\ )/\partial z = 0$. Integration then gives the same result as in Eq. 2.

c) The polarization charge is

$$\rho_p = -\nabla\cdot\bar{P} = -\frac{\partial P_x}{\partial x} = \rho_o \cos(\pi x/a) \tag{5}$$

and it can be seen that in this case, $\bar{J}_p = U\rho_p \bar{i}_y$. This is a special case because in general the polarization current is

$$\nabla\times(\bar{P}\times\bar{v}) \equiv \bar{P}\,\nabla\cdot\bar{v} - \bar{v}\,\nabla\cdot\bar{P} + \bar{v}\cdot\nabla\bar{P} - \bar{P}\cdot\nabla\bar{v} \tag{6}$$

In this example, the first and last terms vanish because the motion is rigid body, while (because there is no y variation), the next to last term $\bar{v}\cdot\nabla\bar{P} = U\partial\bar{P}/\partial y = 0$. The remaining term is simply $\rho_p\bar{v}$.

Prob. 2.9.1 a) With $\bar{M}$ the only source of $\bar{H}$, it is reasonable to presume that $\bar{H}$ only depends on x and it follows from Gauss' law for $\bar{H}$ that

$$\nabla\cdot\bar{H} = -\nabla\cdot\bar{M} \Rightarrow \frac{\partial H_x}{\partial x} = \frac{\rho_o}{\mu_o}\cos(\pi x/a) \Rightarrow H_x = \frac{\rho_o a}{\mu_o \pi}\sin\left(\frac{\pi x}{a}\right) \tag{1}$$

b) A solution to Faraday's law that also satisfies the boundary conditions follows by simply setting the arguments of the curls equal.

$$\bar{E} = -\mu_o\bar{M}\times\bar{v} = \frac{\rho_o a}{\pi}U\sin\left(\frac{\pi x}{a}\right)\bar{i}_z \tag{2}$$

c) The current is zero because $\bar{E}'=0$. To see this, use the results of Eqs. 1 and 2 to evaluate

$$\bar{E}' = \bar{E} + \bar{v}\times\mu_o\bar{H} = \bar{i}_z\left[\frac{\rho_o a}{\pi}U\sin\left(\frac{\pi x}{a}\right) - \frac{\rho_o a}{\pi}U\sin\left(\frac{\pi x}{a}\right)\right] = 0 \tag{3}$$

Prob. 2.11.1 With regions to the left, above and below the movable electrode denoted by (a), (b) and (c) respectively, the electric fields there (with up defined as positive) are

$$E_a = (v_2 - v_1)/b \; ; \; E_b = -v_1/(b-\xi_2) \; ; \; E_c = v_2/\xi_2 \tag{1}$$

On the upper electrode, the total charge is the area $d(a-\xi_1)$ times the charge per unit area on the left section of the electrode, $-\epsilon_o E_a$, plus the area $d\xi_1$ times the charge per unit area on the right section, $-\epsilon_o E_b$. The charge on the lower electrode follows similarly so that the capacitance matrix is

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = d\epsilon_o \begin{bmatrix} \frac{a-\xi_1}{b} + \frac{\xi_1}{b-\xi_2} & -\frac{(a-\xi_1)}{b} \\ -\frac{(a-\xi_1)}{b} & \frac{a-\xi_1}{b} + \frac{\xi_1}{\xi_2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{2}$$

Prob. 2.12.1 Define regions (a) and (b) as between the two coils and inside the inner one respectively and it follows that the magnetic fields are uniform in each region and given by

$$H_a = \frac{i_1}{d} \; ; \; H_b = H_a + \frac{i_2}{d} = \frac{i_1}{d} + \frac{i_2}{d} \tag{1}$$

These fields are defined as positive into the paper. Note that they satisfy Ampere's law and the divergence condition in the volume and the jump and boundary conditions at the boundaries. For the contours as defined, the normal to the surface defining λ_1 is into the paper. The fields are uniform, so the surface integral is carried out by multiplying the flux density, $\mu_o H$, by the appropriate area. For example, λ_1 is found as

$$\lambda_1 = \mu_o \frac{i_1}{d}\pi(a^2 - \xi^2) + \mu_o\left(\frac{i_1}{d} + \frac{i_2}{d}\right)\pi\xi^2 \tag{2}$$

Thus, the flux linkages are

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \frac{\mu_o \pi a^2}{d} & \frac{\mu_o \pi \xi^2}{d} \\ \frac{\mu_o \pi \xi^2}{d} & \frac{\mu_o \pi \xi^2}{d} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \tag{3}$$

Prob. 2.13.1 It is a line integration in the state-space (v_1, v_2, ξ_1, ξ_2) that is called for. The system has already been assembled mechanically, so the displacements (ξ_1, ξ_2) are fixed. The remaining path of integration in the space (v_1, v_2) is carried out by raising v_1 to its final value with $v_2=0$ and then raising v_2 with v_1 fixed (so that $\delta v_1=0$) at its final value. Thus,

$$w' = \int q_1 \delta v_1 + q_2 \delta v_2 = \int_0^{v_1} q_1(v_1', 0, \xi_1, \xi_2)\delta v_1' + \int_0^{v_2} q_2(v_1, v_2', \xi_1, \xi_2)\delta v_2' \quad (1)$$

and with the introduction of the capacitance matrix,

$$w' = \frac{1}{2} C_{11} v_1^2 + C_{21} v_1 v_2 + \frac{1}{2} C_{22} v_2^2 \quad (2)$$

Note that $C_{21}=C_{12}$.

Prob. 2.13.2 Even with the nonlinear dielectric, the electric field between the electrodes is simply v/b. Thus, the surface charge on the lower electrode, where there is free space, is $D= \epsilon_o E= \epsilon_o v/b$, while that adjacent to the dielectric is

$$D = \frac{\epsilon_o v}{b} + v/b\,\alpha_1 \sqrt{\alpha_2^2 + \left(\frac{v}{b}\right)^2} \quad (1)$$

It follows that the net charge is

$$q = d a \epsilon_o \frac{v}{b} + \frac{d \xi v}{\alpha_1 \sqrt{\alpha_2^2 b^2 + v^2}} \quad (2)$$

so that

$$w' = \int_0^v q \delta v = \frac{\epsilon_o}{2}\frac{d}{b}(a) v^2 + \frac{d\xi}{\alpha_1}\left[(\alpha_2^2 b^2 + v^2)^{\frac{1}{2}} - \alpha_2 b\right] \quad (3)$$

Prob. 2.14.1 a) To find the energy, it is first necessary to invert the terminal relations found in Prob. 2.14.1. Cramer's rule yields

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} \dfrac{d}{\mu_o \pi (a^2-\xi^2)} & \dfrac{-d}{\mu_o \pi (a^2-\xi^2)} \\ \dfrac{-d}{\mu_o \pi (a^2-\xi^2)} & \dfrac{d}{\mu_o \pi (a^2-\xi^2)}\left(\dfrac{a}{\xi}\right)^2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \quad (1)$$

Integration of Eq. 2.14.11 in (λ_1, λ_2) space can be carried out along any path. But, in particular, integrate on λ_1 with $\lambda_2=0$. Then, with λ_1 at its final value, integrate on λ with $\delta\lambda_1=0$.

Prob. 2.14.1 (cont.)

$$w = \int_0^{\lambda_1} i_1(\lambda_1', 0)\,d\lambda_1' + \int_0^{\lambda_2} i_2(\lambda_1, \lambda_2')\,d\lambda_2' \tag{2}$$

$$= \frac{1}{2}\left[\frac{d}{\mu_0\pi(a^2-\xi^2)}\right]\lambda_1^2 - \left[\frac{d}{\mu_0\pi(a^2-\xi^2)}\right]\lambda_1\lambda_2 + \frac{1}{2}\left[\frac{d}{\mu_0\pi(a^2-\xi^2)}\left(\frac{a}{\xi}\right)^2\right]\lambda_2^2$$

b) The coenergy is found from Eq. 2.14.12 where the flux linkages as given in the solution to Prob. 2.12.1 can be used directly. Now, the integration is in (i_1, i_2) space, and is carried out as in part (a), but with the i's playing the role of the λ's.

$$w' = \int_0^{i_1} \lambda_1(i_1', 0)\,di_1' + \int_0^{i_2} \lambda_2(i_1, i_2')\,di_2' \tag{3}$$

$$= \frac{1}{2}\left(\frac{\mu_0\pi a^2}{d}\right)i_1^2 + \frac{\mu_0\pi\xi^2}{d}i_1 i_2 + \frac{1}{2}\left(\frac{\mu_0\pi\xi^2}{d}\right)i_2^2$$

Prob. 2.15.1 Following the outlined procedure,

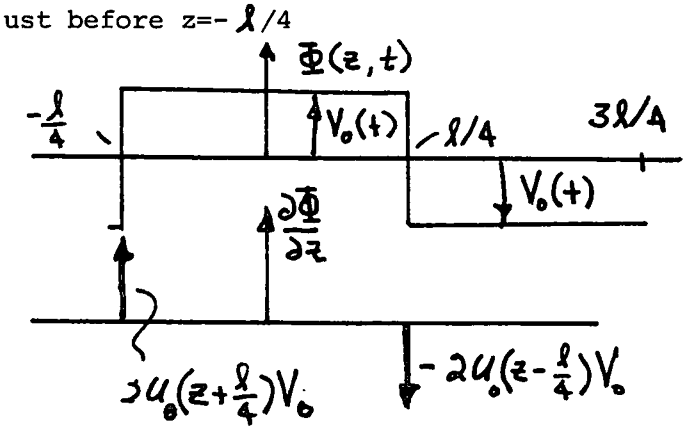
$$\int_z^{z+\ell} \Phi(z,t)\,e^{jk_m z}\,dz = \int_z^{z+\ell} \sum_{n=-\infty}^{+\infty} \tilde{\Phi}_n\,e^{j(k_m - k_n)z}\,dz \tag{1}$$

Each term in the series is integrated to give

$$= \sum_{n=-\infty}^{+\infty} \frac{\tilde{\Phi}_n}{j} e^{j\frac{2\pi}{\ell}(m-n)z}\left[e^{j2\pi(m-n)} - 1\right] \Big/ \frac{2\pi}{\ell}(m-n) \tag{2}$$

Thus, for $m \neq n$, all terms vanish. The term m=n is evaluated by either taking the limit $m \to n$ of Eq. 2 or returning to Eq. 1 to see that the right hand side is simply $\tilde{\Phi}_m \ell$. Thus, solution for $\tilde{\Phi}_m$ gives Eq. 8.

Prob. 2.15.2 One period of the distribution is sketched as a function of z as shown. Note that the function starts just before $z = -\ell/4$ and terminates just before $z = 3\ell/4$. The coefficients follow directly from Eq. 8. Especially for ramp functions, it is often convenient to make use of the fact that



$$\frac{\partial \Phi}{\partial z} \longleftrightarrow -jk_m\tilde{\Phi}_m \tag{1}$$

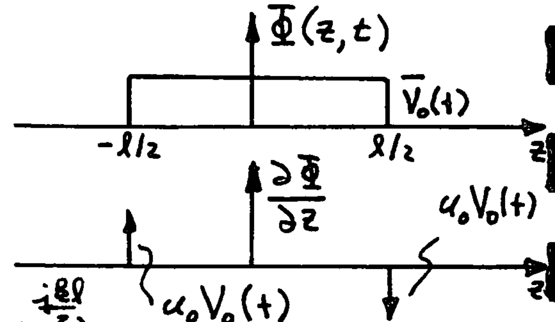
Prob. 2.15.2 (cont.)

and find the coefficients of the derivative of $\Phi(z,t)$, as shown in the sketch. Thus,

$$-jk_m\tilde{\Phi}_m = \frac{1}{l}\int_{-l/4}^{l/4}\frac{\partial\Phi}{\partial z}e^{jk_m z}\,dz = \frac{2V_o}{l}\left(e^{-j\frac{k_m l}{4}} - e^{j\frac{k_m l}{4}}\right) \tag{2}$$

and it follows that the coefficients are as given. Note that m=0 must give $\tilde{\Phi}_m = 0$ because there is no space average to the potential. That the other even components vanish is implicite to Eq. 2.

<u>Prob. 2.15.3</u> The dependence on z of Φ and its spatial derivative are as sketched. Because the transform of $\partial\Phi/\partial z \leftrightarrow -jk\tilde{\Phi}$, the integration over the two impulse functions gives simply



$$-jk\tilde{\Phi} = \int_{-\infty}^{+\infty}\frac{\partial\Phi}{\partial z}e^{jkz}\,dz = 2V_o\frac{\left(e^{-j\frac{kl}{2}} - e^{j\frac{kl}{2}}\right)}{2} \tag{1}$$

Solution of this expression for $\tilde{\Phi}$ results in the given transform. More direct, but less convenient, is the direct evaluation of Eq. 2.15.10.

<u>Prob. 2.15.4</u> Evaluation of the required space average is carried out by fixing attention on one value of n in the infinite series on n and considering the terms of the infinite series on m. Thus,

$$\langle AB\rangle_z = \sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}\tilde{A}_n\tilde{B}_m\frac{1}{l}\int_z^{z+l}\exp -j(k_n+k_m)\,dz \tag{1}$$

$$= \sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}\tilde{A}_n\tilde{B}_m\frac{j}{(n+m)}\exp -3\pi j(n+m)$$

Thus, all terms are zero except the one having n=-m. That term is best evaluated using the original expression to carry out the integration. Thus,

$$\langle AB\rangle_z = \sum_{n=-\infty}^{+\infty}\tilde{A}_n\tilde{B}_{-n} \tag{2}$$

Because the Fourier series is required to be real, $\tilde{B}_{-n} = \tilde{B}_n^*$ and hence the given expression of Eq. 2.15.17 follows.

Prob. 2.16.1 To be formal about deriving transfer relations of Table 2.16.1, start with Eq. 2.16.14

$$\tilde{\Phi} = \tilde{\Phi}_1 \sinh \gamma x + \tilde{\Phi}_2 \cosh \gamma x \tag{1}$$

and require that $\tilde{\Phi}(x=\Delta) \equiv \tilde{\Phi}^\alpha$, $\tilde{\Phi}(x=0) \equiv \tilde{\Phi}^\beta$. Thus,

$$\begin{bmatrix} \sinh \gamma\Delta & \cosh \gamma\Delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{\Phi}_1 \\ \tilde{\Phi}_2 \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^\alpha \\ \tilde{\Phi}^\beta \end{bmatrix} \tag{2}$$

Inversion gives (by Cramer's rule)

$$\begin{bmatrix} \tilde{\Phi}_1 \\ \tilde{\Phi}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\sinh \gamma\Delta} & -\coth \gamma\Delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^\alpha \\ \tilde{\Phi}^\beta \end{bmatrix} \tag{3}$$

Because $\tilde{D}_x = -\epsilon \partial \tilde{\Phi}/\partial x$, it follows for Eq. 1 that

$$\tilde{D}_x = -\epsilon\gamma \left(\tilde{\Phi}_1 \cosh \gamma x + \tilde{\Phi}_2 \sinh \gamma x \right) \tag{4}$$

Evaluation at the respective boundaries gives

$$\begin{bmatrix} \tilde{D}_x^\alpha \\ \tilde{D}_x^\beta \end{bmatrix} = -\epsilon\gamma \begin{bmatrix} \cosh \gamma\Delta & \sinh \gamma\Delta \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\Phi}_1 \\ \tilde{\Phi}_2 \end{bmatrix} \tag{5}$$

Finally, substitution of Eq. 3 for the column matrix on the right in Eq. 5 gives

$$\begin{bmatrix} \tilde{D}_x^\alpha \\ \tilde{D}_x^\beta \end{bmatrix} = -\epsilon\gamma \begin{bmatrix} \cosh \gamma\Delta & \sinh \gamma\Delta \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{1}{\sinh \gamma\Delta} & -\coth \gamma\Delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^\alpha \\ \tilde{\Phi}^\beta \end{bmatrix}$$

$$= -\epsilon\gamma \begin{bmatrix} \coth \gamma\Delta & -\dfrac{1}{\sinh \gamma\Delta} \\ \dfrac{1}{\sinh \gamma\Delta} & -\coth \gamma\Delta \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^\alpha \\ \tilde{\Phi}^\beta \end{bmatrix} \tag{6}$$

which is Eq. (a) of Table 2.16.1.

Prob. 2.16.1 (cont.)

The second form, Eq. (b), is obtained by applying Cramer's rule to the inversion of Eq. 8. Note that the determinant of the coefficients is

$$\mathrm{Det} = -\coth^2\gamma\Delta + \frac{1}{\sinh^2\gamma\Delta} = \frac{1-\cosh^2\gamma\Delta}{\sinh^2\gamma\Delta} = -1 \tag{7}$$

so

$$\begin{bmatrix} \tilde{\Phi}^\alpha \\ \tilde{\Phi}^\beta \end{bmatrix} = \frac{1}{\epsilon\gamma}\begin{bmatrix} -\coth\gamma\Delta & \dfrac{1}{\sinh\gamma\Delta} \\ \dfrac{-1}{\sinh\gamma\Delta} & \coth\gamma\Delta \end{bmatrix}\begin{bmatrix} \tilde{D}_x^\alpha \\ \tilde{D}_x^\beta \end{bmatrix} \tag{8}$$

Prob. 2.16.2 For the limit m=0,k=0, solutions are combined to satisfy the potential constraints by Eq. 2.16.20, and it follows that the electric displacement is

$$\tilde{D}_r = -\epsilon\frac{\partial\tilde{\Phi}}{\partial r} = -\epsilon\,\tilde{\Phi}^\alpha\frac{\left(\frac{1}{r}\right)}{\ln\left(\frac{\alpha}{\beta}\right)} + \epsilon\,\tilde{\Phi}^\beta\frac{\left(\frac{1}{r}\right)}{\ln\left(\frac{\alpha}{\beta}\right)} \tag{1}$$

This is evaluated at the respective boundaries to give Eq. (a) of Table 2.16.2 with f_m and g_m as defined for k=0,m=0.

For $k=0, m\neq 0$, the correct combination of potentials is given by Eq. 2.16.21. It follows that

$$\tilde{D}_r = \epsilon m\left\{\frac{\tilde{\Phi}^\alpha}{\beta}\left[\frac{\left(\frac{\beta}{r}\right)^{m+1}+\left(\frac{r}{\beta}\right)^{m-1}}{\left(\frac{\beta}{\alpha}\right)^m-\left(\frac{\alpha}{\beta}\right)^m}\right] - \frac{\tilde{\Phi}^\alpha}{\alpha}\left[\frac{\left(\frac{r}{\alpha}\right)^{m-1}+\left(\frac{\alpha}{r}\right)^{m+1}}{\left(\frac{\beta}{\alpha}\right)^m-\left(\frac{\alpha}{\beta}\right)^m}\right]\right\} \tag{2}$$

Evaluation of this expression at the respective boundaries gives Eqs. (a) of Table 2.16.2 with entries f_m and g_m as defined for the case k=0,m=0.

For $k\neq 0, m\neq 0$, the potential is given by Eq. 2.16.25. Thus, the electric displacement is

$$\tilde{D}_r = -jk\left\{\tilde{\Phi}^\alpha\frac{[H_m(jk\beta)J_m'(jkr) - J_m(jk\beta)H_m'(jkr)]}{[H_m(jk\beta)J_m(jk\alpha) - J_m(jk\beta)H_m(jk\alpha)]}\right. \tag{3}$$

$$\left.+\tilde{\Phi}^\beta\frac{[J_m(jk\alpha)H_m'(jkr) - H_m(jk\alpha)J_m'(jkr)]}{[H_m(jk\beta)J_m(jk\alpha) - J_m(jk\beta)H_m(jk\alpha)]}\right\}$$

and evaluation at the respective boundaries gives Eqs. (a) of the table with f_m and g_m as defined in terms of H_m and J_m. To obtain g_m in the form given,

Prob. 2.16.2 (cont.)

use the identity in the footnote to the table. These entries can be written in terms of the modified functions, K_m and I_m by using Eqs. 2.16.22.

In taking the limit where the inside boundary goes to zero, it is necessary to evaluate

$$\tilde{D}_r^\alpha = \epsilon\left[f_m(0,\alpha)\,\tilde{\Phi}^\alpha + g_m(\alpha,0)\,\tilde{\Phi}^\beta \right] \tag{4}$$

Because K_m and H_m approach infinity as their arguments go to zero, $g_m(\alpha,0) \rightarrow 0$. Also, in the expression for f_m in terms of the functions H_m and J_m, the first term in the numerator dominates the second while the second term in the denominator dominates the first. Thus, f_m becomes

$$f_m(0,\alpha) \rightarrow jk\frac{H_m(jk\beta)J_m'(jk\alpha)}{-J_m(jk\alpha)H_m(jk\beta)} \tag{5}$$

and with the use of Eqs. 2.16.22, this expression becomes the one given in the table.

In the opposite extreme, where the outside boundary goes to infinity, the desired relation is

$$\tilde{D}_r^\beta = \epsilon\left[g_m(\beta,\infty)\,\tilde{\Phi}^\alpha + f_m(\infty,\beta)\,\tilde{\Phi}^\beta \right] \tag{6}$$

Here, note that I_m and J_m (and hence I_m' and J_m') go to infinity as their arguments become large. Thus, $g_m(\beta,\infty) \rightarrow 0$ and in the expressions for f_m, the second term in the numerator and first term in the denominator dominate to give

$$f_m(\infty,\beta) \rightarrow -jk\frac{J_m(jkx)H_m'(jk\beta)}{J_m(jkx)H_m'(jk\beta)} = -jk\frac{H_m'(jk\beta)}{H_m(jk\beta)} \tag{7}$$
$$= \frac{-k K_m'(k\beta)}{K_m(k\beta)}$$

To invert these results and determine relations in the form of Eqs. (b) of the table, note that the first case, k=0,m=0 involves solutions that are not independent. This reflects the physical fact that it is only the potential difference that matters in this limit and that $(\tilde{\Phi}^\alpha, \tilde{\Phi}^\beta)$ are not really independent variables. Mathematically, the inversion process leads to an infinite determinant.

In general, Cramer's rule gives the inversion of Eqs. (a) as

Prob. 2.16.2 (cont.)

$$F_m(\beta,\alpha) = \epsilon f_m(\alpha,\beta)/\mathrm{Det}\ ;\ F_m(\alpha,\beta) = \epsilon f_m(\beta,\alpha)$$

$$G_m(\beta,\alpha) = -\epsilon g_m(\beta,\alpha)/\mathrm{Det}\ ;\ G_m(\alpha,\beta) = -\epsilon g_m(\alpha,\beta)$$

where $\mathrm{Det} = \epsilon\{f_m(\beta,\alpha)f_m(\alpha,\beta) - g_m(\beta,\alpha)g_m(\alpha,\beta)$

Prob. 2.16.3 The outline for solving this problem is the same as for Prob. 2.16.2. The starting point is Eq. 2.16.36 rather than the three potential distributions representing limiting cases and the general case in Prob. 2.16.2.

Prob. 2.16.4 a) With the z-t dependence $\exp j(\omega t-kz)$, Maxwell's equations become

$$\nabla\cdot\bar{E} = 0 \Rightarrow \frac{\partial\hat{E}_x}{\partial x} = jk\hat{E}_z \tag{1}$$

$$\nabla\cdot\bar{H} = 0 \Rightarrow \frac{\partial\hat{H}_x}{\partial x} = jk\hat{H}_z \tag{2}$$

$$\nabla\times\bar{E} = -\frac{\partial\mu_o\bar{H}}{\partial t} \Rightarrow \begin{cases} jk\hat{E}_y = -j\omega\mu_o\hat{H}_x & (3)\\ -jk\hat{E}_x - \dfrac{\partial\hat{E}_z}{\partial x} = -j\omega\mu_o\hat{H}_y & (4)\\ \dfrac{\partial\hat{E}_y}{\partial x} = -j\omega\mu_o\hat{H}_z & (5)\end{cases}$$

$$\nabla\times\bar{H} = \frac{\partial\epsilon_o\bar{E}}{\partial t} \Rightarrow \begin{cases} jk\hat{H}_y = j\omega\epsilon\hat{E}_x & (6)\\ -jk\hat{H}_x - \dfrac{\partial\hat{H}_z}{\partial x} = j\omega\epsilon\hat{E}_y & (7)\\ \dfrac{\partial\hat{H}_y}{\partial x} = j\omega\epsilon\hat{E}_z & (8)\end{cases}$$

The components $\hat{E}_x$, $\hat{E}_y$, $\hat{H}_x$, $\hat{H}_y$ can be written in terms of $\hat{E}_z$ and $\hat{H}_z$ as follows. Equations 3 and 7 combine to $\left(\gamma^2 \equiv k^2 - (\omega/c)^2\right)$

$$\hat{H}_x = \frac{jk}{\gamma^2}\frac{\partial\hat{H}_z}{\partial x} \tag{9}$$

and Eqs. 4 and 6 give

$$\hat{E}_x = \frac{jk}{\gamma^2}\frac{\partial\hat{E}_z}{\partial x} \tag{10}$$

As a result, Eqs. 6 and 3 give

$$\hat{H}_y = \frac{j\omega\epsilon_o}{\gamma^2}\frac{\partial\hat{E}_z}{\partial x} \tag{11}$$

$$\hat{E}_y = \frac{-j\omega\mu_o}{\gamma^2}\frac{\partial\hat{H}_z}{\partial x} \tag{12}$$

Combining Ampere's and Faraday's laws gives

$$c^2\nabla^2\begin{pmatrix}\bar{H}\\ \bar{E}\end{pmatrix} = \frac{\partial^2}{\partial t^2}\begin{pmatrix}\bar{H}\\ \bar{E}\end{pmatrix} \tag{13}$$

Thus, it follows that

$$\frac{\partial^2}{\partial x^2}\begin{pmatrix}\hat{H}_z\\ \hat{E}_z\end{pmatrix} + \gamma^2\begin{pmatrix}\hat{H}_z\\ \hat{E}_z\end{pmatrix} = 0 \tag{14}$$

Prob. 2.16.4(cont.)

b) Solutions to Eqs. 14 satisfying the boundary conditions are

$$\begin{bmatrix}\hat{H}_z\\ \hat{E}_z\end{bmatrix}=\begin{bmatrix}\hat{H}_z^\alpha\\ \hat{E}_z^\alpha\end{bmatrix}\frac{\sinh\gamma x}{\sinh\gamma\Delta}-\begin{bmatrix}\hat{H}_z^\beta\\ \hat{E}_z^\beta\end{bmatrix}\frac{\sinh\gamma(x-\Delta)}{\sinh\gamma\Delta} \qquad \begin{matrix}(15)\\(16)\end{matrix}$$

c) Use is now made of Eqs. 9 and 10 to obtain

$$\hat{E}_x=\frac{jk}{\gamma}\left\{\hat{E}_z^\alpha\frac{\cosh\gamma x}{\sinh\gamma\Delta}-\hat{E}_z^\beta\frac{\cosh\gamma(x-\Delta)}{\sinh\gamma\Delta}\right\} \qquad (17)$$

$$\hat{H}_x=\frac{jk}{\gamma}\left\{\hat{H}_z^\alpha\frac{\cosh\gamma x}{\sinh\gamma\Delta}-\hat{H}_z^\beta\frac{\cosh\gamma(x-\Delta)}{\sinh\gamma\Delta}\right\} \qquad (18)$$

Also, from Eqs. 3 and 6,

$$\hat{E}_y=-\frac{\omega\mu_o}{k}\hat{H}_x \qquad (19)$$

$$\hat{H}_y=\frac{\omega\epsilon_o}{k}\hat{E}_x \qquad (20)$$

Evaluation of these expressions at the respective boundaries gives the transfer relations summarized in the problem.

d) In the quasistatic limit, times of interest, $1/\omega$, are much longer than the propagation time of an electromagnetic wave in the system. For propagation across the guide, this time is $\Delta/c=\Delta\sqrt{\mu_o\epsilon_o}$. Thus,

$$\Delta\gamma\simeq k\Delta \qquad (21)$$

Note that $k\Delta$ must be larger than τ_{em}/τ, but too large a value of $k\Delta$ means no interaction between the two boundaries. Now, with $\gamma\to k$, $\hat{E}_z=jk\hat{\Phi}$ and $\hat{H}_z=jk\hat{\psi}$, the relations break into the quasi-static transfer relations.

$$\begin{bmatrix}\epsilon_o\hat{E}_x^\alpha\\ \epsilon_o\hat{E}_x^\beta\end{bmatrix}=\epsilon_o k\begin{bmatrix}-\coth k\Delta & \dfrac{1}{\sinh k\Delta}\\ -\dfrac{1}{\sinh k\Delta} & \coth k\Delta\end{bmatrix}\begin{bmatrix}\hat{\Phi}^\alpha\\ \hat{\Phi}^\beta\end{bmatrix} \qquad (22)$$

$$\begin{bmatrix}\mu_o\hat{H}_x^\alpha\\ \mu_o\hat{H}_x^\beta\end{bmatrix}=\mu_o k\begin{bmatrix}-\coth k\Delta & \dfrac{1}{\sinh k\Delta}\\ -\dfrac{1}{\sinh k\Delta} & \coth k\Delta\end{bmatrix}\begin{bmatrix}\hat{\psi}^\alpha\\ \hat{\psi}^\beta\end{bmatrix} \qquad (23)$$

Prob. 2.16.4(cont)

e) Transverse electric(TE) and transverse magnetic(TM) modes between perfectly conducting plates satisfy the boundary conditions

$$\text{(TM)} \quad (H_z = 0) \quad \hat{E}_z^{\alpha\atop\beta} = 0 \tag{24}$$

$$\text{(TE)} \quad (E_z = 0) \quad \hat{H}_x^{\alpha\atop\beta} = 0 \tag{25}$$

where the latter condition is expressed in terms of H_z by using Eqs. 12 and 7. Because the modes separate, it is possible to examine them separately. The electric relations are already in the appropriate form for considering the TM modes. The magnetic ones are inverted to obtain

$$\begin{bmatrix} \mu_0 \hat{H}_z^\alpha \\ \mu_0 \hat{H}_z^\beta \end{bmatrix} = -\frac{\gamma\mu}{jk}\begin{bmatrix} -\coth\gamma\Delta & \frac{1}{\sinh\gamma\Delta} \\ -\frac{1}{\sinh\gamma\Delta} & \coth\gamma\Delta \end{bmatrix}\begin{bmatrix} \hat{H}_x^\alpha \\ \hat{H}_x^\beta \end{bmatrix} \tag{26}$$

With the boundary conditions of Eq. 24 in the electric relations and with those of Eq. 25 in these last relations, it is evident that there can be no response unless the determinant of the coefficients vanishes. In each case this requires that

$$-\coth^2\gamma\Delta + \frac{1}{\sinh^2\gamma\Delta} = 0 \tag{27}$$

This has two solutions.

$$\sinh\gamma\Delta = 0 \quad ; \quad \cosh\gamma\Delta = \pm 1 \tag{28}$$

In either case,

$$\gamma = \frac{jn\pi}{\Delta} \tag{29}$$

It follows from the definition of γ that each mode designated by n must satisfy the dispersion equation

$$\left(\frac{\omega}{c}\right)^2 = k^2 + \left(\frac{n\pi}{\Delta}\right)^2 \tag{30}$$

For propagation of waves through this parallel plate waveguide, k must be real. Thus, all waves attenuate below the cutoff frequency

$$\omega_{cut\,off} = \frac{c\pi}{\Delta} \tag{31}$$

because then all have an imagineary wavenumber,k.

Prob. 2.16.5 Gauss' law and $\bar{E}=-\nabla\Phi$ requires that if there is no free charge

$$\epsilon\nabla^2\Phi + \nabla\epsilon\cdot\nabla\Phi = 0 \qquad (1)$$

For the given exponential dependence of the permittivity, the x dependence of the coefficients in this expression factors out and it again reduces to a constant coefficient expression

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} + 2\gamma\frac{\partial\Phi}{\partial x} = 0 \qquad (2)$$

In terms of the complex amplitude forms from Table 2.16.1, Eq. 2 requires that

$$\frac{d^2\tilde{\Phi}}{dx^2} + 2\gamma\frac{d\tilde{\Phi}}{dx} - k^2\tilde{\Phi} = 0 \qquad (3)$$

Thus, solutions have the form exp px where $p=-\gamma\pm\lambda,\ \lambda\equiv\sqrt{k^2+\gamma^2}$.

The linear combination of these that satisfies the conditions that $\hat{\Phi}$ be $\hat{\Phi}^\alpha$ and $\tilde{\Phi}^\beta$ on the upper and lower surfaces respectively is as given in the problem. The displacement vector is then evaluated as

$$\tilde{D} = -\epsilon_\beta\left\{\tilde{\Phi}^\alpha e^{\gamma(x+\Delta)}\frac{[-\gamma\sinh\lambda x+\lambda\cosh\lambda x]}{\sinh\lambda\Delta} - \tilde{\Phi}^\beta e^{\gamma x}\frac{[-\gamma\sinh\lambda(x-\Delta)+\lambda\cosh\lambda(x-\Delta)]}{\sinh\lambda\Delta}\right. \qquad (4)$$

Evaluation of this expression at the respective surfaces then gives the transfer relations summarized in the problem.

Prob. 2.16.6 The fields are governed by

$$\bar{E} = -\nabla \Phi \tag{1}$$

$$\nabla \cdot \bar{D} = 0 \tag{2}$$

Substitution of Eq. 1 and the constitutive law into Eq. 2 gives a generalization of Laplace's equation for the potential.

$$\epsilon_{ij} \frac{\partial^2 \Phi}{\partial x_i \partial x_j} = 0 \tag{3}$$

Substitution of

$$\Phi = \mathrm{Re}\, \tilde{\Phi}(x)\, e^{-j(k_y y + k_z z)} \tag{4}$$

results in

$$\frac{d^2 \tilde{\Phi}}{dx^2} - jA \frac{d\tilde{\Phi}}{dx} - B\tilde{\Phi} = 0 \tag{5}$$

where

$$A \equiv k_y \frac{(\epsilon_{xy} + \epsilon_{yx})}{\epsilon_{xx}} + k_z \frac{(\epsilon_{xz} + \epsilon_{zx})}{\epsilon_{xx}} ; B = \frac{1}{\epsilon_{xx}} \left[k_y^2 \epsilon_{yy} + k_y k_z (\epsilon_{yz} + \epsilon_{zy}) + k_z^2 \epsilon_{zz} \right]$$

This constant coefficient equation has solutions exp p, where substitution shows that

$$p = j\gamma \pm \lambda \; ; \; \gamma = \frac{A}{2} \; , \; \lambda = \sqrt{B - \frac{A^2}{4}} \tag{6}$$

Thus, solutions take the form

$$\tilde{\Phi} = A_1 e^{j\gamma x} e^{\lambda x} + A_2 e^{j\gamma x} e^{-\lambda x} \tag{7}$$

The coefficients A_1 and A_2 are determined by requiring that $\tilde{\Phi} = \tilde{\Phi}^\alpha$ and $\tilde{\Phi} = \tilde{\Phi}^\beta$ at $x = \Delta$ and $x=0$ respectively. Thus, in terms of the surface potentials, the potential distribution is given by

$$\tilde{\Phi} = \tilde{\Phi}^\alpha e^{j\gamma(x-\Delta)} \frac{\sinh \lambda x}{\sinh \lambda \Delta} + \tilde{\Phi}^\beta e^{j\gamma x} \frac{\sinh \lambda(\Delta - x)}{\sinh \lambda \Delta} \tag{8}$$

The normal electric displacement follows from the x component of the constitutive law,

$$\tilde{D}_x = \epsilon_{xj} \tilde{E}_j = -\epsilon_{xx} \frac{d\tilde{\Phi}}{dx} + j(\epsilon_{xy} k_y + \epsilon_{xz} k_z) \tilde{\Phi} \tag{9}$$

Evaluation using Eq. 8 then gives

Prob. 2.16.6(cont.)

$$D_x=\left\{-\epsilon_{xx}\left[j\gamma e^{j\gamma(x-\Delta)}\frac{\sinh\lambda x}{\sinh\lambda\Delta}+\lambda e^{j\gamma(x-\Delta)}\frac{\cosh\lambda x}{\sinh\lambda\Delta}\right]+j(\epsilon_{xy}k_y+\epsilon_{xz}k_z)e^{j\gamma(x-\Delta)}\frac{\sinh\lambda x}{\sinh\lambda\Delta}\right\}\tilde{\Phi}^\alpha \tag{10}$$

$$+\left\{-\epsilon_{xx}\left[j\gamma e^{j\gamma x}\frac{\sinh\lambda(\Delta-x)}{\sinh\lambda\Delta}-\lambda e^{j\gamma x}\frac{\cosh\lambda(\Delta-x)}{\sinh\lambda\Delta}\right]+j(\epsilon_{xy}k_y+\epsilon_{xz}k_z)e^{j\gamma x}\frac{\sinh\lambda(\Delta-x)}{\sinh\lambda\Delta}\right\}\tilde{\Phi}^\beta$$

The required transfer relations follow by evaluating this expression at the respective boundaries.

$$\begin{bmatrix}\tilde{D}_x^\alpha\\ \\ \tilde{D}_x^\beta\end{bmatrix}=\begin{bmatrix}-\epsilon_{xx}[j\gamma+\lambda\coth\lambda\Delta]+j(\epsilon_{xy}k_y+\epsilon_{xz}k_z) & \dfrac{\epsilon_{xx}\lambda e^{j\gamma\Delta}}{\sinh\lambda\Delta}\\ \\ -\dfrac{\epsilon_{xx}\lambda e^{-j\gamma\Delta}}{\sinh\lambda\Delta} & -\epsilon_{xx}[j\gamma-\lambda\coth\lambda\Delta]+j(\epsilon_{xy}k_y+\epsilon_{xz}k_z)\end{bmatrix}\begin{bmatrix}\tilde{\Phi}^\alpha\\ \\ \tilde{\Phi}^\beta\end{bmatrix} \tag{11}$$

<u>Prob. 2.17.1</u> In cartesian coordinates, $a^\alpha = a^\beta$, so that Eq. 2.17.1 requires that $B_{12}=B_{21}$. Comparison of terms in the canonical and particular transfer relations then shows that

$$B_{12}=e^{\lambda\Delta}/\sinh\lambda\Delta=-B_{21}$$

<u>Prob. 2.17.2</u> Using $\alpha A_{12}=\beta A_{21}$, Table 2.16.2 gives

$$jk\alpha\left[H_m(jk\alpha)J_m'(jk\alpha)-J_m(jk\alpha)H_m'(jk\alpha)\right)$$

$$=-jk\beta\left[H_m(jk\beta)J_m'(jk\beta)-J_m(jk\beta)H_m'(jk\beta)\right) \tag{1}$$

These can only be equal for arbitrary α,β if

$$kx\left[H_m(jkx)J_m'(jkx)-J_m(jkx)H_m'(jkx)\right]=\text{const.} \tag{2}$$

Limit relations, Eqs. 2.16.22 and 2.16.23, are used to evaluate the constant.

$$ku\left[\left(-\frac{2}{\pi}\right)\sqrt{\frac{\pi}{2u}}\,e^{-u}\left(\frac{e^u}{\sqrt{2\pi u}}\left(1-\frac{1}{u}\right)\right)+\frac{1}{\sqrt{2\pi u}}\,e^u\left(\frac{2}{\pi}\right)\sqrt{\frac{\pi}{2u}}\,e^{-u}\left(-1-\frac{1}{u}\right)\right]=\text{const.} \tag{3}$$

Thus, as $u\to\infty$ it is clear that $c=-2/\pi$

Prob. 2.17.3 With the assumption that w is a state function, it follows that

$$\delta W = \frac{\partial W}{\partial \tilde{D}_{nr}^{\alpha}} \delta \tilde{D}_{nr}^{\alpha} + \frac{\partial W}{\partial \tilde{D}_{ni}^{\alpha}} \delta \tilde{D}_{ni}^{\alpha} + \frac{\partial W}{\partial \tilde{D}_{nr}^{\beta}} \tilde{D}_{nr}^{\beta} + \frac{\partial W}{\partial \tilde{D}_{ni}^{\beta}} \delta \tilde{D}_{ni}^{\beta}$$

Because the D's are independent variables, the coefficients must agree with those of the expression for δW in the problem statement. Thus, the relations for the Φ's follow. The reciprocity relations follow from taking cross-derivatives of these energy relations

$$-a^{\alpha} \frac{\partial \tilde{\Phi}_r^{\alpha}}{\partial \tilde{D}_i^{\alpha}} = -a^{\alpha} \frac{\partial \tilde{\Phi}^{\alpha}}{\partial \tilde{D}_r^{\alpha}} \quad (1) \qquad -a^{\alpha} \frac{\partial \tilde{\Phi}_i^{\alpha}}{\partial \tilde{D}_r^{\beta}} = a^{\beta} \frac{\partial \tilde{\Phi}_r^{\beta}}{\partial \tilde{D}_i^{\alpha}} \quad (4)$$

$$-a^{\alpha} \frac{\partial \tilde{\Phi}_r^{\alpha}}{\partial \tilde{D}_r^{\beta}} = a^{\beta} \frac{\partial \tilde{\Phi}_r^{\beta}}{\partial \tilde{D}_r^{\alpha}} \quad (2) \qquad -a^{\alpha} \frac{\partial \tilde{\Phi}_i^{\alpha}}{\partial \tilde{D}_i^{\beta}} = a^{\beta} \frac{\partial \tilde{\Phi}_i^{\beta}}{\partial \tilde{D}_i^{\alpha}} \quad (5)$$

$$-a^{\alpha} \frac{\partial \tilde{\Phi}_r^{\alpha}}{\partial \tilde{D}_r^{\beta}} = a^{\beta} \frac{\partial \tilde{\Phi}_i^{\beta}}{\partial \tilde{D}_r^{\alpha}} \quad (3) \qquad a^{\beta} \frac{\partial \tilde{\Phi}_r^{\beta}}{\partial \tilde{D}_i^{\beta}} = a^{\beta} \frac{\partial \tilde{\Phi}_i^{\beta}}{\partial \tilde{D}_r^{\beta}} \quad (6)$$

The transfer relation written so as to separate the real and imaginary parts, is equivalent to

$$\begin{bmatrix} \tilde{\Phi}_r^{\alpha} \\ \tilde{\Phi}_i^{\alpha} \\ \tilde{\Phi}_r^{\beta} \\ \tilde{\Phi}_i^{\beta} \end{bmatrix} = \begin{bmatrix} -A_{11r} & A_{11i} & A_{12r} & -A_{12i} \\ -A_{11i} & -A_{11r} & A_{12i} & A_{12r} \\ -A_{21r} & A_{21i} & A_{22r} & -A_{22i} \\ -A_{21i} & -A_{21r} & A_{22i} & A_{22r} \end{bmatrix} \begin{bmatrix} \tilde{D}_r^{\alpha} \\ \tilde{D}_i^{\alpha} \\ \tilde{D}_r^{\beta} \\ \tilde{D}_i^{\beta} \end{bmatrix}$$

The reciprocity relations (1) and (6) respectively show that these transfer relations require that $A_{11i}=-A_{11i}$ and $A_{22i}=-A_{22i}$, so that the imaginary

Prob. 2.17.3 (cont.)

parts of A_{11} and A_{22} are zero. The other relations show that $a^{\alpha}A_{12r}=a^{\beta}A_{21r}$ and $a^{\alpha}A_{12i}=-a^{\beta}A_{21i}$ so, $a^{\alpha}A_{12}=a^{\beta}A_{21}^{*}$. Of course, A_{12} and hence, A_{21} are actually real.

Prob. 2.17.4 From Problem 2.17.1, for

$$\begin{bmatrix} \tilde{D}_n^{\alpha} \\ \tilde{D}_n^{\beta} \end{bmatrix} = \begin{bmatrix} -B_{11} & B_{12} \\ -B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^{\alpha} \\ \tilde{\Phi}^{\beta} \end{bmatrix} \tag{1}$$

it is shown that

$$-a^{\alpha}\frac{\partial \tilde{D}_n^{\alpha}}{\partial \tilde{\Phi}^{\beta}} = a^{\beta}\frac{\partial \tilde{D}_n^{\beta}}{\partial \tilde{\Phi}^{\alpha}} \tag{2}$$

which requires that

$$B_{12} = B_{21} \tag{3}$$

For this system $B_{12} = B_{21} = \gamma e^{\gamma\Delta}/\sinh\gamma\Delta$.

Prob. 2.18.1 Observe that in cylindrical coordinates (Appendix A) with $\bar{A}=A_{\theta}\bar{i}_{\theta}$

$$\bar{B} = \nabla\times\bar{A} = -\frac{\partial A_{\theta}}{\partial z}\bar{i}_r + \frac{1}{r}\frac{\partial}{\partial r}(rA_{\theta})\bar{i}_z \tag{1}$$

Thus, substitution of $A_{\theta} = \Lambda(r,z)r^{-1}$ gives

$$\bar{B} = -\frac{1}{r}\frac{\partial \Lambda}{\partial z}\bar{i}_r + \frac{1}{r}\frac{\partial \Lambda}{\partial r}\bar{i}_z \tag{2}$$

as in Table 2.18.1.

Prob. 2.18.2 In spherical coordinates with $\bar{A}=A_{\phi}\bar{i}_{\phi}$ (Appendix A),

$$\bar{B} = \nabla\times\bar{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_{\phi}\sin\theta)\bar{i}_r - \frac{1}{r}\frac{\partial}{\partial r}(rA_{\phi})\bar{i}_{\theta} \tag{1}$$

Thus, substitution of $A_{\phi} = \Lambda(r,\theta)(r\sin\theta)^{-1}$ gives

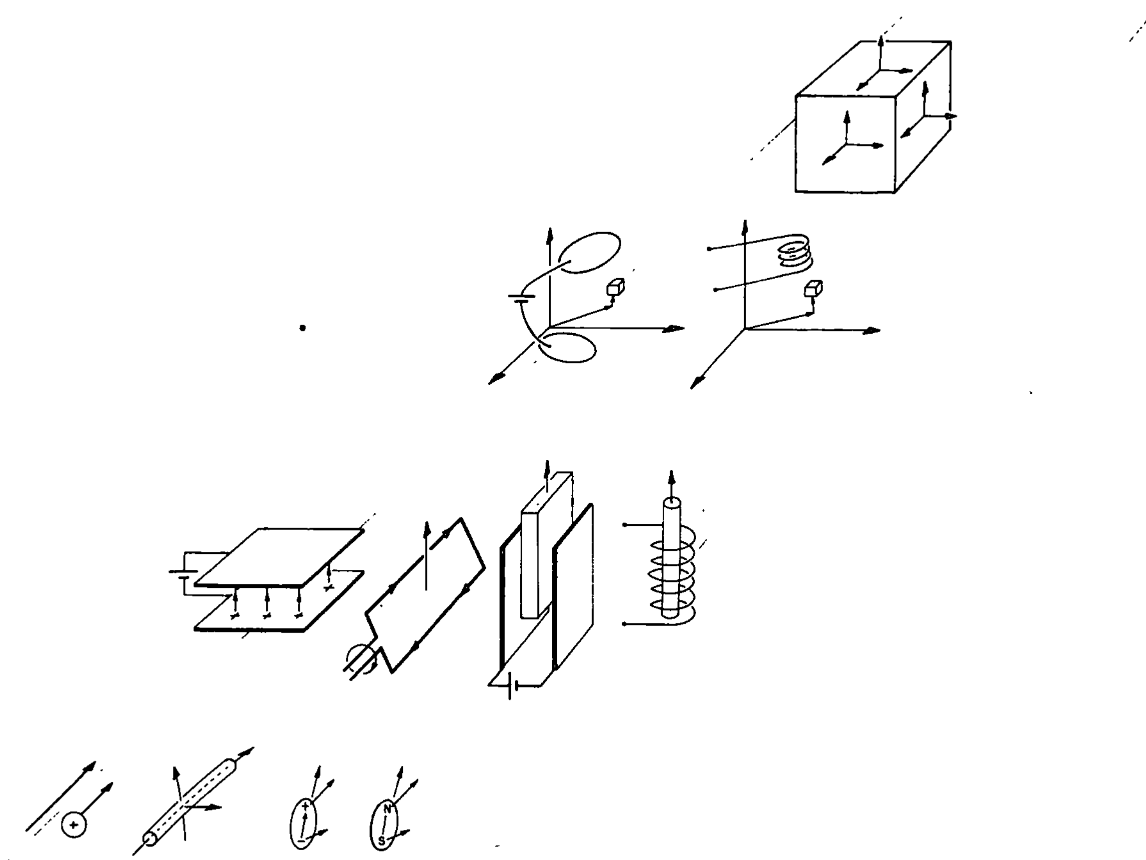
$$\bar{B} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\Lambda}{r}\right)\bar{i}_r - \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\Lambda}{\sin\theta}\right)\bar{i}_{\theta} = \frac{1}{r\sin\theta}\left(\frac{1}{r}\frac{\partial\Lambda}{\partial\theta}\bar{i}_r - \frac{\partial\Lambda}{\partial r}\bar{i}_{\theta}\right) \tag{2}$$

as in Table 2.18.1.

Prob. 2.19.1 The transfer relations are obtained by following the instructions given with Eqs. 2.19.7 through 2.19.12.

3

Electromagnetic Forces, Force Densities and Stress Tensors



Prob. 3.3.1 With inertia included but $\bar{H}=0$, Eqs. 3 become

$$m_+ \frac{d\bar{v}_+}{dt} = -m_+ \nu_+ \bar{v}_+ + q_+ \bar{E} \tag{1}$$

$$m_- \frac{d\bar{v}_-}{dt} = -m_- \nu_- \bar{v}_- - q_- \bar{E}$$

With an imposed $\bar{E} = \mathcal{R}e \exp j\omega t$, the response to these linear equations takes the form $\bar{v}_\pm = \mathcal{R}e\, \hat{\bar{v}}_\pm \exp j\omega t$. Substitution into Eqs. 1 gives

$$\hat{\bar{v}}_\pm = \frac{q_\pm \hat{\bar{E}}}{m_\pm (\nu_\pm + j\omega)} \tag{2}$$

Thus, for the effect of inertia to be ignorable

$$\nu_\pm \gg \omega \tag{3}$$

In terms of the mobility $b_\pm \equiv q_\pm / m_\pm \nu_\pm$, Eq. 3 requires that

$$q_\pm / b_\pm m_\pm \gg \omega = 2\pi f \tag{4}$$

For copper, evaluation gives

$$(1.76\times 10^{11})/(2\pi)(3\times 10^{-3}) = 9.34\times 10^{12}\ \mathrm{Hz} \gg f \tag{5}$$

At this frequency the wavelength of an electromagnetic wave is $\lambda = c/f = 3\times 10^{8}/9.34\times 10^{12}$, which is approaching the optical range ($32\,\mu m$).

Prob. 3.5.1 (a) The cross-derivative of Eq. 9 gives the reciprocity condition

$$\frac{\partial q_1}{\partial v_2} = \frac{\partial^2 w'}{\partial v_1 \partial v_2} = \frac{\partial q_2}{\partial v_1} \tag{1}$$

from which it follows that $C_{12} = C_{21}$.

(b) The coenergy found in Prob. 2.13.1 can be used with Eq. 3.5.9 to find the two forces.

Prob. 3.5.1 (cont.)

$$f_1 = \frac{\partial w'}{\partial \xi_1} = \frac{1}{2} v_1^2 \frac{\partial C_{11}}{\partial \xi_1} + v_1 v_2 \frac{\partial C_{21}}{\partial \xi_1} + \frac{1}{2} v_2^2 \frac{\partial C_{22}}{\partial \xi_1} \qquad (2)$$

$$f_2 = \frac{\partial w'}{\partial \xi_2} = \frac{1}{2} v_1^2 \frac{\partial C_{11}}{\partial \xi_2} + v_1 v_2 \frac{\partial C_{21}}{\partial \xi_2} + \frac{1}{2} v_2^2 \frac{\partial C_{22}}{\partial \xi_2} \qquad (3)$$

The specific dependences of these capacitances on the displacements are determined in Prob. 2.11.1. Thus, Eqs. 2 and 3 become

$$f_1 = d\epsilon_o \left[\frac{1}{2} v_1^2 \left(\frac{1}{b-\xi_2} - \frac{1}{b}\right) + \frac{v_1 v_2}{b} + \frac{1}{2} v_2^2 \left(\frac{1}{\xi_2} - \frac{1}{b}\right)\right] \qquad (4)$$

$$f_2 = d\epsilon_o \left[\frac{1}{2} v_1^2 \frac{\xi_1}{(b-\xi_2)^2} - \frac{1}{2} v_2^2 \frac{\xi_1}{\xi_2^2}\right] \qquad (5)$$

Prob. 3.5.2 (a) The system is electrically linear, so $w' = \frac{1}{2} Cv^2$, where C is the charge per unit voltage on the positive electrode. Note that throughout the region between the electrodes, $E=v/d$. Hence,

$$w' = \frac{1}{2} v^2 \left[\frac{a w \epsilon_o}{d} + \xi \frac{w}{d}(\epsilon - \epsilon_o)\right] \qquad (1)$$

(b) The force due to polarization tending to pull the slab into the region between the electrodes is then

$$f = \frac{\partial w'}{\partial \xi} = wd\,(\epsilon - \epsilon_o)\left(\frac{v}{d}\right)^2 \qquad (2)$$

The quantity multiplying the cross-sectional area of the slab, wd, can alternatively be thought of as a pressure associated with the Kelvin force density on dipoles induced in the fringing field acting over the cross-section (Sec. 3.6) or as the result of the Korteweg-Helmholtz force density (Sec. 3.7). The latter is confined to a surface force density acting over the cross-section dw, at the dielectric-free space interface. Either viewpoint gives the same net force.

Prob. 3.5.3 From Eq. 9 and the coenergy determined in Prob. 2.13.2,

$$f = \frac{\partial w'(v,\xi)}{\partial \xi} = \qquad + \frac{d}{a_1}\left[(a_2^2 b^2 + v^2)^{\frac{1}{2}} - a_2 b\right] \qquad (1)$$

Prob. 3.5.4 (a) Using the coenergy function found in Prob. 2.14.1, the radial surface force density follows as

$$T_r = \frac{1}{2\pi\xi d}\frac{\partial w'}{\partial \xi} = \frac{\mu_o i_1 i_2}{d^2} + \frac{\mu_o i_2^2}{2d^2} \tag{1}$$

(b) A similar calculation using the λ's as the independent variables first requires that $w(\lambda_1, \lambda_2, \xi)$ be found, and this requires the inversion of the inductance matrix terminal relations, as illustrated in Prob. 2.14.1. Then, because the ξ dependence of w is more complicated than of w', the resulting expression is more cumbersome to evaluate.

$$T_r = \frac{-1}{2\pi\xi d}\frac{\partial w}{\partial \xi} = \frac{-1}{2\pi^2\mu_o\xi}\left\{\frac{\xi\lambda_1^2}{(a^2-\xi^2)^2} - \frac{2\xi\lambda_1\lambda_2}{(a^2-\xi^2)^2} + \frac{a^2(2\xi^2-a^2)}{(a^2-\xi^2)^2\xi^3}\lambda_2^2\right\} \tag{2}$$

However, if it is one of the λ's that is contrained, this approach is perhaps worthwhile.

(c) Evaluation of Eq. 2 with $\lambda_2 = 0$ gives the surface force density if the inner ring completely excludes the flux.

$$T_r = \frac{-\lambda_1^2}{2\pi^2\mu_o(a^2-\xi^2)^2} \tag{3}$$

Note that according to either Eq. 1 or 3, the inner coil is compressed, as would be expected by simply evaluating $\bar{J}_f \times \mu_o \bar{H}$. To see this from Eq. 1, note that if $\lambda_2=0$, then $i_1=-i_2$.

Prob. 3.6.1 Force equilibrium for each element of the static fluid is

$$\nabla p = \bar{F} = \nabla\left[\tfrac{1}{2}(\epsilon-\epsilon_o)E^2\right] \tag{1}$$

where the force density due to gravity could be included, but would not contribute to the discussion. Integration of Eq. (1) from the outside interface (a) to the lower edge of the slab (b) (which is presumed well within the electrodes) can be carried out without regard for the details

Prob. 3.6.1 (cont.)

of the field by using Eq. 2.6.1.

$$\int_a^b \nabla p \cdot d\bar{l} = \int_a^b \nabla\left[\tfrac{1}{2}(\epsilon-\epsilon_o)E^2\right]\cdot d\bar{l} \Rightarrow P_b - P_a = \tfrac{1}{2}(\epsilon-\epsilon_o)\left[E_a^2 - E_b^2\right] \quad (2)$$

Thus, the pressure acting upward on the lower extremity of the slab is

$$P_b = \tfrac{1}{2}(\epsilon-\epsilon_o)E^2 \quad (3)$$

which gives a force in agreement with the result of Prob. 3.5.2, found using the lumped parameter energy method.

$$f = wdP_b = wd\tfrac{1}{2}(\epsilon-\epsilon_o)E^2 \quad (4)$$

Prob. 3.6.2 With the charges comprising the dipole respectively at $\bar{r}_+$ and $\bar{r}_-$, the torque is

$$\bar{\Upsilon} = \bar{r}_+ \times q\bar{E}(\bar{r}_+) - \bar{r}_- \times q\bar{E}(\bar{r}_-) \quad (1)$$

Expanding about the position of the negative charge, $\bar{r}_-$,

$$\bar{\Upsilon} \cong (\bar{r}_- + \bar{d})\times\left[q\bar{E}(\bar{r}_-) + q\bar{d}\cdot\nabla\bar{E}\right] - \bar{r}_- \times q\bar{E}(\bar{r}_-) \quad (2)$$

To first order in $\bar{d}$ this becomes the desired expression.

The torque on a magnetic dipole could be found by using an energy argument for a discrete system, as in Sec. 3.5. Forces and displacements would be replaced by torques and angles. However, because of the complete analogy summarized by Eqs. 8-10, $\bar{H} \leftrightarrow \bar{E}$ and $\bar{P} \leftrightarrow \mu_o\bar{M}$ This means that $\bar{p} \leftrightarrow \mu_o\bar{m}$ and so the desired expression follows directly from Eq. 2.

Prob. 3.7.1 Demonstrate that for a constitutive law implying no interaction the Korteweg-Helmholtz force density

$$\bar{F} = \rho_f\bar{E} + \bar{P}\cdot\nabla\bar{E} + \nabla\left(\tfrac{1}{2}\epsilon_o\bar{E}\cdot\bar{E} + W - \bar{E}\cdot\bar{D} - \sum_{i=1}^{m}\frac{\partial W}{\partial\alpha_i}\alpha_i\right) \quad (1)$$

becomes the Kelvin force density. That is, $(\quad) = 0$. Let $\chi_e = c\rho$, $\alpha_1 = \rho$ and evaluate $(\quad)$

$$W = \int \bar{E}\cdot\delta\bar{D} = \frac{D^2}{2\epsilon_o(1+\chi_e)} = \frac{\bar{E}\cdot\bar{D}}{2} \quad (2)$$

Thus,

$$\frac{\partial W}{\partial\rho} = \frac{\partial W}{\partial\chi_e}\frac{\partial\chi_e}{\partial\rho} = c\left[\frac{-D^2}{2\epsilon_o(1+\chi_e)^2}\right] = -\frac{c\,\epsilon_o E^2}{2} \quad (3)$$

Prob. 3.7.1 (cont.)

so that

$$-\frac{\partial W}{\partial \rho}\rho = \chi_e \frac{\epsilon_0}{2} E^2 \tag{4}$$

and

$$(\quad) = \left(\frac{\bar{E}\cdot\bar{D}}{2} + \frac{\epsilon_0}{2}E^2 - \bar{E}\cdot\bar{D} + \frac{\chi_e\epsilon_0}{2}E^2\right)$$

$$= -\frac{\epsilon_0 E^2}{2}(1+\chi_e) + \frac{\epsilon_0}{2}E^2 + \chi_e\frac{\epsilon_0}{2}E^2 \tag{5}$$

<u>Prob. 3.9.1</u> In the expression for the torque, Eq. 3.9.16,

$$\bar{r} = x\bar{i}_x + y\bar{i}_y + z\bar{i}_z \tag{1}$$

so that it becomes

$$\bar{T} = \int_V \left[\bar{i}_x(yF_3 - zF_2) + \bar{i}_y(zF_1 - xF_3) + \bar{i}_z(xF_2 - yF_1)\right]dV \tag{2}$$

Because $F_i = \partial T_{ij}/\partial x_j$

$$\bar{T} = \int_V \left[\bar{i}_x\left(y\frac{\partial T_{3j}}{\partial x_j} - z\frac{\partial T_{2j}}{\partial x_j}\right) + \bar{i}_y\left(z\frac{\partial T_{1j}}{\partial x_j} - x\frac{\partial T_{3j}}{\partial x_j}\right) + \bar{i}_z\left(x\frac{\partial T_{2j}}{\partial x_j} - y\frac{\partial T_{1j}}{\partial x_j}\right)\right]dV$$

$$\begin{aligned} = \int_V \Big[& \bar{i}_x\Big(\frac{\partial}{\partial x_j}yT_{3j} - \underbrace{T_{3j}\frac{\partial y}{\partial x_j}}_{T_{32}} - \frac{\partial}{\partial x_j}zT_{2j} + \underbrace{T_{2j}\frac{\partial z}{\partial x_j}}_{T_{23}}\Big) \\ & + \bar{i}_y\Big(\frac{\partial}{\partial x_j}zT_{1j} - \underbrace{T_{1j}\frac{\partial z}{\partial x_j}}_{T_{13}} - \frac{\partial}{\partial x_j}xT_{3j} + \underbrace{T_{3j}\frac{\partial x}{\partial x_j}}_{T_{31}}\Big) \\ & + \bar{i}_z\Big(\frac{\partial}{\partial x_j}xT_{2j} - \underbrace{T_{2j}\frac{\partial x}{\partial x_j}}_{T_{21}} - \frac{\partial}{\partial x_j}yT_{1j} + \underbrace{T_{1j}\frac{\partial y}{\partial x_j}}_{T_{12}}\Big)\Big]dV \end{aligned} \tag{3}$$

Prob. 3.9.1 (cont.)

Because $T_{ij} = T_{ji}$ (symmetry)

$$\bar{\tau} = \int_V \frac{\partial}{\partial x_j}\left[\bar{i}_x(yT_{3j} - zT_{2j}) + \bar{i}_y(zT_{1j} - xT_{3j}) + \bar{i}_z(xT_{2j} - yT_{1j})\right]dV \qquad (4)$$

From the tensor form of Gauss' theorem, Eq. 3.8.4, this volume integral becomes the surface integral

$$\bar{\tau} = \oint_S \left[\bar{i}_x(yT_{3j} - zT_{2j}) + \bar{i}_y(zT_{1j} - xT_{3j}) + \bar{i}_z(xT_{2j} - yT_{1j})\right]n_j da \qquad (5)$$

$$= \oint_S \bar{r} \times (\bar{\bar{T}} \cdot \bar{n})\, da$$

Prob. 3.10.1 Using the product rule,

$$\bar{F} = \frac{1}{2}\epsilon \nabla(\bar{E} \cdot \bar{E}) = \nabla\left(\frac{1}{2}\epsilon \bar{E} \cdot \bar{E}\right) - \frac{1}{2}\bar{E} \cdot \bar{E} \nabla \epsilon \qquad (1)$$

The first term takes the form $\nabla\pi$ while the second agrees with Eq. 3.7.22 if $\rho_f = 0$.

In index notation,

$$F_i = \frac{1}{2}\epsilon \frac{\partial}{\partial x_i}(E_k E_k) \qquad (2)$$

where ε is a spatially varying function.

$$F_i = \epsilon E_k \frac{\partial E_k}{\partial x_i} \qquad (3)$$

Because $\nabla \times \bar{E} = 0$,

$$F_i = \epsilon E_k \frac{\partial E_i}{\partial x_k} = \frac{\partial}{\partial x_k}(\epsilon E_k E_k) - E_i \frac{\partial \epsilon E_k}{\partial x_k} \qquad (4)$$

Because $\rho_f = \nabla \cdot \epsilon \bar{E} = 0$, the last term is absent. The first term takes the required form $\partial T_{ik}/\partial x_k$.

Prob. 3.10.2 From Eqs. 2.13.11 and 3.7.19,

$$W' = \int \bar{D} \cdot \delta \bar{E} = \int (\alpha_1 E + \alpha_2 E^2)\delta E = \frac{1}{2}\alpha_1 E^2 + \frac{\alpha_2}{4}E^4;\ T_{ij} = E_i D_j - \delta_{ij} W' \qquad (1)$$

Thus, the force density is $(\partial E_i/\partial x_j = \partial E_j/\partial x_i,\ \partial D_j/\partial x_j = 0)$

$$F_i = \frac{\partial T_{ij}}{\partial x_j} = \frac{\partial E_i}{\partial x_j} D_j - \frac{\partial W'}{\partial x_i} = -\frac{1}{2}\bar{E} \cdot \bar{E}\frac{\partial \alpha_1}{\partial x_i} - \frac{1}{4}(\bar{E} \cdot \bar{E})^2 \frac{\partial \alpha_2}{\partial x_i} \qquad (2)$$

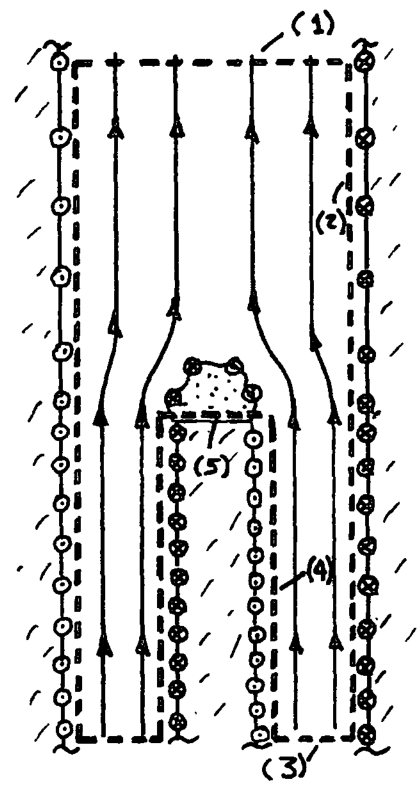
The Kelvin stress tensor, Eq. 3.6.5, differs from Eq. 1b only by the term in δ_{ij}, so the force densities can only differ by the gradient of a pressure.

Prob. 3.10.3

(a) The magnetic field is "trapped" in the region between tubes. For an infinitely long pair of coaxial conductors, the field in the annulus is uniform. Hence, because the total flux $\pi a^2 B_o$ must be constant over the length of the system, in the lower region

$$B_z = \frac{a^2 B_o}{a^2 - b^2} \tag{1}$$

(b) The distribution of surface current is as sketched below. It is determined by the condition that the magnetic flux at the extremities be as found in (a) and by the condition that the normal flux density on any of the perfectly conducting surfaces vanish.



(c) Using the surface force density $\bar{K} \times \langle\bar{B}\rangle$, it is reasonable to expect the net magnetic force in the z direction to be downward.

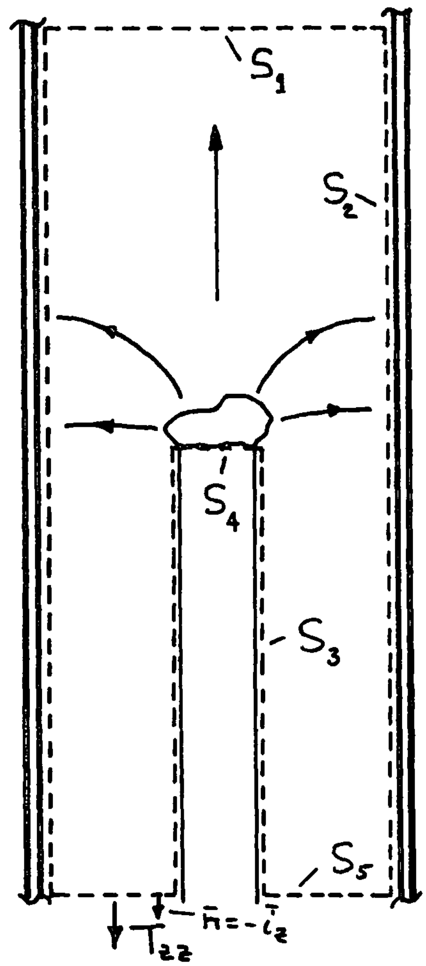
(d) One way to find the net force is to enclose the "blob" by the control volume shown in the figure and integrate the stress tensor over the enclosing surface.

$$f_z = \oint_S T_{zj} n_j da$$

Contributions to this integration over surfaces (4) and (2) (the walls of the inner and outer tubes which are perfectly conducting) vanish because there is no shear stress on a perfectly conducting surface. Surface (5) cuts under the "blob" and hence sustains no magnetic stress. Hence, only surfaces (1) and (3) make contributions, and on them the magnetic flux density is given and uniform. Hence, the net force is

$$f_z = \pi a^2 \left(\frac{B_o^2}{2\mu_o}\right) - \pi(a^2 - b^2)\frac{B_o^2 a^4}{2\mu_o (a^2 - b^2)^2} = -\frac{\pi a^2 B_o^2}{2\mu_o}\frac{b^2}{(a^2 - b^2)} \tag{2}$$

Note that, as expected, this force is negative.



Prob. 3.10.4 The electric field is sketched in the figure. The force on the cap should be upward. To find this force use the surface S shown to enclose the cap. On S_1 the field is zero. On S_2 and S_3 the electric shear stress is zero because it is an equipotential and hence can support no tangential $\bar{E}$. On S_4 the field is zero. Finally, on S_5 the field is that of infinite coaxial conductors.

$$\bar{E} = \bar{i}_r \frac{V_o}{\ln\left(\frac{a}{b}\right)} \frac{1}{r} \tag{1}$$

Thus, the normal electric stress is

$$T_{zz} = -\frac{\epsilon_o}{2} E_r^2 = -\frac{1}{2} \frac{\epsilon_o V_o^2}{\ln^2\left(\frac{a}{b}\right)} \frac{1}{r^2} \tag{2}$$

and the integral for the total force reduces to

$$f_z = \oint_S T_{zj} n_j \, da = -\int_b^a T_{zz} 2\pi r \, dr = \frac{V_o^2 \epsilon_o 2\pi}{2 \ln^2 \frac{a}{b}} \ln \frac{a}{b} = \frac{\pi V_o^2 \epsilon_o}{\ln\left(\frac{a}{b}\right)} \tag{3}$$

Prob. 3.10.5

$$F_i = (\rho_p + \rho_f) E_i = \frac{\partial \epsilon_o E_j}{\partial x_j} E_i = \frac{\partial}{\partial x_j}(\epsilon_o E_i E_j) - \epsilon_o E_j \frac{\partial E_i}{\partial x_j} \tag{1}$$

Because $\frac{\partial E_i}{\partial x_j} = \frac{\partial E_j}{\partial x_i}$, the last term becomes

$$-\epsilon_o E_j \frac{\partial E_i}{\partial x_j} = -\epsilon_o E_j \frac{\partial E_j}{\partial x_i} = -\frac{\partial}{\partial x_i}\left(\frac{1}{2} \epsilon_o E_k E_k\right) \tag{2}$$

Thus

$$F_i = \frac{\partial}{\partial x_j}\left(\epsilon_o E_i E_j - \frac{1}{2} \delta_{ij} \epsilon_o E_k E_k\right) \tag{3}$$

where the quantity in brackets is T_{ij}. Because T_{ij} is the same as any of the T_{ij}'s in Table 3.10.1 when evaluated in free space, use of a surface S surrounding the object to evaluate Eq. 3.9.4 will give a total force in agreement with that predicted by the correct force densities.

Prob. 3.10.6

Showing that the identity holds is a matter of simply writing out the components in cartesian coordinates. The i'th component of the force density is then written using the identity to write $\bar{J}\times\bar{B}$ where $\bar{J} = \nabla\times\bar{H}$.

$$F_i = \frac{\partial H_i}{\partial x_j}B_j - \frac{\partial H_j}{\partial x_i}B_j + \sum_{k=1}^{m}\frac{\partial W}{\partial \alpha_k}\frac{\partial \alpha_k}{\partial x_i} - \frac{\partial}{\partial x_i}\left(\sum_{k=1}^{m}\alpha_k\frac{\partial W}{\partial \alpha_k}\right) \tag{1}$$

In the first term, B_j is moved inside the derivative and the condition $\partial B_j/\partial x_j = \nabla\cdot\bar{B} = 0$ exploited. The third term is replaced by the magnetic analogue of Eq. 3.7.26.

$$F_i = \frac{\partial}{\partial x_j}(H_iB_j) - \frac{\partial H_j}{\partial x_i}B_j + B_j\frac{\partial H_j}{\partial x_i} - \frac{\partial}{\partial x_i}(B_jH_j) + \frac{\partial W}{\partial x_i} - \frac{\partial}{\partial x_i}\sum_{k=1}^{m}\alpha_k\frac{\partial W}{\partial \alpha_k} \tag{2}$$

The second and third terms cancel, so that this expression can be rewritten

$$F_i = \frac{\partial}{\partial x_j}\left[H_iB_j - \delta_{ij}\left(W' + \sum_{k=1}^{m}\alpha_k\frac{\partial W}{\partial \alpha_k}\right)\right];\ W' \equiv \bar{B}\cdot\bar{H} - W \tag{3}$$

and the stress tensor identified as the quantity in brackets.

Problem 3.10.7 The i'th component of the force density is written using the identity of Prob. 2.10.5 to express $\bar{J}_f \times \mu_o\bar{H} = (\nabla\times\bar{H})\times\mu_o\bar{H}$

$$F_i = \mu_o\left(\frac{\partial H_i}{\partial x_j}H_j\right) - \mu_o\frac{\partial H_j}{\partial x_i}H_j + \mu_o M_j\frac{\partial H_i}{\partial x_j} \tag{1}$$

This expression becomes

$$F_i = \frac{\partial}{\partial x_j}(\mu_o H_iH_j) - H_i\frac{\partial}{\partial x_j}(\mu_o H_j) - \frac{\partial}{\partial x_i}\left(\frac{1}{2}\mu_o H_jH_j\right) + \frac{\partial}{\partial x_j}(\mu_o M_jH_i) - H_i\frac{\partial}{\partial x_j}(\mu_o M_j) \tag{2}$$

where the first two terms result from the first term in F_i, the third term results from taking the H_j inside the derivative and the last two terms are an expansion of the last term in F_i. The second and last term combine to give $\nabla\cdot\mu_o(\bar{H}+\bar{M}) \equiv \nabla\cdot\bar{B} = 0$. Thus, with $\bar{B} = \mu_o(\bar{H}+\bar{M})$, the expression takes the proper form for identifying the stress tensor.

$$F_i = \frac{\partial}{\partial x_j}\left[\mu_o(M_j + H_j)H_i - \delta_{ij}\frac{1}{2}\mu_o H_kH_k\right] \tag{3}$$

<u>Prob. 3.10.8</u> The integration of the force density over the volume of the dielectric is broken into two parts, one over the part that is well between the plates and therefore subject to a uniform field v/b, and the other enclosing what remains to the left. Observe that throughout this latter volume, the force density acting in the ξ direction is zero. That is, the force density is confined to the interfaces, where it is singular and constitutes a surface force density acting normal to the interfaces. The only region where the force density acts in the ξ direction is on the interface at the right. This is covered by the first integral, and the volume integration can be replaced by an integration of the stress over the enclosing surface. Thus,

$$f = ad\left[-\frac{1}{2}\epsilon_o\left(\frac{v}{b}\right)^2 + \frac{1}{2}\epsilon\left(\frac{v}{b}\right)^2\right] \tag{1}$$

in agreement with the result of Prob. 2.13.2 found using the energy method.

<u>Prob. 3.11.1</u> With the substitution $\bar{V} = -\gamma\bar{n}$ (suppress the subscript E), Eq. 1 becomes

$$-\oint_C \gamma\bar{n}\times d\bar{l} = \int_S\left[-\bar{n}\gamma\nabla\cdot\bar{n} - \bar{n}(\bar{n}\cdot\nabla\gamma) + \bar{n}\cdot(\gamma\bar{n}\nabla)\right]da \tag{1}$$

where the first two terms on the right come from expanding $\nabla\cdot\psi\bar{A} = \psi\nabla\cdot\bar{A} + \bar{A}\cdot\nabla\psi$. Thus, the first two terms in the integrand of Eq. 4 are accounted for. To see that the last term in the integrand on the right in Eq. 1 accounts for remaining term in Eq. (4) of the problem, this term is written out in Cartesian coordinates.

$$\begin{aligned}\bar{n}\cdot(\gamma\bar{n}\nabla) = {} & \bar{i}_x\left[n_x\frac{\partial\gamma n_x}{\partial x} + n_y\frac{\partial\gamma n_y}{\partial x} + n_z\frac{\partial\gamma n_z}{\partial x}\right] \\ & + \bar{i}_y\left[n_x\frac{\partial\gamma n_x}{\partial y} + n_y\frac{\partial\gamma n_y}{\partial y} + n_z\frac{\partial\gamma n_z}{\partial y}\right] \\ & + \bar{i}_z\left[n_x\frac{\partial\gamma n_x}{\partial z} + n_y\frac{\partial\gamma n_y}{\partial z} + n_z\frac{\partial\gamma n_z}{\partial z}\right]\end{aligned} \tag{2}$$

Prob. 3.11.1 (cont.)

Further expansion gives

$$\bar{n}\cdot(\gamma\,\bar{n}\nabla) =$$

$$\bar{i}_x\left[n_x^2\frac{\partial\gamma}{\partial x}+n_y^2\frac{\partial\gamma}{\partial x}+n_z^2\frac{\partial\gamma}{\partial x}\right]+\bar{i}_x\left[n_x\gamma\frac{\partial n_x}{\partial x}+n_y\gamma\frac{\partial n_y}{\partial x}+n_z\gamma\frac{\partial n_z}{\partial x}\right] \tag{3}$$

$$+\ \bar{i}_y\left[n_x^2\frac{\partial\gamma}{\partial y}+n_y^2\frac{\partial\gamma}{\partial y}+n_z^2\frac{\partial\gamma}{\partial y}\right]+\bar{i}_y\left[n_x\gamma\frac{\partial n_x}{\partial y}+n_y\gamma\frac{\partial n_y}{\partial y}+n_z\gamma\frac{\partial n_z}{\partial y}\right]$$

$$+\ \bar{i}_z\left[n_x^2\frac{\partial\gamma}{\partial z}+n_y^2\frac{\partial\gamma}{\partial z}+n_z^2\frac{\partial\gamma}{\partial z}\right]+\bar{i}_z\left[n_x\gamma\frac{\partial n_x}{\partial z}+n_y\gamma\frac{\partial n_y}{\partial z}+n_z\gamma\frac{\partial n_z}{\partial z}\right]$$

Note that $n_x^2 + n_y^2 + n_z^2 = 1$. Thus, the first third and fifth terms become $\nabla\gamma$.

The second term can be written as

$$\frac{\gamma}{2}\frac{\partial}{\partial x}\left(n_x^2+n_y^2+n_z^2\right)=\frac{\gamma}{2}\frac{\partial}{\partial x}(1)=0 \tag{4}$$

The fourth and sixth terms are similarly zero. Thus, these three terms vanish and Eq. 3 is simply $\nabla\gamma$. Thus, Eq. 1 becomes

$$-\oint_C \gamma\bar{n}\times d\bar{l} = \int_S\left\{-\bar{n}\gamma\nabla\cdot\bar{n}+\left[\nabla\gamma-\bar{n}(\bar{n}\cdot\nabla\gamma)\right]\right\}da \tag{5}$$

With the given alternative ways to write these terms, it follows that Eq. 5 is consistent with the last two terms of Eq. 3.11.8.

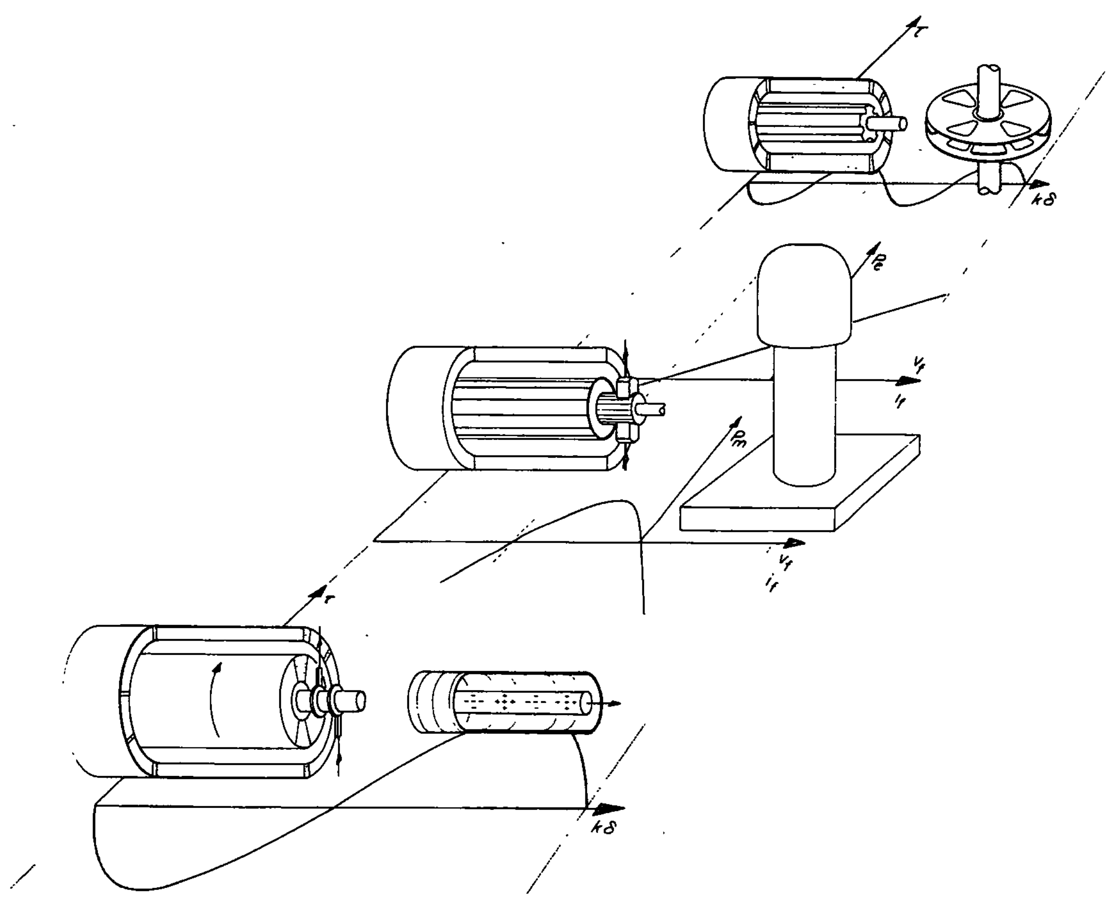
Prob. 3.11.2 Use can be made of Eq. 4 from Prob. 3.11.1 to convert the integral over the surface to one over a contour C enclosing the surface.

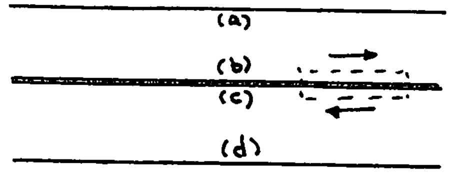
$$\bar{f} = -\int_C \gamma_E\,\bar{n}\times d\bar{l} \tag{1}$$

If the surface, S, is closed, then the contour, C, must vanish and it is clear that the net contribution of the integration is zero. The double-layer can not produce a net force on a closed surface.

4

Electromechanical Kinematics: Energy-Conversion Models and Processes





Prob. 4.3.1 With the positions as shown in the sketch, the required force is

$$f_z = \frac{A}{2} \mathrm{Re}\, \tilde{B}_x^b [\tilde{H}_z^b - \tilde{H}_z^c]^* \tag{1}$$

With the objective of finding $\tilde{B}_x^b$, first observe that the boundary conditions are.

$$\tilde{H}_z^a = \tilde{K}^s \; ; \; -\tilde{H}_z^b + \tilde{H}_z^c = \tilde{K}^r \; ; \; \tilde{B}_x^b = \tilde{B}_x^c \; ; \; -\tilde{H}_z^d = \tilde{K}^s \tag{2}$$

and the transfer relations of Table 2.16.1 applied to the respective regions require that

$$\begin{bmatrix} \tilde{B}_x^a \\ \tilde{B}_x^b \end{bmatrix} = \mu_0 k \begin{bmatrix} -\coth kd & \dfrac{1}{\sinh kd} \\ \dfrac{-1}{\sinh kd} & \coth kd \end{bmatrix} \begin{bmatrix} \dfrac{\tilde{K}^s}{jk} \\ \dfrac{\tilde{H}_z^b}{jk} \end{bmatrix} ; \quad \begin{bmatrix} \tilde{B}_x^c \\ \tilde{B}_x^d \end{bmatrix} = \mu_0 k \begin{bmatrix} -\coth kd & \dfrac{1}{\sinh kd} \\ \dfrac{-1}{\sinh kd} & \coth kd \end{bmatrix} \begin{bmatrix} \dfrac{\tilde{H}_z^c}{jk} \\ \dfrac{-\tilde{K}^s}{jk} \end{bmatrix} \tag{3}$$

Here, Eqs. 2a and 2d have already been used, as has also the relation $\tilde{H}_z = jk\tilde{\Psi}$

In view of Eq. 2c, Eqs. 3 are used to write

$$\tilde{B}_x^b = \mu_0 k \left[\frac{-\tilde{K}^s}{jk \sinh kd} + \frac{\tilde{H}_z^b \coth kd}{jk} \right] = \tilde{B}_x^c = \mu_0 k \left[-\frac{\tilde{H}_z^c \coth kd}{jk} - \frac{\tilde{K}^s}{jk \sinh kd} \right] \tag{4}$$

and it is concluded that

$$\tilde{H}_z^c = -\tilde{H}_z^b \tag{5}$$

This relation could be argued from the symmetry. In view of Eq. 2b, it follows that

$$\tilde{H}_z^b = -\frac{\tilde{K}^r}{2} \tag{6}$$

so that the required normal flux on the rotor surface follows from Eq. 2b as

$$\tilde{B}_x^b = \mu_0 k \left[\frac{-\tilde{K}^s}{\sinh kd \; jk} - \coth kd \frac{\tilde{K}^r}{2jk} \right] \tag{7}$$

Finally, evaluation of Eq. 1 gives

$$f_z = -\frac{A}{2} \mathrm{Re}\, \tilde{B}_x^b (\tilde{K}^r)^* = -\frac{\mu_0 A}{2} \mathrm{Re} \frac{j \tilde{K}^s (\tilde{K}^r)^*}{\sinh kd} \tag{8}$$

This result is identical to Eq. 4.3.4a, so the results for parts (b) and (c) will be the same as Eqs. 4.3.9a.

Prob. 4.3.2 Boundary conditions on the stator and rotor surfaces are

$$\tilde{H}_z^s = \tilde{K}^s \tag{1}$$

$$\tilde{B}_x^r = \tilde{B}^r \tag{2}$$

where

$$\tilde{K}^s = -j K_o^s e^{j\omega t} \tag{3}$$

$$\tilde{B}^r = B_o^r e^{jk(Ut+\delta)} \tag{4}$$

From Eq. (a) of Table 2.16.1, the air gap fields are therefore related by

$$\begin{bmatrix} \tilde{B}_x^s \\ \tilde{B}_x^r \end{bmatrix} = \mu_o k \begin{bmatrix} -\coth kd & \dfrac{1}{\sinh kd} \\ \dfrac{-1}{\sinh kd} & \coth kd \end{bmatrix} \begin{bmatrix} \dfrac{\tilde{K}^s}{jk} \\ \dfrac{\tilde{H}_z^r}{jk} \end{bmatrix} \tag{5}$$

In terms of these complex amplitudes, the required force is

$$f_z = \frac{A}{4} \mathrm{Re}\, \tilde{B}_x^r \tilde{H}_z^{r*} \tag{6}$$

From Eq. 5b,

$$\tilde{H}_z^r = jk \tanh kd \left(\frac{\tilde{B}_x^r}{\mu_o k} + \frac{\tilde{K}_z^s}{jk \sinh kd} \right) \tag{7}$$

Introduced into Eq. 6, this expression gives

$$f_z = \frac{A}{4} \frac{1}{\cosh kd} \mathrm{Re}\, \tilde{K}_z^{s*} \tilde{B}_x^r \tag{8}$$

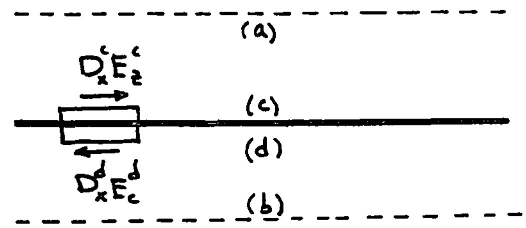
For the particular distributions of Eqs. 3 and 4,

$$\begin{aligned} f_z &= \frac{A}{4} \frac{1}{\cosh kd} \mathrm{Re} \left(j K_o^s e^{-j\omega t} \right) \left(B_o^r e^{jk(Ut+\delta)} \right) \\ &= -\frac{A}{4} \frac{1}{\cosh kd} K_o^s B_o^r \sin\left[(kU-\omega)t + k\delta \right] \end{aligned} \tag{9}$$

Under synchronous conditions, this becomes

$$f_z = -\frac{A K_o^s B_o^r}{4 \cosh kd} \sin k\delta$$

<u>Prob. 4.3.3</u> With positions as designated in the sketch, the total force per unit area is



$$\langle f_z \rangle_z = \langle D_x^c E_z^c - D_x^d E_z^d \rangle_z A \qquad (1)$$

$$= \frac{1}{2}\mathrm{Re}\left(\tilde{D}_x^c \tilde{E}_z^{c*} - \tilde{D}_x^d \tilde{E}_z^{d*}\right) A$$

With the understanding that the surface charge on the sheet is a given quantity, boundary conditions reflecting the continuity of tangential electric field at the three surfaces and that Gauss' law be satisfied through the sheet are

$$\tilde{\Phi}^a \text{ given}\ ;\ \tilde{\Phi}^b \text{ given}\ ;\ \tilde{\Phi}^c = \tilde{\Phi}^d\ ;\ \tilde{D}_x^c - \tilde{D}_x^d = \tilde{\sigma}_f \text{ given} \qquad (2)-(4)$$

Bulk relations are given by Table 2.16.1. In the upper region

$$\begin{bmatrix} \tilde{D}_x^a \\ \tilde{D}_x^c \end{bmatrix} = \epsilon_o k \begin{bmatrix} -\coth kd & \dfrac{1}{\sinh kd} \\ -\dfrac{1}{\sinh kd} & \coth kd \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^a \\ \tilde{\Phi}^c \end{bmatrix} \qquad (5)$$

and in the lower

$$\begin{vmatrix} \tilde{D}_x^d \\ \tilde{D}_x^b \end{vmatrix} = \epsilon_o k \begin{bmatrix} -\coth kd & \dfrac{1}{\sinh kd} \\ -\dfrac{1}{\sinh kd} & \coth kd \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^d \\ \tilde{\Phi}^b \end{bmatrix} \qquad (6)$$

In view of Eq. 4, Eq. 1 becomes

$$\langle f_z \rangle_z = \frac{1}{2}\mathrm{Re}\left[-jk\tilde{\sigma}_f \tilde{\Phi}^{c*}\right] A \qquad (7)$$

so what is now required is the amplitude $\tilde{\Phi}^c$. The surface charge, given by Eq. 4, as the difference $\tilde{D}_x^c - \tilde{D}_x^d$, follows in terms of the potentials from taking the difference of Eqs. 5b and 6a. The resulting expression is solved for

$$\tilde{\Phi}^c = \frac{\tilde{\sigma}_f}{2\epsilon k \coth kd} + \frac{\tilde{\Phi}^a + \tilde{\Phi}^b}{2\cosh kd} \qquad (8)$$

Substituted into Eq. 7 (where the self terms in $\tilde{\sigma}_f \tilde{\sigma}_f^*$ are imaginary and can therefore be dropped) the force is expressed in terms of the given excitations.

Prob. 4.3.3(cont.)2

$$\langle f_z \rangle_z = \frac{1}{2} k \,\mathrm{Re}\left[-j\tilde{\sigma}_f \frac{\tilde{\tilde{\Phi}}^{a*} + \tilde{\tilde{\Phi}}^{b*}}{2\cosh kd}\right] A \tag{9}$$

b) Translation of the given excitations into complex amplitudes gives

$$\begin{aligned} \tilde{\sigma}_f &= -\sigma_o e^{j\omega t} e^{jk\delta} \\ \tilde{\tilde{\Phi}}^a &= V_o e^{j\omega t} \\ \tilde{\tilde{\Phi}}^b &= \pm V_o e^{j\omega t} \end{aligned} \tag{10}$$

Thus, with the even excitation, where $\Phi^a = \Phi^b$

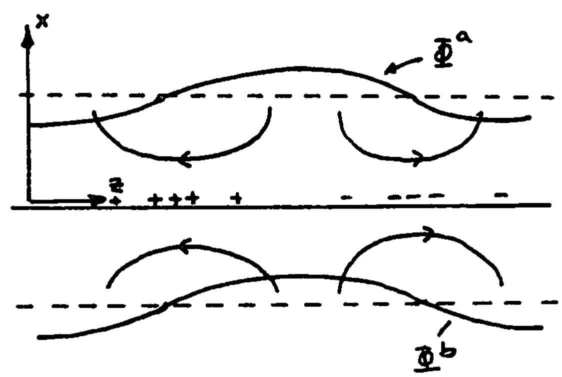
$$\langle f_z \rangle_z = -\frac{k V_o \sigma_o A}{2\cosh kd} \sin k\delta \tag{11}$$

and with the odd excitation, $\langle f_z \rangle_z = 0$.

c) This is a specific case from part (b) with $\omega = 0$ and $\delta = \lambda/4$. Thus,

$$\langle f_z \rangle_z = -\frac{k V_o \sigma_o A}{2\cosh kd} \tag{12}$$

The sign is consistent with the sketch of charge distribution on the sheet and electric field due to the potentials on the walls sketched.



Prob. 4.4.1 a) In the rotor, the magnetization, $\bar{M}$, is specified. Also, it is uniform, and hence has no curl. Thus, within the rotor,

$$\nabla \times \bar{B} \equiv \nabla \times [\mu_o(\bar{H}+\bar{M})] = \nabla \times \mu_o \bar{H} = 0 \tag{1}$$

Also, of course, $\bar{B}$ is solenoidal.

$$\nabla \cdot \bar{B} = 0 \Rightarrow \bar{B} = \nabla \times \bar{A} \tag{2}$$

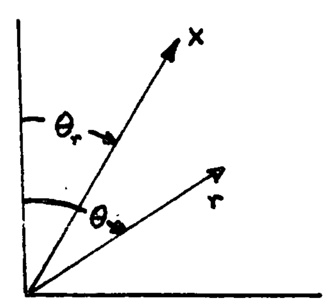
So, the derivation of transfer relations between $\bar{B}$ and $\bar{A}$ is the same as in Sec. 2.19 so long as $\mu_o\bar{H}$ is identified with $\bar{B}$.

b) The condition on the jump in normal flux density is as usual. However, with $\bar{M}$ given, Ampere's law requires that $\bar{n} \times [\![\bar{H}]\!] = \bar{K}_f$ and this can be rewritten using the definition of $\bar{B}$, $\bar{B} = \mu_o(\bar{H}+\bar{M})$. Thus, the boundary condition becomes

$$\bar{n} \times [\![\bar{B}]\!] = \mu_o \bar{K}_f + \mu_o \bar{n} \times [\![\bar{M}]\!] \tag{3}$$

where the jump in tangential $\bar{B}$ is related to the given surface current and <u>given</u> jump in magnetization.

c) With these background statements, the representation of the fields, solution for the torque and determination of the electrical terminal relation follows the usual pattern. First, represent the boundary conditions in terms of the given form of excitation. The magnetization can be written in complex notation, perhaps most efficiently, with the following reasoning. Use x as a <u>cartesian</u> coordinate rotated to the rotor axis angle, as shown in the figure. Then, if the gradient is pictured for the moment in cartesian coordinates, it can be seen that the uniform vector field $M_o\bar{i}_x$ is represented by



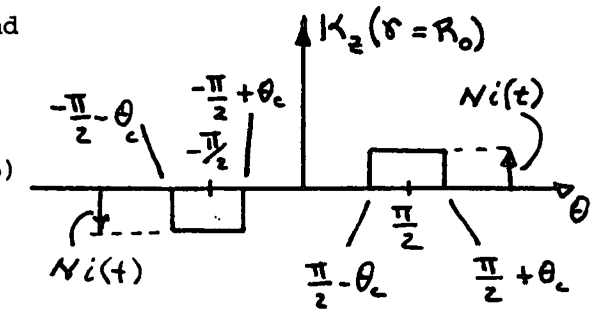
$$\bar{M} = -\nabla \Theta \; ; \; \Theta = -M_o x \tag{4}$$

Prob. 4.4.1(cont.)

Observe that x = r cos(θ - θ_r) and it follows from Eq. 4 that $\bar{M}$ is written in the desired Fourier notation as

$$\bar{M} = \nabla M_o r \cos(\theta-\theta_r) = \nabla\left\{\frac{M_o r}{2}\left[e^{j(\theta-\theta_r)} + e^{-j(\theta-\theta_r)}\right]\right\} \tag{5}$$
$$= \frac{M_o}{2}\left\{\bar{i}_r\left[e^{-j\theta_r}e^{j\theta} + e^{j\theta_r}e^{-j\theta}\right] + \bar{i}_\theta\left[j e^{-j\theta_r}e^{j\theta} - j e^{j\theta_r}e^{-j\theta}\right]\right\}$$

Next, the stator currents are represented in complex notation. The distribution of surface current is as shown in the figure and represented in terms of a Fourier series.



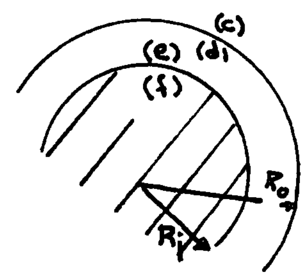
$$\bar{H} = \sum_{m=-\infty}^{+\infty} \tilde{\bar{H}}_m e^{-jm\theta} \tag{6}$$

The coefficients are given by (Eq.2.15.8)

$$\tilde{K}_{zn}(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_z(\theta,t)e^{jn\theta}d\theta = \frac{2Ni}{\pi n} j \sin\left(\frac{n\pi}{2}\right) \sin(n\theta_o) \tag{7}$$

Thus, because superposition can be used throughout, it is possible to determine the fields by considering the boundary conditions as applying to the complex Fourier amplitudes.

Boundary conditions reflecting Eq. 2 at each of the interfaces (designated as shown in the sketch) are,



$$\tilde{A}^c_n = \tilde{A}^d_n \tag{8}$$

$$\tilde{A}^e_n = \tilde{A}^f_n \tag{9}$$

while those representing Eq. 3 at each interface are

$$-\tilde{B}^d_{\theta n} = \mu_o \tilde{K}_{zn} \tag{10}$$

$$\tilde{B}^e_{\theta n} - \tilde{B}^f_{\theta n} = -\mu_o\tilde{M}_{\theta n} = -\left[j e^{-j\theta_r}\delta_{-1n} - j e^{j\theta_r}\delta_{1n}\right]\frac{M_o\mu_o}{2} \tag{11}$$

That $\bar{H}$=0 in the infinitely permeable stator is reflected in Eq. 10. Thust, Eq. 8 is not required to determine the fields in the gap and in the rotor.

Prob. 4.4.1(cont.)

In the gap and within the rotor, the transfer relations (Eqs. (c) of Table 2.19.1) apply

$$\begin{bmatrix} \tilde{B}^d_{\theta m} \\ \tilde{B}^e_{\theta m} \end{bmatrix} = \begin{bmatrix} f_m(R_i,R_o) & g_m(R_o,R_i) \\ g_m(R_i,R_o) & f_m(R_o,R_i) \end{bmatrix} \begin{bmatrix} \tilde{A}^d_m \\ \tilde{A}^e_m \end{bmatrix} \tag{12}$$

$$\tilde{B}^f_{\theta m} = f_m(0,R_i)\tilde{A}^f_m \tag{13}$$

Before solving these relations for the Fourier amplitudes, it is well to look ahead and see just which ones are required. To determine the torque, the rotor can be enclosed by any surface within the air-gap, but the one just inside the stator has the advantage that the tangential field is specified in terms of the driving current, Eq. 10. For that surface (using Eq. 3.9.17 and the orthogonality relation for space averaging the product of Fourier series, Eq. 2.15.17),

$$\begin{aligned} \tau &= R_o(2\pi R_o d)\langle T^d_{\theta r}\rangle_\theta = 2\pi R^2 d \langle H^d_r B^d_\theta \rangle_\theta \\ &= 2\pi R_o^2 d \sum_{m=-\infty}^{+\infty} \tilde{H}^d_{rm} \tilde{B}^{e*}_{\theta m} \end{aligned} \tag{14}$$

Because $\tilde{B}^e_{\theta m}$ is known, it is $\tilde{H}^d_{rm}$ that is required where $\tilde{H}^d_{rm} = -jm\tilde{A}^d_m/\mu_o R_o$.

Subtract Eq. 13 from Eq. 12b and use the result to evaluate Eq. 11. Then, in view of Eq. 9 the first of the following two relations follow.

$$\begin{bmatrix} g_m(R_i,R_o) & f_m(R_o,R_i)-f_m(0,R_i) \\ f_m(R_i,R_o) & g_m(R_o,R_i) \end{bmatrix} \begin{bmatrix} \tilde{A}^d_m \\ \tilde{A}^e_m \end{bmatrix} = \begin{bmatrix} -\tilde{M}_{om} \\ \mu_o \tilde{K}_{zm} \end{bmatrix} \tag{15}$$

The second relation comes from Eqs. 12a and 10. From these two equations in two unknowns the required amplitude follows

$$\tilde{A}^d_m = \frac{-\tilde{M}_{om}\, g_m(R_o,R_i) - \mu_o \tilde{K}_{zm}[f_m(R_o,R_i) - f_m(0,R_i)]}{D_m} \tag{16}$$

where $D_m \equiv g_m(R_i,R_o)g_m(R_o,R_i) - f_m(R_i,R_o)[f_m(R_o,R_i) - f_m(0,R_i)]$

Prob. 4.4.1(cont.)

Evaluation of the torque, Eq. 14, follows by substitution of $\tilde{H}^d_{rm}$ as determined by Eq. 16 and $\tilde{B}^d_{\theta m}$ as given by Eq. 10.

$$T_z = 2\pi R_o^2 d \sum_{m=-\infty}^{+\infty} \Big\{ -\frac{jm\, g_m(R_o,R_i)\tilde{M}_{\theta m}\tilde{K}^*_{zm}}{R_o D_m} - \frac{jm}{R_o}\mu_o \frac{\tilde{K}_{zm}\tilde{K}^*_{zm}}{D_m}\left[f_m(R_o,R_i)-f_m(0,R_i)\right]\Big\} \tag{17}$$

The second term involves products of the stator excitation amplitudes and it must therefore be expected that this term vanishes. To see that this is so, observe that $\tilde{K}_{zm}\tilde{K}^*_{zm}$ is positive and real and that f_m and g_m are even in m. Because of the m appearing in the series it then follows that the m term cancels with the -m term in the series. The first term is evaluated by using the expressions for $\tilde{M}_{\theta m}$ and $\tilde{K}^*_{zm}$ given by Eqs. 10 and 11. Because there are only two Fourier amplitudes for the magnetization, the torque reduces to simply

$$T_z = -4\mu_o R_o d M_o \sin\theta_o K \sin\theta_r N i(t) \tag{18}$$

where

$$K = g_1(R_o,R_i)\Big/\left\{g_1(R_i,R_o)g_1(R_o,R_i) - f_1(R_i,R_o)\left[f_1(R_o,R_i)-f_1(0,R_i)\right]\right\}$$

From the definitions of g_m and f_m, it can be shown that $K=R_i^2/R_o$, so that the final answer is simply

$$T_z = -4\mu_o R_i^2 d M_o \sin\theta_o \sin\theta_r N i(t) \tag{19}$$

Note that this is what is obtained if a dipole moment is defined as the product of the uniform volume magnetization multiplied over the rotor volume and directed at the angle θ_r.

$$|\bar{m}| = \pi R_i^2 d M_o \tag{20}$$

in a uniform magnetic field associated with the m=1 and m=-1 modes,

$$|\bar{H}| = \frac{4N i(t)}{\pi}\sin\theta_o \tag{21}$$

with the torque evaluated as simply $\bar{T} = \mu_o \bar{m}\times\bar{H}$. (Eq. 2, Prob. 3.6.2)

Prob. 4.4.1(cont.)

The flux linked by turns at the position θ having the span $R_o d\theta$ is

$$\Phi_\lambda = [N R_o(d\theta)][\,A^d(\theta) - A^d(\theta+\pi)]d \tag{22}$$

Thus, the total flux is

$$\lambda = \int_{\frac{\pi}{2}-\theta_o}^{\frac{\pi}{2}+\theta_o} \Phi_\lambda d\theta = \int_{\frac{\pi}{2}-\theta_o}^{\frac{\pi}{2}+\theta_o} d N R_o \sum_{m=-\infty}^{+\infty} \tilde{A}_m^d (1 - e^{-jm\pi}) e^{-jm\theta} d\theta \tag{23}$$

The exponential is integrated to give

$$\lambda = 4 d N R_o \sum_{m=-\infty}^{+\infty} \tilde{A}_m^d \frac{j}{m} e^{-jm\pi} \sin\left(\frac{m\pi}{2}\right) \sin m\theta_o \tag{24}$$

where the required amplitude, $\tilde{A}_m^d$, is given by Eq. 16. Substitution shows that

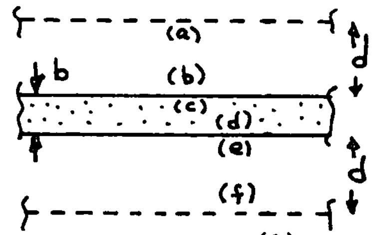
$$\lambda = L i(t) + A_r \mu_o M_o \cos\theta_r \tag{25}$$

where

$$L = \frac{8}{\pi} N^2 \mu_o R_o d \sum_{\substack{m=-\infty \\ (odd)}}^{+\infty} \left(\frac{\sin m\theta_o}{m}\right)^2 \frac{[f_m(0, R_i) - f_m(R_o, R_i)]}{[g_m(R_i, R_o)/R_i]}$$

and

$$A_r = 4 N R_i^2 d \sin\theta_o$$



Prob. 4.6.1 With locations as indicated by the sketch, the boundary conditions are written in terms of complex amplitudes as

$$\tilde{\Phi}^a = \tilde{V}_o;\ \tilde{\Phi}^b = \tilde{\Phi}^c;\ \tilde{D}_x^b = \tilde{D}_x^c;\ \tilde{\Phi}^d = \tilde{\Phi}^e;\ \tilde{D}_x^d = \tilde{D}_x^e;\ \tilde{\Phi}^f = \tilde{V}_o \tag{1}$$

Because of the axial symmetry, the analysis is simplified by recognizing that

$$\tilde{\Phi}^f = \tilde{\Phi}^a;\quad \tilde{D}_x^f = -\tilde{D}_x^a \tag{2}$$

This makes it possible to write the required force as

$$f_z = A\left\langle E_z^a D_x^a - E_z^f D_x^f \right\rangle_z = A\,\mathrm{Re}\left(-jk\tilde{\Phi}^{a*}\tilde{D}_x^a\right) = A\,\mathrm{Re}\left(-jk\tilde{V}_o^*\tilde{D}_x^a\right) \tag{3}$$

The transfer relations for the beam are given by Eq. 4.5.18, which becomes

$$\begin{bmatrix} \tilde{\Phi}^c \\ \tilde{\Phi}^d \end{bmatrix} = \frac{1}{\epsilon_o k}\begin{bmatrix} -\coth kb & \dfrac{1}{\sinh kb} \\ \dfrac{-1}{\sinh kb} & \coth kb \end{bmatrix}\begin{bmatrix} \tilde{D}_x^c \\ \tilde{D}_x^d \end{bmatrix} + \sum_{i=0}^{\infty} \frac{\tilde{\rho}_i}{\epsilon_o(\nu_i^2 + k^2)}\begin{bmatrix} (-1)^i \\ 1 \end{bmatrix} \tag{4}$$

These also apply to the air-gap, but instead use the inverse form from Table 2.16.1.

$$\begin{bmatrix} \tilde{D}_x^a \\ \tilde{D}_x^b \end{bmatrix} = \epsilon_o k\begin{bmatrix} -\coth kd & \dfrac{1}{\sinh kd} \\ \dfrac{-1}{\sinh kd} & \coth kd \end{bmatrix}\begin{bmatrix} \tilde{\Phi}^a \\ \tilde{\Phi}^b \end{bmatrix} \tag{5}$$

From the given distribution of ρ it follows that only one Fourier mode is required (because of the boundary conditions chosen for the modes).

$$\Pi_i = \begin{cases} 1 & i = 0 \\ 0 & i \neq 0 \end{cases} \Rightarrow \tilde{\rho}_i = \begin{cases} \tilde{\rho}_o & i = 0 \\ 0 & i \neq 0 \end{cases};\ \nu_i = 0 \tag{6}$$

With the boundary and symmetry conditions incorporated, Eqs. 4 and 5 become

$$\begin{bmatrix} \tilde{D}_x^a \\ \tilde{D}_x^b \end{bmatrix} = \epsilon_o k\begin{bmatrix} -\coth kd & \dfrac{1}{\sinh kd} \\ \dfrac{-1}{\sinh kd} & \coth kd \end{bmatrix}\begin{bmatrix} \tilde{V}_o \\ \tilde{\Phi}^b \end{bmatrix} \tag{7}$$

Prob. 4.6.1 (cont)

$$\begin{bmatrix}\tilde{\Phi}^b\\ \\ \tilde{\Phi}^b\end{bmatrix}=\frac{1}{\epsilon_o k}\begin{bmatrix}-\coth kb & \frac{1}{\sinh kb}\\ \\ \frac{-1}{\sinh kb} & \coth kb\end{bmatrix}\begin{bmatrix}\tilde{D}_x^b\\ \\ \tilde{D}_x^b\end{bmatrix}+\frac{\tilde{\rho}_o}{\epsilon_o k^2} \tag{8}$$

These represent four equations in the three unknowns $\tilde{D}_x^a, \tilde{D}_x^b, \tilde{\Phi}^b$. They are redundent because of the implied symmetry. The first three equations can be written in the matrix form (9)

$$\begin{bmatrix}-1 & 0 & \frac{\epsilon_o k}{\sinh kd}\\ \\ 0 & -1 & \epsilon_o k\coth kd\\ \\ 0 & -\frac{1}{\epsilon_o k}\left(\coth kb+\frac{1}{\sinh kb}\right) & -1\end{bmatrix}\begin{bmatrix}\tilde{D}_x^a\\ \\ \tilde{D}_x^b\\ \\ \tilde{\Phi}^b\end{bmatrix}=\begin{bmatrix}\epsilon_o k\coth kd\,\tilde{V}_o\\ \\ \frac{\epsilon_o k\tilde{V}_o}{\sinh kd}\\ \\ -\frac{\tilde{\rho}_o}{\epsilon_o k^2}\end{bmatrix}$$

In using Cramer's rule for finding $\tilde{D}_x^a$ (required to evaluate Eq. 3) note that terms proportional to $\tilde{V}_o$ will make no contribu on when inserted into Eq. 3 (all coefficients in Eq. 9 are real), so there is no need to write these terms out. Thus,

$$\tilde{D}_x^a=[\;]\tilde{V}_o+\frac{\tilde{\rho}_o G}{k}\;;\; G=\left[\sinh kd+\cosh kd\left(\coth kb+\frac{1}{\sinh kb}\right)\right]^{-1} \tag{10}$$

and Eq. 3 becomes

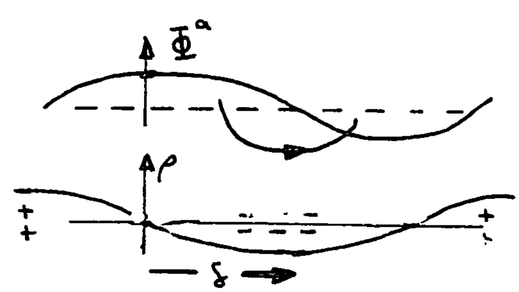
$$f_z = A G\,\mathrm{Re}\left[-j\tilde{V}_o^{*}\tilde{\rho}_o\right] \tag{11}$$

b) In the particular case where

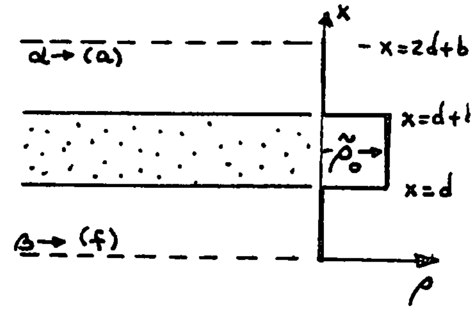
$$\tilde{V}_o = V_o e^{j\omega t}\;;\;\tilde{\rho}_o=-\rho_o e^{j(\omega t+k\delta)} \tag{12}$$

the force given by Eq. 11 reduces to

$$f_z=-AGV_o\rho_o\sin k\delta \tag{13}$$



The sketch of the wall potential and the beam charge when t=0 suggests that indeed the force should be zero if δ and be negative if $0<k\delta<\pi$



Prob. 4.6.1 (cont.)

c) With the entire region represented by the relations of Eq. 4, the charge distribution to be represented by the modes is that of the sketch. With $\Delta \equiv b + 2d$ and $\Pi_i = \cos \frac{i\pi}{\Delta} x$, Eq. 4.5.17 gives the mode amplitudes.

$$\tilde{\rho}_i = \frac{2}{\Delta}\int_d^{d+b} \tilde{\rho}_o \cos\frac{i\pi}{\Delta}x\, dx = \frac{2\tilde{\rho}_o}{i\pi}\left[\sin\frac{i\pi}{\Delta}(d+b) - \sin\frac{i\pi d}{\Delta}\right]\Bigg|_{i\neq 0};\ \tilde{\rho}_i = \frac{\tilde{\rho}_o b}{\Delta}\Bigg|_{i=0} \tag{14}$$

So, with the transfer relations of Eq. 4.5.18 applied to the entire region,

$$\begin{bmatrix}\tilde{V}_o \\ \tilde{V}_o\end{bmatrix} = \frac{1}{\epsilon_o k}\begin{bmatrix} -\coth k\Delta & \frac{1}{\sinh k\Delta} \\ -\frac{1}{\sinh k\Delta} & \coth k\Delta\end{bmatrix}\begin{bmatrix}\tilde{D}_x^a \\ \tilde{D}_x^f\end{bmatrix} + \sum_{i=0}^{\infty}\frac{\tilde{\rho}_i}{\epsilon_o\left[\left(\frac{i\pi}{\Delta}\right)^2 + k^2\right]}\begin{bmatrix}(-1)^i \\ 1\end{bmatrix} \tag{15}$$

Symmetry requires that $\tilde{D}_x^a = -\tilde{D}_x^f$, which is consistent with both of Eqs. 15 reducing to the same thing. That is, the modal amplitudes are zero for i odd. From either equation it follows that

$$\tilde{V}_o = \frac{1}{\epsilon_o k}\left[-\coth k\Delta - \frac{1}{\sinh k\Delta}\right]\tilde{D}_x^a + \sum_{i=0}\frac{\tilde{\rho}_i}{\epsilon_o\left[\left(\frac{i\pi}{\Delta}\right)^2 + k^2\right]} \tag{16}$$

The terms multiplying V_o are not written out because they make no contribution to the force.

$$\tilde{D}_x^a = (\quad)\tilde{V}_o + \frac{\epsilon_o k \sinh k\Delta}{(\cosh k\Delta + 1)}\sum_{\substack{i=0 \\ (\text{even})}}^{\infty}\frac{\tilde{\rho}_i}{\epsilon\left[\left(\frac{i\pi}{\Delta}\right)^2 + k^2\right]} \tag{17}$$

Thus, the force is evaluated using as surfaces of integration surfaces at (a) and (f).

$$f_z = \frac{1}{2}A\,\mathrm{Re}\left(-jk\tilde{\Phi}^{a*}\tilde{D}_x^a + jk\tilde{\Phi}^{f*}\tilde{D}_x^f\right) = Ak\,\mathrm{Re}\left(-j\tilde{V}_o^*\tilde{D}_x^a\right) \tag{18}$$

$$= -\frac{Ak^2\epsilon_o \sinh k\Delta}{\cosh k\Delta + 1}\,\mathrm{Re}\sum_{\substack{i=0 \\ \text{even}}}^{\infty}\frac{j\tilde{V}_o^*\tilde{\rho}_i}{\epsilon\left[\left(\frac{i\pi}{\Delta}\right)^2 + k^2\right]}$$

Prob. 4.6.1 (cont)

In terms of the z-t dependence given by Eq. 12, this force is

$$f_z = -\frac{A k^2 \epsilon_o \sinh k\Delta}{\cosh k\Delta + 1}\left\{\frac{b}{\Delta\epsilon k^2} + \sum_{\substack{i=2\\ \text{even}}}^{\infty} \frac{2[\sin\frac{i\pi}{\Delta}(d+b) - \sin\frac{i\pi d}{\Delta}]}{\epsilon i\pi[(\frac{i\pi}{\Delta})^2 + k^2]}\right\} V_o \rho_o \sin k\delta \quad (19)$$

Prob. 4.8.1 a)The relations of Eq. 9 are applicable in the case of the planar layer provided the coefficients F_m and G_m are identified by comparing Eq. 8 to Eq. (b) of Table 2.19.1.

$$F_m(\beta,\alpha) = -F_m(\alpha,\beta) \to -\frac{\coth k\Delta}{k} \; ; \; G_m(\alpha,\beta) = -G_m(\beta,\alpha) \to \frac{1}{k \sinh k\Delta} \quad (1)$$

Thus, the transfer relations are as given in the problem.

b) The given forms of A_p and J_z are substituted into Eq. 4.8.3a to show that

$$\frac{d^2\Pi_i}{dx^2} + \nu_i^2 \Pi_i = 0 \quad (2)$$

where

$$\nu_i^2 = \frac{\mu \tilde{J}_i^2}{\tilde{A}_i} - k^2 \quad (3)$$

Solutions to Eq. that have zero derivatives on the boundaries (and hence make H_{yp}=0 on the α and β surfaces) are

$$\Pi_i = \cos\nu_i x \; ; \; \nu_i = \frac{i\pi}{\Delta} \; , \; i = 0, 1, 2, \ldots \quad (4)$$

From Eq. 3 it then follows that

$$\tilde{A}_i \Pi_i = \frac{\mu \tilde{J}_i \cos\nu_i x}{(k^2 + (\frac{i\pi}{\Delta})^2)} \quad (5)$$

Substitution into the general transfer relation found in part (a) then gives the required transfer relation from part (b).

In view of the Fourier modes selected to represent the x dependence, Eq. 4, the Fourier coefficients are

$$\underset{i\neq 0}{\tilde{J}_i} = \frac{2}{\Delta}\int_0^\Delta \tilde{J}_z(x)\cos\left(\frac{\pi i}{\Delta}x\right)dx \; ; \; \tilde{J}_o = \frac{1}{\Delta}\int_0^\Delta \tilde{J}_z(x)dx \quad (6)$$

Prob. 4.9.1 Because of the step function dependence of the current density on y, it is generally necessary to use a Fourier series representation (rather than complex amplitudes). The positions just below the stator current sheet and just above the infinitely permeable "rotor" material are designated by (a) and (b) respectively. Then, in terms of the Fourier amplitudes, the force per unit y-z area is

$$T_y = \langle H_y^a B_x^a \rangle_y = \sum_{m=-\infty}^{+\infty} \tilde{H}_{ym}^{*} \tilde{B}_{xm} = \sum_{m=-\infty}^{+\infty} j \tilde{K}_m^{s*} k_m \tilde{A}_m^a \tag{1}$$

The stator excitation is represented as a Fourier series by writing it as

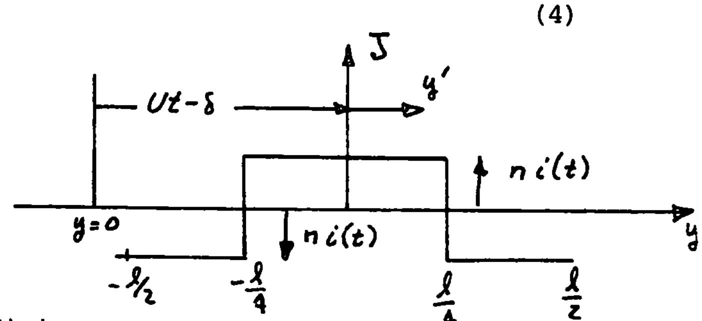
$$K_z^s = \frac{\tilde{K}^s}{2} e^{-jk_1 y} + \frac{\tilde{K}^{s*}}{2} e^{jk_1 y} = \sum_{m=-\infty}^{+\infty} \tilde{K}_m^s e^{-jk_m y}; \quad \tilde{K}_m^s = \frac{1}{2}\left(\tilde{K}^s \delta_{1m} + \tilde{K}^{s*} \delta_{-1m}\right) \tag{2}$$

The "rotor" current density is written so as to be consistent with the adaptation of the transfer relations of Prob. 4.8.1 to the Fourier representation.

$$J = \sum_{m=-\infty}^{+\infty} \sum_{p=0}^{\infty} \tilde{J}_{mp}(t) \cos \nu_p x \, e^{-jk_m y} \tag{3}$$

Here, the expansion on p accounting for the x dependence reduces to just the p=0 term, so Eq. 3 becomes

$$J = \sum_{m=-\infty}^{+\infty} \tilde{J}_{mo}(t) e^{-jk_m y} \tag{4}$$



The coefficients J_{mo} are determined by the y dependence, sketched in the figure. First, expand in terms of the series

$$J = \sum_{m=-\infty}^{+\infty} \tilde{J}'_{mo} e^{-jk_m y'} \tag{5}$$

where $y' = y - (Ut - \delta)$. This gives the coefficients

$$\tilde{J}'_{mo} = \frac{2ni(t)}{\pi m} \sin\left(\frac{m\pi}{2}\right) \tag{6}$$

Thus, the coefficients in the y dependent Fourier series, Eq.4, become

$$\tilde{J}_{mo} = \frac{2ni(t)}{\pi m} \sin \frac{m\pi}{2} e^{jk_m(Ut-\delta)} \tag{7}$$

Boundary conditions at the (a) and (b) surfaces require that $\tilde{H}_{ym}^b = 0$ and $\tilde{H}_{ym}^a = -\tilde{K}_m^s$. Thus, the first equation in the transfer relation found in Problem 4.8.1 becomes

Prob. 4.9.1(cont.)

$$\tilde{A}_m^a = \frac{\mu_o}{k_m}\coth k_m d\ \tilde{K}_m^s + \frac{\mu_o \tilde{J}_{mo}}{k_m^2} \tag{8}$$

Thus, Eq. 1 can be evaluated. Note that the "self" terms drop out because the coefficient of $\tilde{K}_m^s\ \tilde{K}_m^{s*}$ is odd in m (the m'th term is cancelled by the -m'th term)

$$T_y = \sum_{m=-\infty}^{+\infty} \frac{j\mu_o}{2\pi m k_m}\left[\tilde{K}^{s*}\delta_{1m} + \tilde{K}^s\delta_{-1m}\right]\sin\left(\frac{m\pi}{2}\right)e^{jk_m(Ut-\delta)} \tag{9}$$

This expression reduces to

$$T_y = \frac{2\mu_o n\, i(t)}{\pi k_1}\left[\frac{\tilde{K}^s e^{-jk_1(Ut-\delta)} - \tilde{K}^{s*}e^{jk_1(Ut-\delta)}}{2j}\right] \tag{10}$$

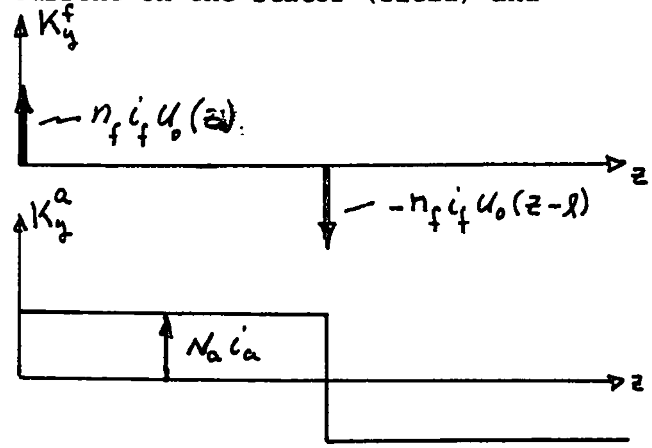
If the stator current is the pure traveling wave

$$K^s = K_o^s \cos(\omega t - k_1 y) \Rightarrow \tilde{K}^s = K_o^s e^{j\omega t} \tag{11}$$

and Eq. (10) reduces to

$$T_y = n\, i(t)\,\frac{\mu_o \ell}{\pi^2} K_o^s \sin\left(\frac{2\pi\delta}{\ell}\right) \tag{12}$$

<u>Prob. 4.10.1</u> The distributions of surface current on the stator (field) and rotor (armature) are shown in the sketches. These are represented as Fourier series having the standard form



$$K_y^f = \sum_{m=-\infty}^{+\infty} \tilde{K}_m^f e^{-jk_m z} \tag{1}$$

with coefficients given by

$$\tilde{K}_m^f = \frac{1}{2\ell}\int_0^{2\ell} K_y^f e^{jk_m z}\,dz \tag{2}$$

It follows that the Fourier amplitudes are

$$\tilde{K}_m^f = \frac{n_f i_f}{2\ell}\left(1 - e^{-jm\pi}\right) \tag{3}$$

and

$$\tilde{K}_m^a = j\frac{N_a i_a}{m\pi}\left(1 - e^{jm\pi}\right) \tag{4}$$

Boundary conditions at the stator (f) and rotor (a) surfaces are $(\bar{H} = -\nabla\psi)$

$$H_z^f = K_y^f \Rightarrow \tilde{\psi}_m^f = \tilde{K}_m^f / jk_m \tag{5}$$

$$H_z^a = -K_y^a \Rightarrow \tilde{\psi}_m^a = -\tilde{K}_m^a / jk_m \tag{6}$$

Fields in the air-gap are represented by the flux-potential transfer relations (Table 2.16.1)

$$\begin{bmatrix} \tilde{B}_{xm}^f \\ \\ \tilde{B}_{xm}^a \end{bmatrix} = \mu_o \frac{m\pi}{\ell} \begin{bmatrix} -\coth\left(\frac{m\pi b}{\ell}\right) & \dfrac{1}{\sinh\left(\frac{m\pi b}{\ell}\right)} \\ \\ \dfrac{-1}{\sinh\left(\frac{m\pi b}{\ell}\right)} & \coth\left(\frac{m\pi b}{\ell}\right) \end{bmatrix} \begin{bmatrix} \dfrac{\tilde{K}_m^f}{jk_m} \\ \\ -\dfrac{\tilde{K}_m^a}{jk_m} \end{bmatrix} \tag{7}$$

The force is found by evaluating the Maxwell stress over a surface that encloses the rotor with the air-gap part of the surface adjacent to the rotor (where fields are denoted by (a)).

$$f_z = 2\ell d \langle B_x^a H_z^a \rangle_z = -2\ell d \langle B_x^a K_y^a \rangle_z = -2\ell d \sum_{m=-\infty}^{+\infty} \tilde{B}_{xm}^a (\tilde{K}_m^a)^* \tag{8}$$

Prob. 4.10.1(cont.)

In view of the transfer relations, Eqs. 7, this expression becomes

$$f_z = -j2\ell d\mu_o \sum_{m=-\infty}^{+\infty} \frac{\left(\tilde{K}_m^{a}\right)^{*}\tilde{K}_m^{f}}{\sinh\left(\frac{m\pi b}{\lambda}\right)} \tag{9}$$

In turn, the surface currents are given in terms of the terminal currents by Eqs. 3 and 4. Note that the self-field term makes no contribution because the sum is over terms that are odd in m. That is, for the self-field contribution, the m'th term in the series is cancelled by the -m'th term.

$$f_z = \mu_o d N_a i_a n_f i_f \sum_{m=-\infty}^{+\infty} - \frac{\left(1-e^{-jm\pi}\right)\left(1-e^{-jm\pi}\right)}{m\pi \sinh\left(\frac{m\pi b}{\ell}\right)} \tag{10}$$

This expression reduces to the standard form

$$f_z = -G_m i_a i_f \tag{11}$$

where

$$G_m = \mu_o d N_a n_f \sum_{m\text{ odd}}^{\infty} \frac{8}{\pi} \frac{1}{m \sinh\left(\frac{m\pi b}{\ell}\right)} \tag{12}$$

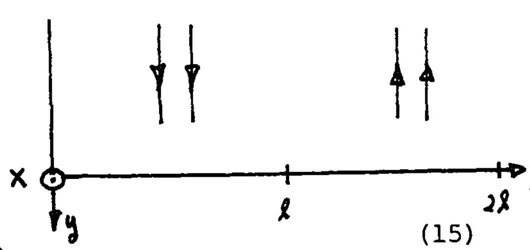
To find the armature terminal relation, Faraday's integral law is written for a contour that is fixed in space and passes through the brushes and instantaneously contiguous conductors.

$$\oint_C (\bar{E} - \bar{v}\times\mu_o\bar{M})\cdot d\bar{l} = -\int_S \frac{\partial \bar{B}}{\partial t}\cdot\bar{n}\,da \tag{13}$$

In the conductors, $\bar{M}=0$ and Ohm's law requires that

$$\bar{E} = \frac{\bar{J}}{\sigma} - \bar{v}\times\mu_o\bar{H} \tag{14}$$

The armature winding is wound as in Fig. 4.10.3a with the axes and position of the origin as sketched to the right. Thus, Eq. 13 becomes



$$-v_a + \int_{wire} \frac{\bar{J}}{\sigma}\cdot d\bar{l} - \int_{wire} U B_x \bar{i}_y \cdot d\bar{l} = -\frac{d}{dt}\int_S B_x da \tag{15}$$

Each of the solid conductors in Fig. 4.10.3 carries half of the current. Thus, the second term in Eq. 15 becomes

$$\int_{wire} \frac{\bar{J}}{\sigma}\cdot d\bar{l} = \frac{A_a J_y}{A_a\sigma} l_a = \frac{i_a}{2}\frac{l_a}{\sigma A_a} = R_a i_a\,;\; R_a \equiv \frac{l_a}{2\sigma A_a} \tag{16}$$

Prob. 4.10.1(cont.)

The third "speed-voltage" term in Eq. 15 becomes

$$\int_{wire} U B_x \bar{i}_y \cdot d\bar{l} = d \int_0^{l} U B_x N_a d - d \int_{l}^{2l} U B_x N_a dz \tag{17}$$

and this becomes

$$\int_{wire} U B_x \bar{i}_y \cdot d\bar{l} = d U N_a \left\{ \int_0^{l} \sum_{-\infty}^{+\infty} \tilde{B}_{xm}^a e^{-jk_m z} - \int_{l}^{2l} \sum_{-\infty}^{+\infty} \tilde{B}_{xm}^a e^{-jk_m z} \right\}$$

$$= -4j d U N_a \sum_{\substack{m=-\infty \\ (odd)}}^{+\infty} \frac{\tilde{B}_{xm}^a}{k_m} \tag{18}$$

From the bulk transfer relations, Eq. 7b, this becomes

$$\int_{wire} U B_x \bar{i}_y \cdot d\bar{l} = -4j d U N_a \sum_{\substack{m=-\infty \\ (odd)}}^{+\infty} \frac{\mu_o m\pi}{l k_m} \left\{ \frac{-\frac{n_f i_f}{2l}(1-e^{-jm\pi})}{\sinh\left(\frac{m\pi b}{l}\right) j k_m} - \frac{\coth\left(\frac{m\pi b}{l}\right)}{k_m} \frac{N_a i_a}{m\pi}(1-e^{-jm\pi}) \right\} \tag{19}$$

The second term makes no contribution because it is odd in m. Thus, the speed-voltage term reduces to

$$\int_{wire} U B_x \bar{i}_y \cdot d\bar{l} = G_m U i_f \tag{20}$$

where G_m is the same as defined by Eq. 12.

To evaluate the right hand side of Eq. 15, observe that the flux linked by turns in the range z'+dz' to z' is

$$\left(d \int_{z'}^{z'+l} B_x^a \, dz\right) N_a dz' \tag{21}$$

so that altogether the flux linked is

$$\int_S B_x da = \int_0^{l} \left[d \int_{z'}^{z'+l} B_x^a dz \right] N_a dz' \tag{22}$$

Expressed in terms of the Fourier series, this becomes

$$\int_S B_x da = -\frac{4 N_a d l^2}{\pi^2} \sum_{\substack{-\infty \\ (odd)}}^{+\infty} \frac{\tilde{B}_{xm}^a}{m^2} \tag{23}$$

The normal flux at the armature is expressed in terms of the terminal currents by using Eqs. 15b and 3 and 4.

$$\int_S B_x da = -\frac{4 N_a d l \mu_o}{\pi} \sum_{\substack{-\infty \\ (odd)}}^{+\infty} \frac{1}{m} \left\{ \frac{-\left(\frac{n_f i_f}{2l}\right)(1-e^{-jm\pi})}{\sinh\left(\frac{m\pi b}{l}\right) j k_m} - \frac{\coth\left(\frac{m\pi b}{l}\right) N_a i_a}{m\pi \, k_m}(1-e^{-jm\pi}) \right\} \tag{24}$$

Prob. 4.10.1(cont.)

The first term in this expression is odd in m and makes no contribution. Thus, it reduces to simply

$$\int_S B_x \, da = L_a i_a \tag{25}$$

where

$$L_a \equiv \frac{16 N_a^2 d \ell^2 \mu_o}{\pi^3} \sum_{\substack{1 \\ (odd)}}^{\infty} \frac{\coth\left(\frac{m\pi b}{\ell}\right)}{m^3} \tag{26}$$

So, the armature terminal relation is in the classic form

$$v_a = R_a i_a + L_a \frac{d i_a}{dt} - G_m U i_f \tag{27}$$

where R_a, L_a and G_m are defined by Eqs. 16, 26 and 12.

The use of Faraday's law for the field winding is similar but easier because it is not in motion. Equation 13 written for a path through the field winding becomes

$$-v_f + R_f i_f = -\frac{d}{dt} \int B_x^f \, da \tag{28}$$

The term on the right is written in terms of the Fourier series and the integral carried out to obtain

$$\int B_x^f \, da = d n_f \int_0^{\ell} B_x^f \, dz = d n_f \int_0^{\ell} \sum_{-\infty}^{+\infty} \tilde{B}_{xm}^f e^{-jk_m z} \, dz \tag{29}$$

Substitution of Eqs. 3 and 4 gives

$$\int B_x^f da = d n_f \mu_o \sum_{-\infty}^{+\infty} \left(e^{-jm\pi} - 1\right) \left\{ \frac{-n_f i_f (1 - e^{-jm\pi})}{j k_m 2\ell} \coth\left(\frac{m\pi b}{\ell}\right) - \frac{N_a i_a (1 - e^{jm\pi})}{(j k_m)^2 \sinh\left(\frac{m\pi b}{\ell}\right)} \right\} \tag{30}$$

The last term vanishes because it is odd in m. Thus,

$$\int B_x^f da = L_f i_f \; ; \; L_f \equiv \frac{4 \mu_o d n_f^2}{\pi} \sum_{\substack{1 \\ (odd)}}^{\infty} \frac{\coth m\pi b/\ell}{m} \tag{31}$$

and the field terminal relation, Eq. 28, becomes

$$v_f = R_f i_f + L_f \frac{d i_f}{dt} \tag{32}$$

Prob. 4.12.1 The divergence and curl relations for $\bar{E}$ require that

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{\partial E_z}{\partial z} = 0 \tag{1}$$

$$\frac{\partial E_r}{\partial z} + \frac{\partial E_z}{\partial r} = 0 \tag{2}$$

Because $E_r=0$ on the z axis, the first term in Eq. 2, the condition that the curl be zero, is small in the neighborhood of the z axis. Thus,

$$\frac{\partial E_z}{\partial r} \approx 0 \Rightarrow E_z \simeq E_z(z) \tag{3}$$

and Eq. 1 requires that

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_r) = -\frac{dE_z}{dz} \tag{4}$$

Integration of this expression on r can be carried out because the right-hand side is only a function of z. Because $E_r=0$ at r=0, it follows that

$$E_r = -\frac{1}{2}r\, dE_z/dz \tag{5}$$

Now, if it is recognized that $E_z = -d\Phi/dz$ without approximation, it follows that Eq. 5 is the required expression for E_r.

Prob. 4.13.1 Using the same definitions of surface variables and potential difference as used in the text, $(\hat{\xi}^s \equiv \hat{\xi}_o,\ \hat{\xi}^r \equiv \hat{\xi}_o e^{jk\delta})$

$$V = Re\,\hat{V}_o e^{j\omega t};\ \xi_s = Re\,\hat{\xi}^s e^{-jkz};\ \xi_r = Re\,\hat{\xi}^r e^{j(2\omega t - kz)} \tag{1}$$

At each of the electrode surfaces, the constant potential boundary condition requires that

$$n \times \bar{E} = 0 \Rightarrow E_z = -E_x \frac{\partial \xi}{\partial z} \tag{2}$$

For example, at the rotor surface,

$$E_z(x=0) + \frac{\partial E_z}{\partial x}\bigg|_{x=0} \xi_r = -E_x \frac{\partial \xi_r}{\partial z} \Rightarrow E_z^r = -\frac{\partial}{\partial z}(E_x^r \xi) \tag{3}$$

where the irrotational nature of $\bar{E}$ is exploited to write the second equation. Thus, the conditions at the perturbed electrode surfaces are related to those in fictitious planes x=0 and x=d for the rotor and stator respectively as

$$E_z^r = -\frac{\partial}{\partial z}(E_x^r \xi^r) \Rightarrow \Phi^r = -E_x^r \xi^r \tag{4}$$

$$E_z^s = -\frac{\partial}{\partial z}(E_x^s \xi^s) \Rightarrow \Phi^s = -E_x^s \xi^s + Re\,\hat{V}_o e^{j\omega t} \tag{5}$$

First, find the net force on a section of the rotor having length ℓ in the y direction and $2\pi/k$ in the z direction at some arbitrary instant in time.

$$f_z = \epsilon_o \ell \frac{2\pi}{k} \langle E_x E_z \rangle_z \tag{6}$$

The periodicity condition, together with the fact that there is no material in the air-gap, and hence no force density there, require that Eq. 6 can be integrated in any x plane and the same answer will be obtained. Although not physically meaningful, the integration is mathematically correct if carried out in the plane x=0 (the rotor plane). For convenience, that is what will be done here.

By way of finding the quantities required to evaluate Eqs. 4 and 5, it follows from Eqs. 1 that

$$E_x^s \xi^s = \frac{1}{4d}\left(\hat{V}_o e^{j\omega t} + \hat{V}_o^* e^{-j\omega t}\right)\left(\hat{\xi}^s e^{-jkz} + \hat{\xi}^{s*} e^{jkz}\right) \tag{7}$$

$$= \frac{1}{4d}\left\{\left[\hat{V}_o \hat{\xi}^s e^{j(\omega t - kz)} + (\hat{V}_o \hat{\xi}^s)^* e^{-j(\omega t - kz)}\right] + \left[\hat{V}_o \hat{\xi}^{s*} e^{j(\omega t + kz)} + (\hat{V}_o \hat{\xi}^{s*})^* e^{-j(\omega t + kz)}\right]\right\}$$

Prob. 4.13.1 (cont.)

and that

$$E_x^r \xi^r = \frac{1}{4d}\Big\{\big[\hat{V}_o \hat{\xi}^r e^{j(3\omega t - kz)} + (\hat{V}_o \hat{\xi}^r)^* e^{-j(3\omega t - kz)}\big] + \big[(\hat{V}_o^* \hat{\xi}^r) e^{j(\omega t - kz)} + (\hat{V}_o^* \hat{\xi}^r)^* e^{-j(\omega t - kz)}\big]\Big\} \tag{8}$$

Thus, these last two equations can be written in the complex amplitude form

$$E_x^s \xi^s = \frac{1}{2d} \mathrm{Re}\big[(\hat{V}_o \hat{\xi}^s e^{j\omega t}) e^{-jkz} + (\hat{V}_o^* \hat{\xi}^s e^{-j\omega t}) e^{-jkz}\big] \tag{9}$$

$$E_x^r \xi^r = \frac{1}{2d} \mathrm{Re}\big[(\hat{V}_o \hat{\xi}^r e^{3j\omega t}) e^{-jkz} + (\hat{V}_o^* \hat{\xi}^r e^{j\omega t}) e^{-jkz}\big] \tag{10}$$

The transfer relations, Eqs. a of Table 2.16.1, relate variables in this form evaluated in the fictitious stator and rotor planes.

$$\begin{bmatrix} \tilde{E}_x^s \\ \tilde{E}_x^r \end{bmatrix} = k \begin{bmatrix} -\coth kd & \dfrac{1}{\sinh kd} \\ \dfrac{-1}{\sinh kd} & \coth kd \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^s \\ \tilde{\Phi}^r \end{bmatrix} \tag{11}$$

It follows that

$$E_x^r = \mathrm{Re}\Big\{\frac{\hat{V}_o}{d} e^{j\omega t} - \frac{k}{2d \sinh kd}\big[\hat{V}_o \hat{\xi}^s e^{j\omega t} + \hat{V}_o^* \hat{\xi}^s e^{-j\omega t}\big] e^{-jkz} + \frac{k \coth kd}{2d}\big[\hat{V}_o \hat{\xi}^r e^{3j\omega t} + \hat{V}_o^* \hat{\xi}^r e^{j\omega t}\big] e^{-jkz} \tag{12}$$

Also, from Eq. 4,

$$E_z^r = \mathrm{Re} \, -\frac{jk}{2d}\big[\hat{V}_o \hat{\xi}^r e^{3j\omega t} + \hat{V}_o^* \hat{\xi}^r e^{j\omega t}\big] e^{-jkz} \tag{13}$$

Thus, the space average called for with Eq. 6 becomes

$$f_z = \frac{\epsilon_o \ell 2\pi}{k} \frac{1}{2} \mathrm{Re}\big[\tilde{E}_x^r (\tilde{E}_z^r)^*\big] \tag{14}$$

which, with the use of Eqs. 12 and 13, is

$$f_z = \frac{\epsilon_o \ell \pi}{4k} \mathrm{Re}\Big\{\frac{-jk^2}{d^2 \sinh kd}\big[\hat{V}_o \hat{V}_o^* \hat{\xi}^s \hat{\xi}^{r*} e^{-2j\omega t} + \hat{V}_o \hat{V}_o^* \hat{\xi}^s \hat{\xi}^{r*} e^{-4j\omega t} + \hat{V}_o \hat{V}_o^* \hat{\xi}^s \hat{\xi}^{r*} + \hat{V}_o^* \hat{V}_o \hat{\xi}^s \hat{\xi}^{r*} e^{-2j\omega t}\big] - \frac{jk^2}{d^2} \coth kd \big[2\hat{V}_o \hat{\xi}^r \hat{V}_o^* \hat{\xi}^{r*} + (\hat{\xi}^r)^2 (\hat{V}_o^2 e^{2j\omega t} + \hat{V}_o^{*2} e^{-2j\omega t})\big] \tag{15}$$

The self terms (in $\hat{\xi}^r \cdot \hat{\xi}^r$) either are imaginary or have no time average. The terms in $\hat{\xi}^r \cdot \hat{\xi}^s$ also time-average to zero, except for the term that is

Prob. 4.13.1 (cont.)

independent of time. That term makes the only contribution to the time-average expression $(\hat{\xi}^s = \hat{\xi}_o, \hat{\xi}^r = \hat{\xi}_o e^{jk\delta})$

$$\langle f_z \rangle_t = \frac{\epsilon_o \ell \pi k}{4d^2 \sinh kd} |\hat{V}_o|^2 \, \mathrm{Re}\, j \hat{\xi}_o \hat{\xi}_o^* e^{-jk\delta} \tag{16}$$

In the long-wave limit $kd \ll 1$, this result becomes

$$\langle f_z \rangle_t = \frac{\epsilon_o \ell \pi}{4 d^3} |\hat{V}_o|^2 |\xi_o|^2 \sin k\delta \tag{17}$$

which is in agreement with Eq. 4.13.12.

Prob. 4.13.2 For purposes of making a formal quasi-one-dimensional expansion, field variables are normalized such that

$$H_x = H_o \underline{H}_x \quad , x = d\,\underline{x} \quad , \xi = d\,\underline{\xi} \quad , f_z = \mu_o\left(\frac{d}{\lambda}\right)H_o^2\,\underline{f}_z \tag{1}$$
$$H_z = H_o\left(\frac{d}{\lambda}\right)\underline{H}_z \quad , z = \lambda\,\underline{z} \quad , \psi = H_o d\,\underline{\psi} \quad ,$$

The MQS conditions that the field intensity be irrotational and solenoidal in the air gap then require that

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = 0 \tag{2}$$

$$\frac{\partial H_x}{\partial x} + \left(\frac{d}{\lambda}\right)^2\frac{\partial H_z}{\partial z} = 0 \tag{3}$$

If all field quantities are expanded as series in $\gamma \equiv (d/\lambda)^2$,

$$H_x = \sum_{i=0}^{\infty} H_{xi}\gamma^i \quad , H_z = \sum_{i=0}^{\infty} H_{zi}\gamma^i \tag{4}$$

then, the equations become

$$\frac{\partial H_{xi}}{\partial z} = \frac{\partial H_{zi}}{\partial x} \quad , \frac{\partial H_{xi}}{\partial x} = -\frac{\partial H_{z(i-1)}}{\partial z} \tag{5}$$

The lowest order field follows from the first two equations

$$\frac{\partial H_{xo}}{\partial x} = 0 \tag{6}$$

$$\frac{\partial H_{zo}}{\partial x} = \frac{\partial H_{xo}}{\partial z} \tag{7}$$

It follows that

$$H_x \cong H_{xo} = f(z,t) \tag{8}$$

$$H_z \cong H_{zo} = x\frac{\partial f}{\partial z} + g(z,t) \tag{9}$$

Boundary conditions at the stator and rotor surfaces respectively are

$$H_z^a = K_y(z,t) = K_o \sin(\omega t - kz) \tag{10}$$

$$\bar{n}\times\bar{H}(x=\xi) = 0 \tag{11}$$

In terms of the magnetic potential, these conditions are

$$\psi^a = -\left(\frac{\lambda}{d}\right)\frac{1}{2\pi}\cos\left[2\pi(t - z)\right] \tag{12}$$

$$\psi(x=\xi) = 0 \tag{13}$$

where variables are normalized such that $H_o = K_o$, $t = \tau\,\underline{t}$ $(\tau \equiv 2\pi/\omega)$.

Prob. 4.13.2(cont.)

Integration of $\bar{H}=-\nabla\Psi$ between the rotor and stator surfaces shows that

$$-H_o d\,\underline{\Psi}^a = \int_{-1+\underline{\xi}}^{D} H_o\,\underline{H}_x\,d\,d\,\underline{x} \tag{14}$$

In view of Eq. 8,

$$\Psi^a = -\int_{-1+\xi}^{D} H_x dx = -(1-\xi)f \qquad \underline{15}$$

and so the integration function f(z,t) is determined.

$$f(z,t) = \frac{-\Psi^a}{1-\xi} = \frac{\lambda}{d}\frac{1}{2\pi}\frac{\cos[2\pi(t-z)]}{1-\xi} \qquad \underline{16}$$

From Eqs. 8 and 9 it follows that

$$H_z^a \cong H_{zo}^a = \left[x\frac{\partial f}{\partial z} + g\right]_{x=0} = K_y = \sin[2\pi(t-z)] \qquad \underline{17}$$

so that

$$g = K_y \tag{18}$$

Actually, this result is not required to find the force, but it does complete the job of finding the zero order fields as given by Eqs. 8 and 9.

To find the force at any instant, it is necessary to carry out an integration of the magnetic shear stress over the lower surface of the stator.

$$\langle f_z\rangle_z = \int_0^1 H_x^a H_z^a\,dz \qquad \underline{19}$$

Evaluation gives

$$\langle f_z\rangle_z = \int_0^1 f(z,t)\left(\frac{\lambda}{d}\right)K_y\,dz \qquad \underline{20}$$

$$= \int_0^1\left\{\frac{\lambda}{d}\frac{1}{2\pi}\frac{\cos[2\pi(t-z)]}{1-\xi_o\cos[4\pi(Ut-(z-\delta)]}\right\}\left\{\frac{\lambda}{d}\sin[2\pi(t-z)]\right\}dz$$

$$= \left(\frac{\lambda}{d}\right)^2\frac{1}{4\pi}\int_0^1\frac{\sin[4\pi(t-z)]\,dz}{1-\xi_o\cos[4\pi(Ut-(z-\delta))]} \equiv F(t,\delta)$$

The time average force (per unit area then follows as

$$\langle\langle f_z\rangle_z\rangle_t = \int_0^1 F(t,\delta)\,dt \qquad \underline{21}$$

In the small amplitude limit, this integration reduces to $(\xi \ll 1)$

Prob. 4.13.2(cont.)

$$\langle f_z \rangle_z = \frac{1}{4\pi}\left(\frac{\lambda}{d}\right)^2 \int_0^1 \sin[4\pi(t-z)][1+\xi_0 \cos[4\pi(Ut-(z-\delta))]dz \qquad \underline{\underline{22}}$$

$$= \frac{1}{4\pi}\left(\frac{\lambda}{d}\right)^2 \int_0^1 \xi_0 \sin[4\pi(t-z)]\cos[4\pi(Ut-(z-\delta))]dz$$

$$= -\frac{1}{4\pi}\left(\frac{\lambda}{d}\right)^2 \xi_0 \int_0^1 \sin^2[4\pi(t-z)]\sin\{4\pi[(U-1)t+\delta]\}dz$$

$$= -\frac{1}{8\pi}\left(\frac{\lambda}{d}\right)^2 \xi_0 \sin[(U-1)t+\delta]$$

Thus, the time average force is in general zero. However, for the synchronous condition, where $\underline{U} \equiv [U/\lambda][2\pi/\omega] = 1$,

it follows that the time average force per unit area is

$$\langle\langle f_z \rangle_z\rangle_t = -\frac{1}{8\pi}\left(\frac{\lambda}{d}\right)^2 \xi_0 \sin\delta \qquad \underline{\underline{23}}$$

In dimensional form, this expression is

$$\langle f_z \rangle_z = \frac{-\mu_0 K_0^2 k \xi_0}{4(kd)^2}\sin(2k\delta) \qquad (24)$$

and the same as the long wave limit of Eq. 4.3.27, which as kd $\rightarrow$ 0, becomes

$$\langle f_z \rangle_z = -\frac{\mu_0 K_0^2 k \xi_0}{4\sinh^2 kd}\sin 2k\delta \rightarrow -\frac{\mu_0 K_0^2 k \xi_0}{4(kd)^2}\sin 2k\delta \qquad (25)$$

In fact it is possible to carry out the integration called for with Eq. 20 provided interest is in the synchronous condition. In that case and Eq. 20 reduces to $(G \equiv (d/\lambda)^2 4\pi F)$

$$G = \frac{1}{4\pi}\int_a^{a+4\pi} \frac{\sin\varsigma\cos(4\pi\delta) - \cos\varsigma\sin 4\pi\delta}{1-\xi_0\cos\varsigma}d\varsigma \qquad \underline{\underline{26}}$$

where

$$\varsigma \equiv 4\pi(t-z) + 4\pi\delta, \quad a \equiv 4\pi(t+\delta) - 4\pi$$

In turn, this expression becomes

$$G = \frac{\cos 4\pi\delta}{4\pi}\int_a^{a+4\pi}\frac{\sin\varsigma}{1-\xi_0\cos\varsigma}d\varsigma - \frac{\sin 4\pi\delta}{4\pi}\int_a^{a+4\pi}\frac{\cos\varsigma\, d\varsigma}{1-\xi_0\cos\varsigma} \qquad \underline{\underline{27}}$$

The first integral vanishes, as can be seen from

$$\int_a^{a+4\pi}\frac{\sin\varsigma}{1-\xi_0\cos\varsigma}d\varsigma = -\int_a^{a+4\pi}\frac{d(\cos\varsigma)}{1-\xi_0\cos\varsigma} = \frac{1}{\xi_0}\ln\left[1-\xi_0\cos\varsigma\right]_a^{a+4\pi} = \frac{1}{\xi_0}\ln[1] = 0 \qquad (28)$$

Prob. 4.13.2(cont.)

By use of integral tables, the remaining integral can be carried out.

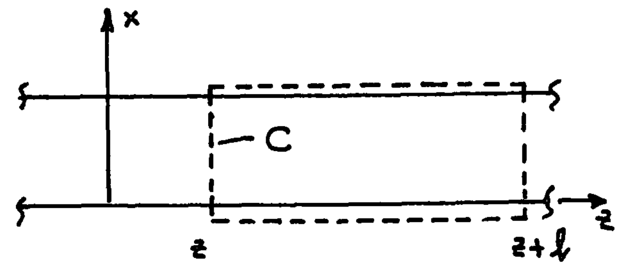
$$G = -\frac{\sin 4\pi\delta}{\xi_o\sqrt{1-\xi_o^2}}\left(1 - \sqrt{1-\xi_o^2}\right) \tag{29}$$

In dimensional form, the force per unit area therefore becomes

$$\langle f_z\rangle_t = -\frac{\mu_o K_o^2 \sin 2k\delta}{2\xi_o k\sqrt{1-\left(\frac{\xi_o}{d}\right)^2}}\left[1 - \sqrt{1-\left(\frac{\xi_o}{d}\right)^2}\right] \tag{30}$$

Note that under synchronous conditions, the instantaneous force is independent of time, so no time-average is required. Also, in the limit $\xi_o/d \ll 1$, this expression reduces to Eq. 25.

<u>Prob. 4.14.1</u> Ampere's law and the condition that $\bar{H}$ is solenoidal take the quasi-one-dimensional forms



$$\frac{\partial H_x}{\partial x} = 0 \tag{1}$$

$$\frac{\partial H_z}{\partial x} = \frac{\partial H_x}{\partial z} \tag{2}$$

and it follows that

$$H_x = H_x(z) \tag{3}$$

$$H_z = x\frac{\partial H_x}{\partial z} + f(z,t) \tag{4}$$

The integral form of Ampere's law becomes

$$\oint_C \bar{H}\cdot d\ell = [H_x(z+\ell) - H_x(z)]b = \int_S \bar{J}\cdot\bar{n}\,da = \begin{cases} -n_f i_f - 2N_a(z-\ell/2) \; ; \; 0<z<\ell \\ n_f i_f + 2N_a(z-3\ell/2) \; ; \; \ell<z<2\ell \end{cases} \tag{5}$$

Because the model represents one closed on itself, $\bar{H}(z+\ell) = -\bar{H}(z)$ and it follows that Eqs. 5 become

$$H_x(z) = \begin{cases} \frac{n_f i_f}{2b} + \frac{N_a i_a}{b}\left(z - \frac{\ell}{2}\right) \; ; \; 0<z<\ell \\ -\frac{n_f i_f}{2b} - \frac{N_a i_a}{b}\left(z - \frac{3\ell}{2}\right) \; ; \; \ell<z<2\ell \end{cases} \tag{6}$$

and it follows that

$$\frac{\partial H_x}{\partial z} = \begin{cases} \pm N_a i_a/b \; ; \; \substack{0\\ \ell} < z < \substack{\ell\\ 2\ell} \\ \pm\frac{n_f i_f}{b} u_o\left(\substack{z\\ z-\ell}\right) \end{cases} \tag{7}$$

At the rotor surface, where x=0,

$$H_z = \pm N_a i_a \; ; \; \substack{0\\ \ell} < z < \substack{\ell\\ 2\ell} \tag{8}$$

and so Eq. 7 can be used to deduce that

$$H_z = \begin{cases} \pm N_a i_a\left(\frac{x}{b} - 1\right) \; ; \; \substack{0\\ \ell} < z < \substack{\ell\\ 2\ell} \\ \pm n_f i_f \frac{x}{b} u_o\left(\substack{z\\ z-\ell}\right) \mp N_a i_a\left(\frac{x}{b} - 1\right) \; ; \; z = \substack{0\\ \ell} \end{cases} \tag{9}$$

The force follows from an integration of the stress tensor over the surface of a volume enclosing the rotor with depth d in the y direction and one periodicity length, 2ℓ in the z direction.

$$f = d\int_0^{2\ell} \mu_o H_x H_z \, dz \tag{10}$$

Prob. 4.14.1 (cont.)

This expression is evaluated.

$$f=\mu_o d\left\{\int_0^{\ell}\left[\frac{n_f i_f}{2b}+\frac{N_a i_a}{b}\left(z-\frac{\ell}{2}\right)\right]N_a i_a\left(\frac{x}{b}-1\right)dz+\int_{0^-}^{0^+}\left(-n_f i_f\frac{x}{b}u_o(z)\right)\frac{\ell}{2}\frac{N_a i_a}{b}dz\right. \tag{11}$$

$$\left.+\int_{\ell}^{2\ell}-\left[\frac{n_f i_f}{2b}+\frac{N_a i_a}{b}\left(z-\frac{3\ell}{2}\right)\right]\left[-N_a i_a\left(\frac{x}{b}-1\right)\right]dz+\int_{\ell^-}^{\ell^+}\left[n_f i_f\frac{x}{b}u_o(x-\ell)\right]\left[-\frac{\ell}{2}\frac{N_a i_a}{b}\right]dz\right\}$$

$$=\mu_o d\left[N_a i_a n_f i_f\frac{\ell}{b}\left(\frac{x}{b}-1\right)-N_a i_a n_f i_f\frac{\ell}{b}\frac{x}{b}\right]$$

$$=-N_a n_f i_a i_f \mu_o d\frac{\ell}{b}$$

This detailed calculation is simplified if the surface of integration is pushed to x=0, where the impulses do not contribute and the result is the same as given by Eq. 11.

$$f=-G_m i_f i_a\ ;\quad G_m\equiv\mu_o\frac{d\ell}{b}N_a n_f \tag{12}$$

Note that this agrees with the result from Prob. 4.10.1, where in the long-wave limit ($b/\ell \ll 1$)

$$G_m\rightarrow\frac{\mu_o d\ell N_a n_f}{b}\sum_{n=1}^{\infty}\frac{8}{\pi^2}\frac{1}{m^2} \tag{13}$$

because

$$\sum_{n=1}^{\infty}\frac{1}{m^2}\rightarrow\frac{\pi^2}{8} \tag{14}$$

To determine the field terminal relation, use Faraday's integral law

$$-v_f+\int_{\text{wire}}\bar{E}\cdot d\ell=-\frac{d}{dt}\lambda_f\ ;\quad \lambda_f\equiv\int_S\bar{B}\cdot\bar{n}\,da \tag{15}$$

Using the given fields, this expression becomes

$$\lambda_f=n_f\Phi \tag{16}$$

$$\Phi=d\int_0^{\ell}\mu_o H_x\,dz=d\int_0^{\ell}\left[\frac{n_f i_f}{2b}+\frac{N_a i_a}{b}\left(z-\frac{\ell}{2}\right)\right]\mu_o\,dz=L_f i_f\ ;\ L_f=\mu_o d n_f^2\ell/2b$$

This results compares to Eq. 31 of Prob. 4.10.1 where in this limit

$$L_f\rightarrow\frac{4\mu_o\ell d n_f^2}{8b}\sum_{\substack{n=1\\(\text{odd})}}^{\infty}\frac{8}{\pi^2 m^2} \tag{17}$$

The field winding is fixed, so Ohm's law is simply $\bar{J}=\sigma\bar{E}$ and therefore Eq. 15 becomes

$$-v_f+\int_{\text{wire}}\frac{\bar{J}}{\sigma}\cdot d\bar{\ell}=-L_f\frac{di_f}{dt} \tag{18}$$

Because

Prob. 4.14.1 (cont.)

$$R_f \equiv \frac{1}{\sigma}\frac{2n_f d}{A_{wire}} \tag{19}$$

the field equation is

$$v_f = i_f R_f + L_f \frac{di_f}{dt} \tag{20}$$

For the armature the integration is again in the laboratory frame of reference.

The flux linked is

$$\lambda_a = \int_0^{\ell} \phi(z) N_a \, dz \tag{21}$$

where

$$\phi = \int_z^{z+\ell} \mu_o H_x \, dz' = \int_z^{\ell} \mu_o\left[\frac{n_f i_f}{2b} + \frac{N_a i_a}{b}\left(z' - \frac{\ell}{2}\right)\right]dz' + \int_{\ell}^{z+\ell} \mu_o\left[-\frac{n_f i_f}{2b} - \frac{N_a i_a}{b}\left(z' - \frac{3\ell}{2}\right)\right]dz' \tag{22}$$

$$= \frac{\mu_o d}{b}\left[\frac{1}{2} n_f i_f (\ell - 2z) - N_a i_a z(z-\ell)\right]$$

Thus,

$$\lambda_a = L_a i_a; \quad L_a \equiv \frac{1}{6}\frac{\mu_o d \ell^3}{b} N_a^2 \tag{23}$$

This compares to the result from Prob. 4.10.1

$$L_a = \frac{N_a^2 d \ell^3 \mu_o}{6b} \sum_{-\infty (odd)}^{+\infty} \frac{6 \cdot 16}{\pi^4 m^4} \; ; \; \sum_{\substack{-\infty \\ odd}}^{+\infty} \frac{1}{m^4} = \frac{\pi^4}{6 \cdot 16} \tag{24}$$

For the moving conductors, Ohm's law requires that

$$E_y = \frac{i_a}{A_{wire}\sigma} - v_z \mu_o H_x \tag{25}$$

and so Faraday's law becomes

$$-v_a + d\int_0^{\ell} N_a E_y \, dz - d\int_{\ell}^{2\ell} N_a E_y \, dz = -\frac{d}{dt} L_a i_a \tag{26}$$

or

$$-v_a + dN_a\left\{\frac{2\ell i_a}{A_{wire}\sigma} + \int_0^{\ell} -v_z \mu_o\left[\frac{n_f i_f}{2b} + \frac{N_a i_a}{b}\left(z - \frac{\ell}{2}\right)\right]dz\right. \tag{27}$$

Thus

$$\left. - \int_{\ell}^{2\ell} -v_z \mu_o\left[-\frac{n_f i_f}{2b} - \frac{N_a i_a}{b}\left(z - \frac{3\ell}{2}\right)\right]dz\right\} = -L_a \frac{di_a}{dt} \tag{28}$$

and finally

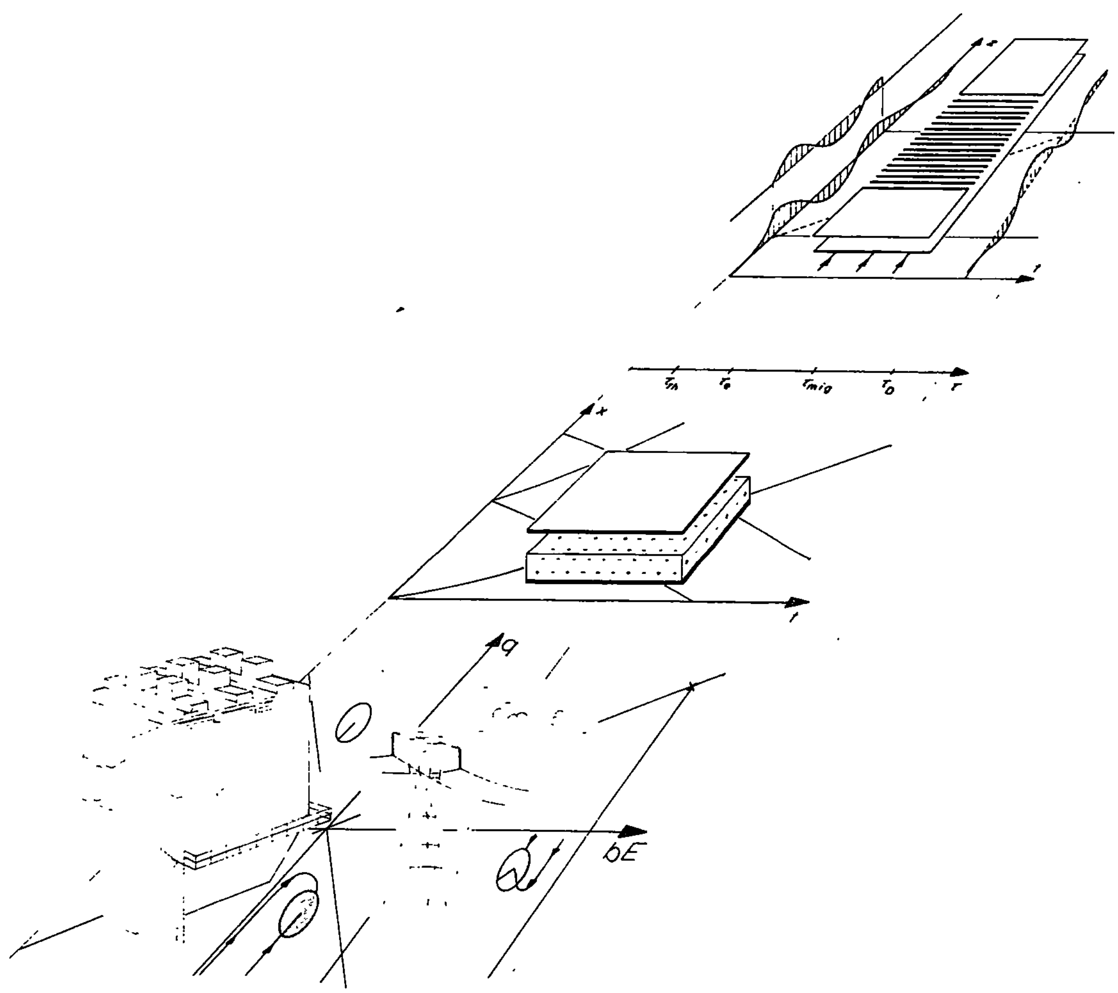
$$v_a = i_a R_a - G_m v_z i_f + L_a \frac{di_a}{dt} \tag{29}$$

where

$$R_a = \frac{2\ell d N_a}{A_{wire}\sigma}$$

5

Charge Migration, Convection and Relaxation



Prob. 5.3.1 In cartesian coordinates (x,y)

$$\begin{bmatrix} \bar{E} \\ \bar{v} \end{bmatrix} = \left[\bar{i}_x \frac{\partial}{\partial y} - \bar{i}_y \frac{\partial}{\partial x} \right] \begin{bmatrix} A_E \\ A_v \end{bmatrix} \tag{1}$$

Thus, the characteristic equation, Eq. 5.3.4, becomes

$$\frac{dx}{dt} = \frac{\partial}{\partial y}\left(A_v \pm b_i A_E \right) \tag{2}$$

$$\frac{dy}{dt} = -\frac{\partial}{\partial x}\left(A_v \pm b_i A_E \right) \tag{3}$$

The ratio of these expressions is

$$\frac{dx}{dy} = -\frac{\frac{\partial}{\partial y}\left(A_v \pm b_i A_E \right)}{\frac{\partial}{\partial x}\left(A_v \pm b_i A_E \right)} \tag{4}$$

which, multiplied out, becomes

$$\frac{\partial}{\partial x}\left(A_v \pm b_i A_E \right) dx + \frac{\partial}{\partial y}\left(A_v \pm b_i A_E \right) = 0 \tag{5}$$

If A_v and A_E are independent of time, the quantity $A_v \pm b_i A_E$ is a perfect differential. That is,

$$A_v \pm b_i A_E = \text{constant} \tag{6}$$

is a solution to Eq. 5.3.4. Along these lines ρ_i = constant.

Prob. 5.3.2 In axisymmetric cylindrical coordinates (r,z), Eq. (h) of Table 2.18.1 can be used to represent the solenoidal $\bar{E}$ and $\bar{v}$.

$$\begin{bmatrix}\bar{E}\\ \bar{v}\end{bmatrix} = \left[-\bar{i}_r\frac{1}{r}\frac{\partial}{\partial z} + \bar{i}_z\frac{1}{r}\frac{\partial}{\partial r}\right]\begin{bmatrix}\Lambda_E\\ \Lambda_v\end{bmatrix} \tag{1}$$

In terms of Λ_E and Λ_v, Eq. 5.3.4 becomes

$$\frac{dr}{dt} = -\frac{1}{r}\frac{\partial}{\partial z}\left(\Lambda_v \pm b_i\Lambda_E\right) \tag{2}$$

$$\frac{dz}{dt} = \frac{1}{r}\frac{\partial}{\partial r}\left(\Lambda_v \pm b_i\Lambda_E\right) \tag{3}$$

The ratio of these two expressions gives

$$\frac{dr}{dz} = \frac{-\frac{\partial}{\partial z}\left(\Lambda_v \pm b_i\Lambda_E\right)}{\frac{\partial}{\partial r}\left(\Lambda_v \pm b_i\Lambda_E\right)} \tag{4}$$

and hence

$$\frac{\partial}{\partial r}\left(\Lambda_v \pm b_i\Lambda_E\right)dr + \frac{\partial}{\partial z}\left(\Lambda_v \pm b_i\Lambda_E\right)dz = 0 \tag{5}$$

Provided Λ_v and Λ_E are independent of time, this is a perfect differential. Hence

$$\Lambda_v \pm b_i\Lambda_E = \text{constant} \tag{6}$$

represents the characteristic lines along which ρ_i is a constant.

Prob. 5.4.1 Integration of the given electric field and flow velocity result in $A_E = Vy/d$ and $A_v = -(4U/d)[(x^2/2) - (x^3/3d)]$. It follows from the result of Prob. 5.3.1 that the characteristic lines are $A_v + bA_E$=constant, or the relation given in the problem statement. The characteristic originating at x=0 reaches the upper electrode at $y=y_1$ where y_1 is obtained from the characteristics by first evaluating the constant by setting x=0 and y=0 (constant = 0) and then evaluating the characteristic at x=d and $y=y_1$.

$$y_1 = \frac{2}{3} Ud/(bV/d) \tag{1}$$

Because the current density to the upper electrode is $nqbE_x$ and all characteristics reaching the electrode to the right of $y=y_1$ carry a uniform charge density, nq, the current per unit length is simply the product of the uniform current density and the length $(a-y_1)$. This is the given result.

Prob. 5.4.2 From the given distributions of electric potential and velocity potential, it follows that

$$\bar{E} = -VR^2\left[-\frac{2}{r^3}\cos\theta\,\bar{i}_r - \frac{1}{r^3}\sin\theta\,\bar{i}_\theta\right] \tag{2}$$

$$\bar{v} = UR\left[\left(\frac{1}{R} - \frac{R^2}{r^3}\right)\cos\theta\,\bar{i}_r - \frac{1}{r}\left(\frac{r}{R} + \frac{1}{2}\frac{R^2}{r^2}\right)\sin\theta\,\bar{i}_\theta\right] \tag{3}$$

From the spherical coordinate relations, Eqs. 5.3.8, it in turn is deduced that

$$\Lambda_E = \frac{VR^2\sin^2\theta}{r} \tag{4}$$

$$\Lambda_v = \frac{UR^2}{2}\left(\frac{r^2}{R^2} - \frac{R}{r}\right)\sin^2\theta \tag{5}$$

so the characteristic lines are (Eq. 5.3.13b)

$$\Lambda_v + b\Lambda_E = \frac{UR^2}{2}\left(\frac{r^2}{R^2} - \frac{R}{r}\right)\sin^2\theta + \frac{bVR^2}{r}\sin^2\theta = \text{constant} \tag{6}$$

Normalization makes it evident that the trajectories depend on only one parameter.

$$\left[\left(\frac{r}{R}\right)^2 - \frac{R}{r}\left(1 - \frac{2Vb}{UR}\right)\right]\sin^2\theta = C \tag{7}$$

The critical points are determined by the requirement that both the r and θ components of the force vanish.

Prob. 5.4.2(cont.)

$$b\frac{2VR^2}{r^3}\cos\theta + U\left(1-\frac{R^3}{r^3}\right)\cos\theta = 0 \tag{8}$$

$$b\frac{VR^2}{r^3}\sin\theta - \frac{UR}{r}\left(\frac{r}{R}+\frac{1}{2}\frac{R^2}{r^2}\right)\sin\theta = 0 \tag{9}$$

From the first expression,

$$\text{either}\quad \theta = \pi/2 \qquad \text{or}\quad \left(\frac{r}{R}\right)^3 = 1 - b\frac{2V}{RU} \tag{10}$$

while from the second expression,

$$\text{either}\quad 0,\pi \qquad \text{or}\quad \left(\frac{r}{R}\right)^3 = -\frac{1}{2}\left(1-\frac{2bV}{RU}\right) \tag{11}$$

For $V > 0$ and positive particles, the root of Eq. 10b is not physical. The roots of physical interest are given by Eqs. 10a and 11b. Because $r/R > 1$, the singular line (point) is physical only if $bV/RU > 3/2$.

Because there is no normal fluid velocity on the sphere surface, the characteristic lines have a direction there determined by $\bar{E}$ alone. Hence, the sphere can only accept charge over some part of its southern hemisphere. Just how much of this hemisphere is determined by the origins of the incident lines. Do they originate at infinity where the charge density enters, or do they come from some other part of the spherical surface? The critical point determines the answer to this question.

Characteristic lines typical of having no critical point in the volume and of having one are shown in the figure. For the lines on the right, $bV/RU=1$ so there is no critical point. For those on the left, $bV/RU = 3 > 3/2$.

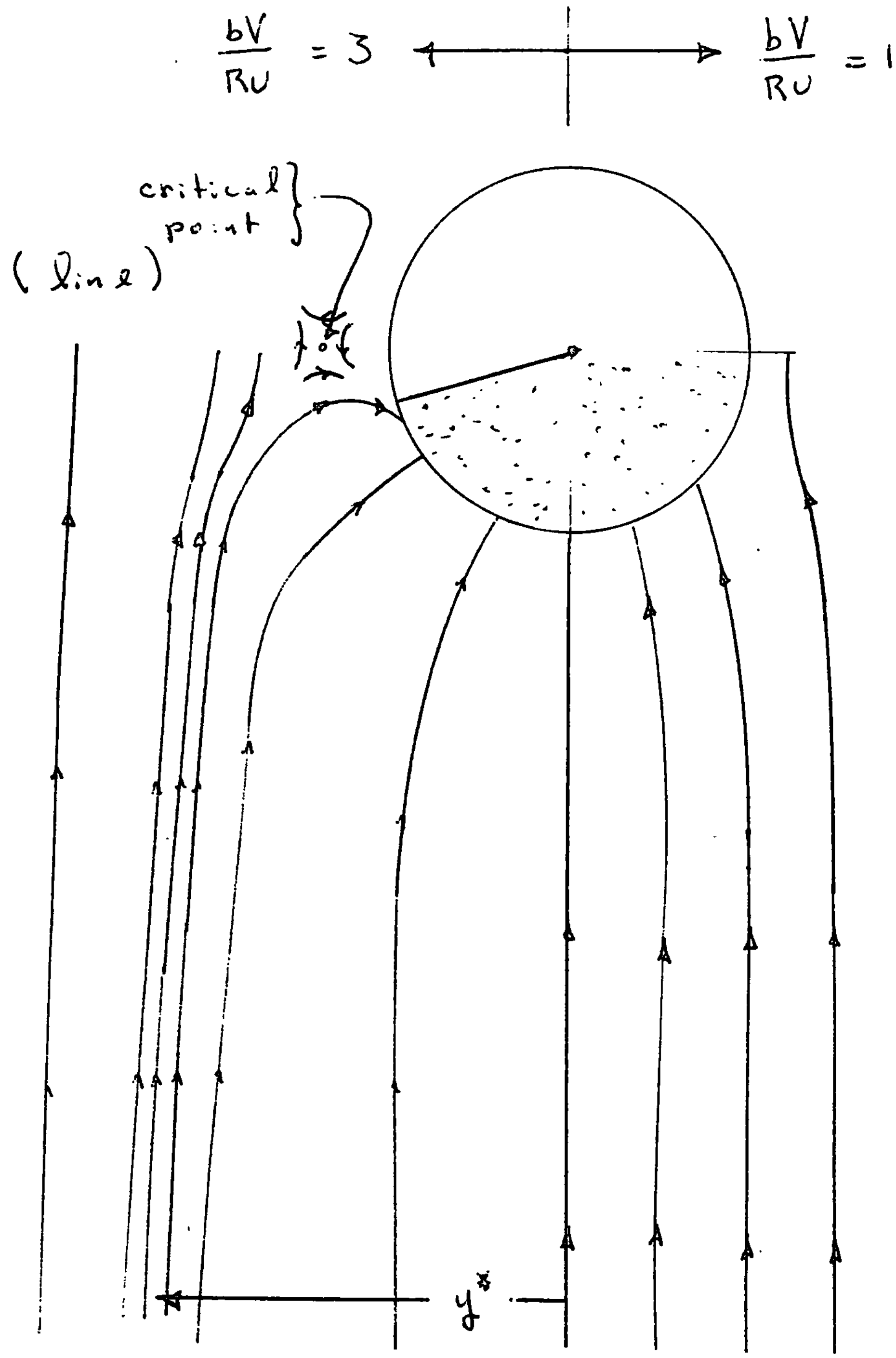
If the critical point is outside the sphere ($bV/RU > 3/2$) then the "window" having area $\pi(y^*)^2$ through which particles enter and ultimately impact the sphere is determined by the characteristic line passing through the critical point

$$\frac{r}{R} = \left[\frac{1}{2}\left(\frac{2Vb}{RU}-1\right)\right]^{1/3}, \quad \theta = \pm\frac{\pi}{2} \tag{12}$$

Thus, in Eq. 7,

$$C = \frac{3}{2}(2)^{1/3}\left(\frac{2bV}{RU}-1\right)^{2/3} \tag{13}$$

Prob. 5.4.2(cont.)



In the limit $r \to \infty$, $\theta \to \pi/2$

$$C \to \left(\frac{r}{R}\right)^2 \sin^2\theta = (y^*)^2/R^2 \tag{14}$$

so, for bV/RU > 3/2,

$$i = \rho U (y^*)^2 \pi = \frac{3\pi R^2}{2} \rho U (2)^{1/3} \left(\frac{2bV}{RU} - 1\right)^{2/3} \tag{15}$$

Prob. 5.4.2(cont.)

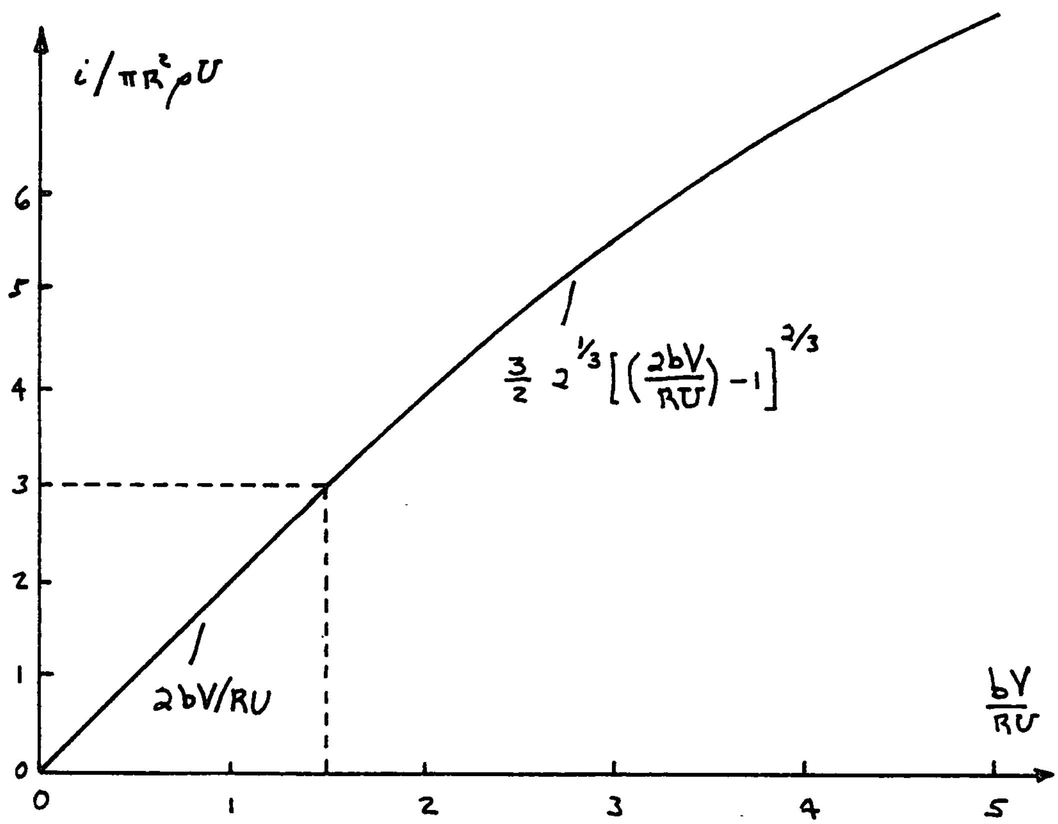
For $bV/RU < 3/2$, the entire southern hemisphere collects, and the window for collection is defined (not by the singular point, which no longer exists in the volume) by the line passing through the equator, $\theta = \pi/2, r/R = 1$

$$(y^*/R)^2 = 2bV/RU \tag{16}$$

Thus, in this range the current is

$$i = \frac{2bV}{RU}\pi R^2 \rho U \tag{17}$$

In terms of normalized variables, the current therefore has the voltage dependence summarized in the figure.



Prob. 5.4.3 (a) The critical points form lines in three dimensions. They occur where the net force is zero. Thus, they occur where the θ component balances

$$U\left(1+\frac{a^2}{r^2}\right)\sin\theta = 0 \Rightarrow \theta = 0 \text{ or } \pi$$

and where the r component is zero

$$-U\left(1-\frac{a^2}{r^2}\right)\cos\theta + bV\frac{1}{r\ln\left(\frac{R_o}{a}\right)} = 0$$

Because the first of these fixes the angle, the second can be evaluated to give the radius

$$\frac{r}{a} = \frac{\underline{V}}{2\cos\theta} + \sqrt{\left(\frac{\underline{V}}{2}\right)^2 + 1} \quad ; \quad \underline{V} \equiv \frac{bV}{aU\ln\left(\frac{R_o}{a}\right)} ; \cos\theta = \pm 1$$

Note that this critical point exists if charge and conductor have the same polarity $(\underline{V}>0)$ at $\theta = 0$ and if $(\underline{V}<0)$ at $\theta = \pi$.

(b) It follows from the given field and flow that

$$A_E = \frac{V\theta}{\ln(R_o/a)} \quad ; \quad A_v = -U\left(r-\frac{a^2}{r}\right)\sin\theta$$

and hence the characteristic lines are

$$A_v + bA_E = -U\left(r-\frac{a^2}{r}\right)\sin\theta + \frac{bV\theta}{\ln(R_o/a)} = \text{const.}$$

These are sketched for the two cases in the figure.

(c) There are two ways to compute the current to the conductor when the voltage is negative. First, the entire surface of the conductor collects with a current density $-\rho bE_r$ that is uniform over its surface. Hence, because the charge density is uniform along a characteristic line, and all striking the conductor surface carry this density,

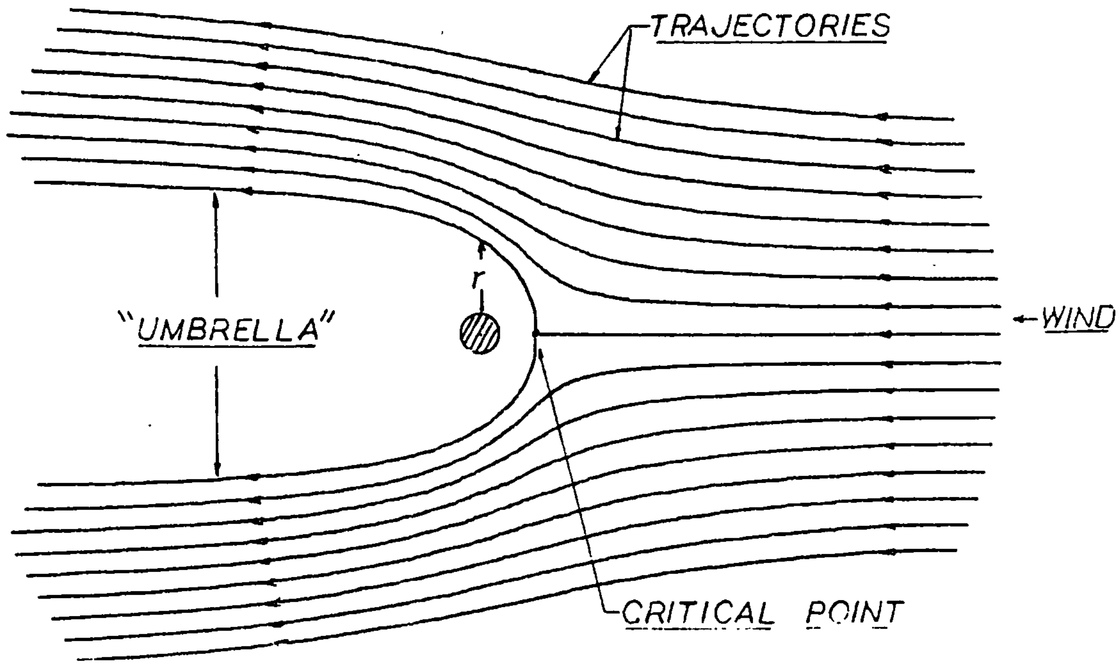
$$i = (2\pi aw)\rho bE_r = 2\pi aw\rho b\left[V/a\ln(R_o/a)\right] \quad ; \quad V<0$$

and i is zero for V > 0. Second, the window at infinity, y*, can be found by evaluating (const.) for the line passing through the critical point. This must be the same constant as found for $r\to\infty$ to the right.

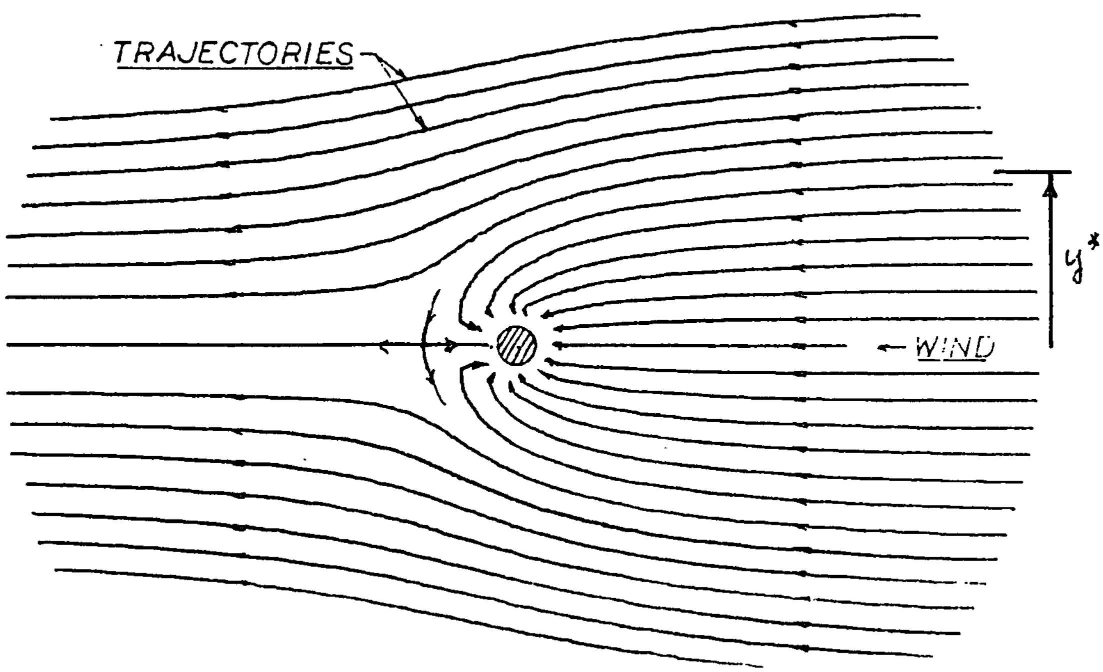
$$\text{const.} = -Uy^* = bV\pi/\ln(R_o/a)$$

It follows that $i = (2y^*w)\rho U$, which is the same current as given above.

Prob. 5.4.3 (cont.)



Positive Particle Trajectories for a Positive Conductor in the Stationary Flow Case (Repelled Particles)



Negative Particle Trajectories for a Positive Conductor in the Stationary Flow Case (Attracted Particles)

Prob. 5.4.4 In terms of the stream function from Table 2.18.1, the velocity is represented by 2Cxy. The volume rate of flow is equal to ℓ times the difference between the stream function evaluated on the electrodes to left and right, so it follows that $-4Ca^2\ell = Q_v$. Thus, the desired stream function is

$$A_v = -\frac{Q_v}{2a^2\ell}\, xy \tag{1}$$

The electric potential is $\Phi = V_o xy/a^2$. Thus, $\bar{E} = -V_o(y\bar{i}_x + x\bar{i}_y)/a^2$ and it follows that the electric stream function is

$$A_E = V_o(x^2-y^2)/2a^2 \tag{2}$$

(b) The critical lines (points) are given by

$$\bar{v} + b\bar{E} = -\frac{Q_v}{2a^2\ell}(x\bar{i}_x - y\bar{i}_y) - \frac{bV_o}{a^2}(y\bar{i}_x + x\bar{i}_y) = 0 \tag{3}$$

Thus, elimination between these two equations gives

$$\frac{-Q_v^2}{4\ell^2(bV_o)^2}\, y = y \tag{4}$$

so that the only lines are at the origin where both the velocity and the electric field vanish.

(c) Force lines follow from the stream functions as

$$\frac{-Q_v}{2a^2\ell}\, xy + \frac{bV_o}{2a^2}(x^2-y^2) = \text{constant} \tag{5}$$

The line entering at the right edge of the throat is given by

$$-\frac{Q_v}{\ell}\, xy + bV_o(x^2-y^2) = -\frac{Q_v}{\ell}a^2 + \frac{bV_o}{c^2}(c^4-a^4) \tag{6}$$

and it reaches the plane x=0 at

$$y^2 = \frac{Q_v}{\ell b V_o}a^2 - \frac{(c^4-a^4)}{c^2} \tag{7}$$

Clearly, force lines do not terminate on the left side of the collection electrode, so the desired current is given by

$$i = -\int_0^{y_1} \ell\rho b E_x(0,y)\,dy = \frac{\ell\rho b V_o}{2a^2}\, y_1^2 \tag{8}$$

where y_1 is equal to a unless the line from $(c, a^2/c)$ strikes to the left of a, in which case y_1 follows from evaluation of Eq. 7, provided that it

Prob. 5.4.4(cont.)

is positive. For still larger values of bV_o, $i=0$.

Thus, at low voltage, where the full width is collecting, $i = \ell\rho bV_o/2$.

This current gives way to a new relation as the force line from the right edge of the throat just reaches (0,a).

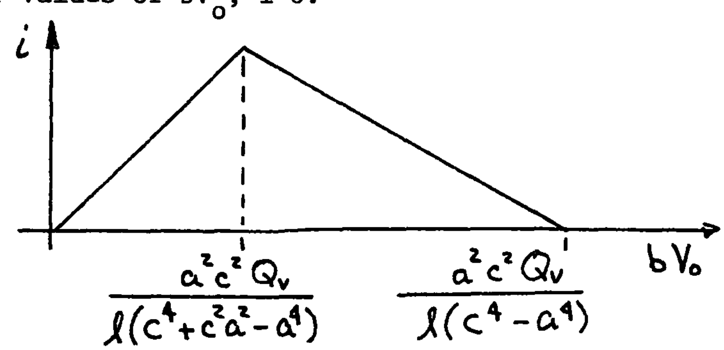
$$bV_o = \frac{Q_v a^2 c^2}{\ell(c^4 + c^2 a^2 - a^4)} \tag{9}$$

$$i = \frac{\ell\rho bV_o}{2a^2}\left[\frac{Q_v}{\ell bV_o}a^2 - \frac{(c^4-a^4)}{c^2}\right] = \frac{\rho Q_v}{2} - \frac{\ell\rho bV_o}{2a^2c^2}(c^4-a^4) \tag{10}$$

Thus, as bV_o is raised, the current diminishes until $y_1=0$, which occurs at

$$bV_o = \frac{\rho\, Q_v a^2 c^2}{\ell\rho\,(c^4 - a^4)} \tag{11}$$

For greater values of bV_o, $i=0$.



Prob. 5.5.1 With both positive and negative ions, the charging current is, in general, the sum of the respective positive and negative ion currents. These two contributions act against each other, and final particle charges other than zero and $\pm q_c$ result. These final charges are those at which the two contributions are equal. The diagram is divided into 12 charging regimes by the coordinate axes q and E_o and the four lines

$$E_o = U_o/b_+ \tag{1}$$

$$E_o = -U_o/b_- \tag{2}$$

$$q = \pm q_c = \pm 12\pi\epsilon_o R^2 E_o \tag{3}$$

In each regime, the charging rate is given by the sum of the four possible current components

$$i_1^{\pm} = \pm 3|I_{\pm}|\left(1 \mp \frac{q}{|q_c|}\right)^2 \tag{4}$$

$$i_2^{\pm} = -12\frac{|I_{\pm}|}{|q_c|}q \tag{5}$$

where $I_{\pm} \equiv \pi R^2 b_{\pm}\rho_{\pm}E_o$, as in the unipolar cases.

In regimes (a), (b), (c) and (d), only i_2^- is acting, driving the particle charge down to the $\pm q_c$ lines. Similarly, in regimes (m), (n), (o) and (p), only i_2^+ is charging the particle, driving q up to the lower $\pm q_c$ lines.

In regimes (e), (i), (h) and (l), the current is $i_1^+ + i_1^-$; the equilibrium charge, defined by

$$i_1^+(q_1) + i_1^-(q_1) = 0 \tag{6}$$

is

$$q_1 = |q_c|\left\{\frac{\frac{|I_+|}{|I_-|}+1}{\frac{|I_+|}{|I_-|}-1} \mp \left[\frac{\left(\frac{|I_+|}{|I_-|}+1\right)^2}{\left(\frac{|I_+|}{|I_-|}-1\right)^2} - 1\right]^{1/2}\right\} \tag{7}$$

where the upper sign holds for $|I_+|>|I_-|$ while the lower one holds for $|I_+|<|I_-|$. In other words, the root of the quadratic which gives $|q_1|<|q_c|$ is taken. Note that q_1 depends linearly on $|E_o|$; the sign of q_1 is that of $|I_+|-|I_-|$. This is seen clearly in the limit $|I_+|\to 0$ or $|I_-|\to 0$.

Prob. 5.5.1 (cont.)

In regime (j), i_1^+ is the only current; in regime (g), i_1^- is the only contribution. In both cases, the particle charge is brought to zero and respectively into regime (f) (where the current is $i_2^- + i_1^+$) or into regime (k) (where the current is $i_2^+ + i_1^-$). The final charge in these regimes is q_2, given by

$$i_2^{\mp}(q_2) + i_1^{\pm}(q_2) = 0 \tag{8}$$

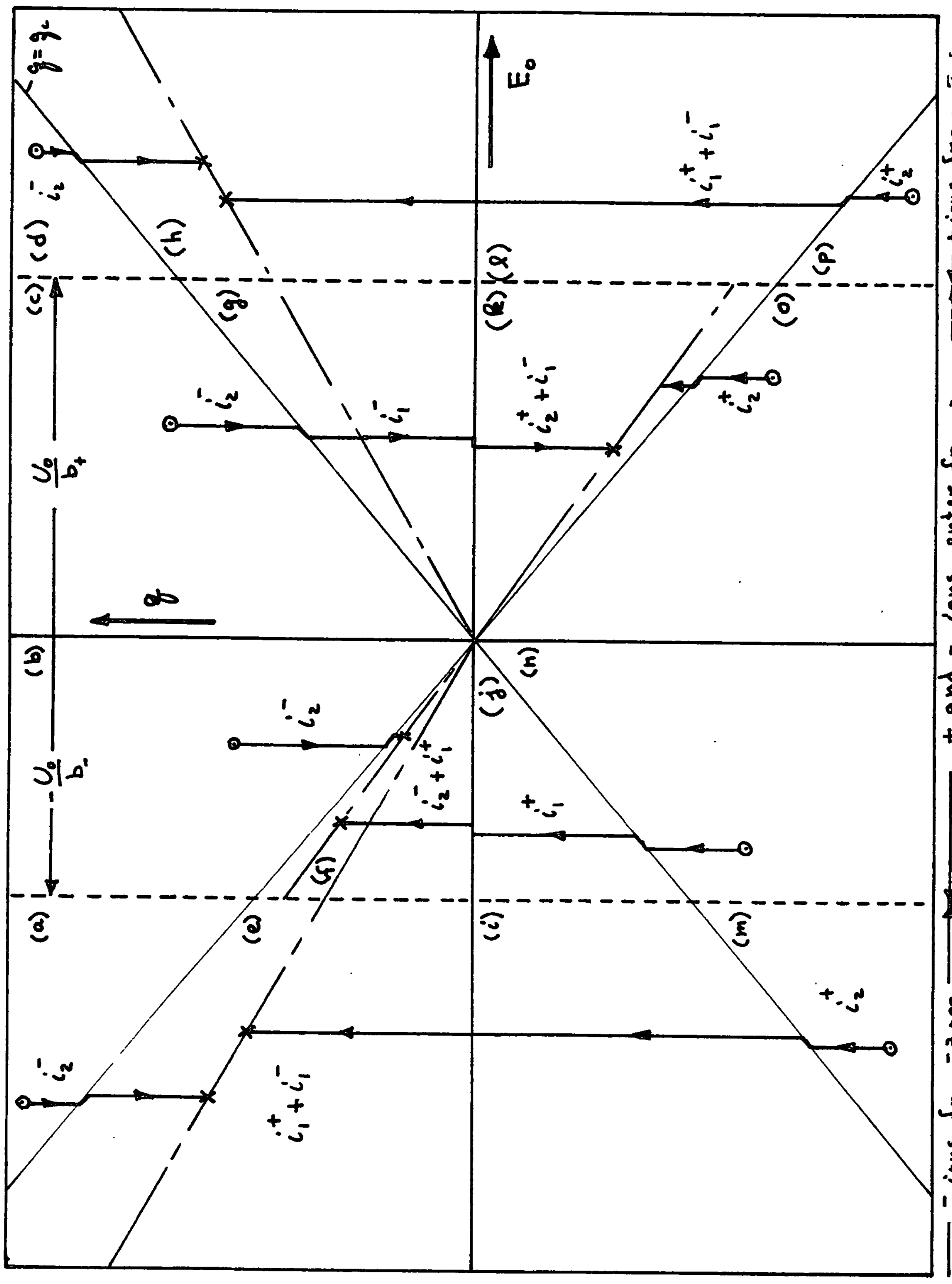
which can be used to find q_2.

$$q_2 = \mp |q_c| \left\{ \left(1 + 2\frac{|I_\pm|}{|I_\mp|}\right) - \left[\left(1 + 2\frac{|I_\pm|}{|I_\mp|}\right)^2 - 1\right]^{1/2} \right\} \tag{9}$$

Here, the upper and lower signs apply to regimes (k) and (f) respectively. Note that q_2 depends linearly on E_o and hence passes straight through the origin.

In summary, as a function of time the particle charge, q, goes to q_1 for $E_o < -U_o/b_-$ or $E_o > U_o/b_+$ and goes to q_2 for $-U_o/b_- < E_o < U_o/b_+$. In the diagram, a shift from the vertical at a regime boundary denotes a change in the functional form of the charging current. Of course, the current itself is continuous there.

Prob. 5.5.1 (cont.)



Prob. 5.5.2 (a) In view of Eq. (k) of Table 2.18.1

$$v_r = \frac{1}{r^2 \sin\theta}\frac{\partial \Lambda_v}{\partial \theta} = -U\left(1 - \frac{R^3}{r^3}\right)\cos\theta \tag{1}$$

$$v_\theta = \frac{-1}{r \sin\theta}\frac{\partial \Lambda_v}{\partial r} = U\left(1 + \frac{R^3}{2r^3}\right)\sin\theta \tag{2}$$

and it follows by integration that

$$\Lambda_v = -\frac{U}{2}\left(r^2 - \frac{R^3}{r}\right)\sin^2\theta \tag{3}$$

Thus, because Λ_E remains Eq. 5.5.4, it follows that the characteristic lines, Eq. 5.3.13b, take the normalized form

$$-\frac{1}{2}\left(\underline{r}^2 - \frac{1}{\underline{r}}\right)\sin^2\theta \pm \underline{E}\left(\frac{1}{\underline{r}} + \frac{1}{2}\underline{r}^2\right)\sin^2\theta \mp 3\underline{q}\cos\theta = \text{const.} \tag{4}$$

where as in the text, $q_c \equiv 12\pi\epsilon_0 R^2 E$, and $\underline{r} = r/R$, $\underline{E} \equiv b_{\pm}E/U$ and $\underline{q} = \underline{E}\, q/q_c$.

Note that $\underline{E}/q_c$ is independent of E and, provided U > 0, is always positive. Without restricting the analysis, U can be taken as positive. Then, $\underline{E}$ can be taken as a normalized imposed field and $\underline{q}$ (which is actually independent of E because $\underline{E}/q_c$ is independent of E) can be taken as a normalized charge on the drop.

(b) Critical points occur where

$$\bar{v} \pm b_{\pm}\bar{E} = 0 \tag{5}$$

The components of this equation, evaluated using Eq. 5.5.3 for $\bar{E}$ and Eqs. 1 and 2 for $\bar{v}$, are

$$-\left(1 - \frac{1}{\underline{r}^3}\right)\cos\theta \pm \underline{E}\left(\frac{2}{\underline{r}^3} + 1\right)\cos\theta \pm \frac{3\underline{q}}{\underline{r}^2} = 0 \tag{6}$$

$$\left(1 + \frac{1}{2\underline{r}^3}\right)\sin\theta \pm \underline{E}\left(\frac{1}{\underline{r}^3} - 1\right)\sin\theta = 0 \tag{7}$$

One set of solutions to these simultaneous equations for (r, θ) follows by recognizing that Eq. 7 is satisfied if

Prob. 5.5.2 (cont.)

$$\sin\theta = 0 \Rightarrow \theta = \binom{0}{\pi} \Rightarrow \cos\theta = \pm 1 \equiv k \tag{8}$$

Then, Eq. (6) becomes an expression for r.

$$k\left[-(r^3-1) \pm E(2+r^3)\right] \pm 3qr = 0 \tag{9}$$

This cubic expression for r has up to three roots that are of interest. These roots must be real and greater than unity to be of physical interest. Rather than attempting to deal directly with the cubic, Eq. 9 is solved for the normalized charge, q,

$$q = \frac{k}{3}\left[(\pm 1 - E)r^2 - \frac{1}{r}(\pm 1 + 2E)\right] \tag{10}$$

The objective is to determine the charging current (and hence current of mass) to the drop when it has some location in the charge-imposed field plane (q, E). Sketches of the right-hand side of Eq. 10 as a function of r, fall in three categories, associated with the three regimes of this plane $E<-\frac{1}{2}$, $-\frac{1}{2}<E<1$, $1<E$ as shown in Fig. P5.5.2a.

The sketches make it possible to establish the number of critical points and their relative positions. Note that the extremum of the curves comes at

$$r_m = \left[\frac{1+2E}{2(E-1)}\right]^{1/3} \gtrless 1 \quad ; \quad \begin{cases} 1 < E \\ E < -\frac{1}{2} \end{cases} \tag{11}$$

For example, in the range $1<E$ this root is greater than unity and it is clear that on the $\theta=0$ axis

$$-q^* < q \Rightarrow \text{no roots}; -E<q<-q^* \Rightarrow 2\,\text{roots};\ q<-E \Rightarrow 1\,\text{root} \tag{12}$$

where

$$q^* \equiv \begin{cases} \frac{1}{2}(1+2E)^{2/3}\left[2(E-1)\right]^{1/3}; & 1 < E \\ \frac{1}{2}(-2E-1)^{2/3}\left[2(1-E)\right]^{1/3}; & E < -\frac{1}{2} \end{cases} \tag{13}$$

With the aid of these sketches, similar reasoning discloses critical points on the z axis, as shown in Fig. P5.5.2b. Note that $q = E \Rightarrow q = q_c$.

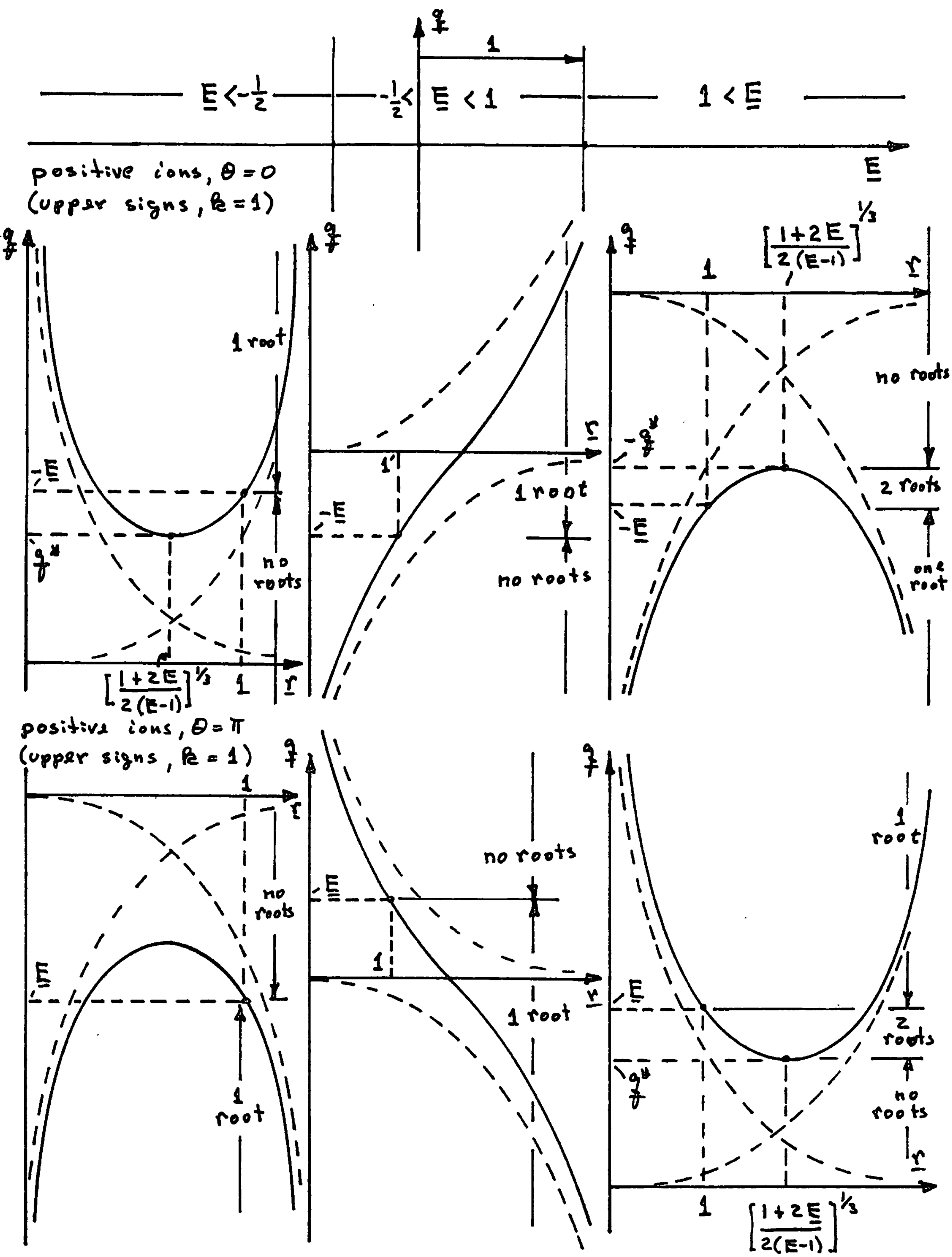

E < -1/2
-1/2 < E < 1
1 < E
1
E
positive ions, θ = 0
(upper signs, k = 1)
1 root
-E
g*
no roots
[(1+2E)/(2(E-1))]^(1/3)
1
r
1
1 root
no roots
no roots
2 roots
one root
positive ions, θ = π
(upper signs, k = 1)
no roots
1 root
no roots
1 root
2 roots
no roots

Fig. P5.5.2a

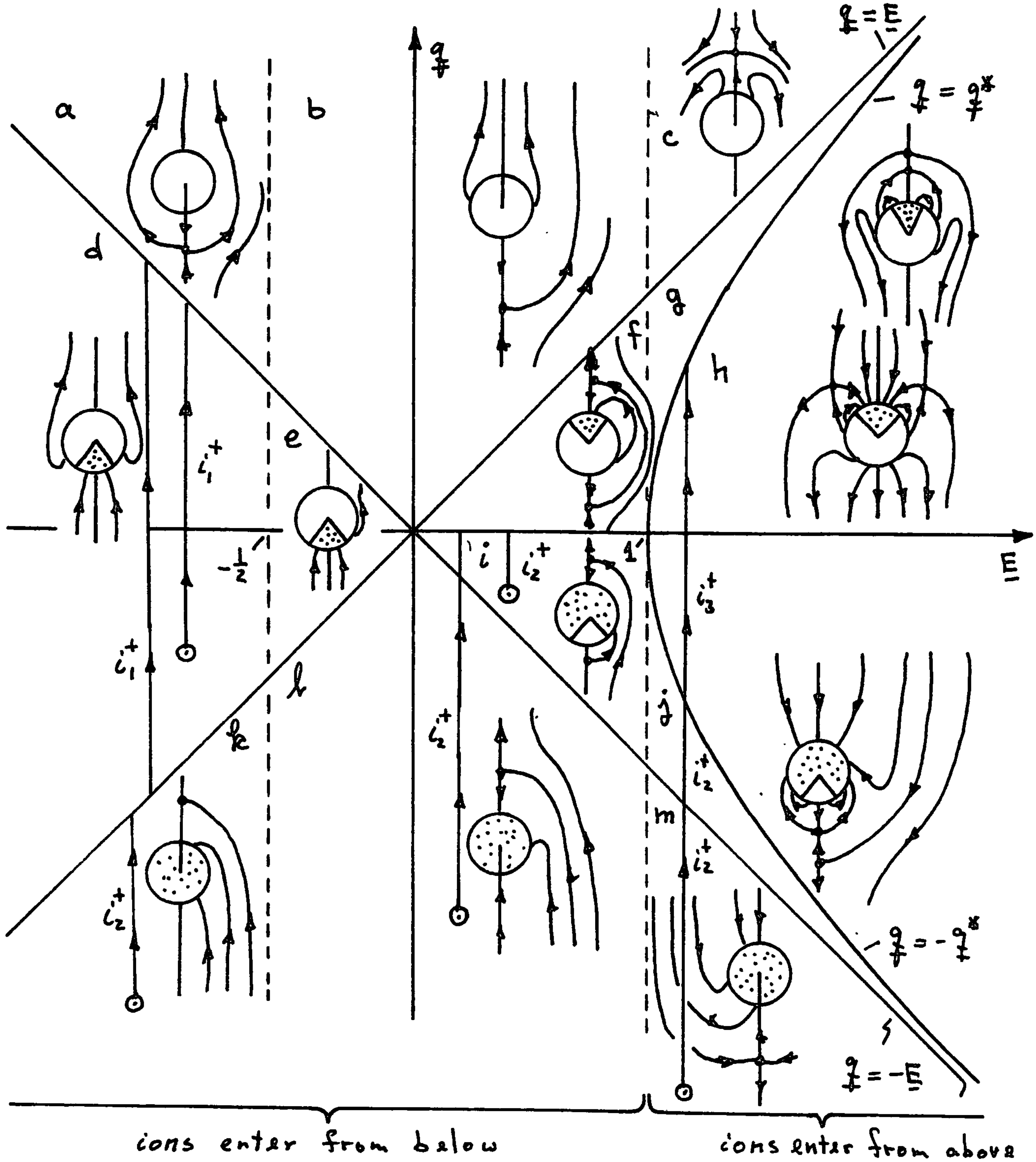


Fig. P5.5.2b Regimes of charging and critical points for positive ions.

Prob. 5.5.2 (cont.)

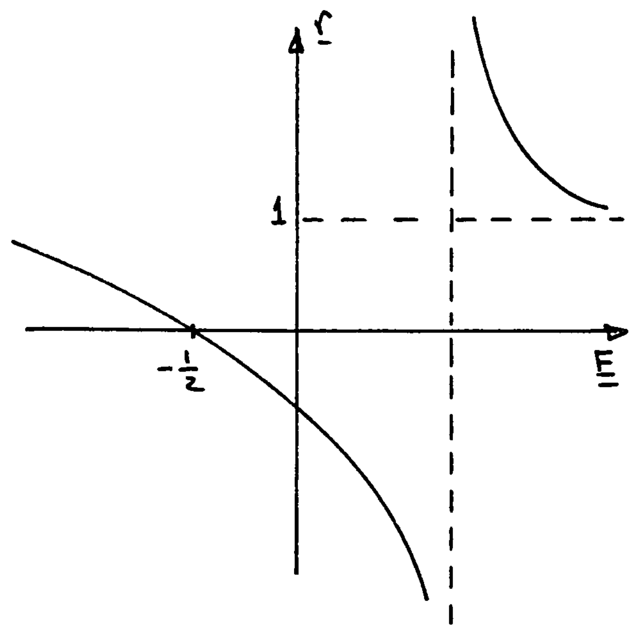
Any possible off-axis roots of Eqs. 6 and 7 are found by first considering solutions to Eq. 7 for $\sin\theta \neq 0$. Solution for $\underline{r}$ then gives

$$\underline{r} = \frac{\left(\frac{1}{2} \pm \underline{E}\right)^{1/3}}{(\underline{E} \mp 1)^{1/3}} \tag{14}$$

This expression is then substituted into Eq. 6, which can then be solved for $\cos\theta$

$$\cos\theta = -\frac{\underline{q}}{\underline{q}^*} \tag{15}$$

A sketch if Eq. 14 as a function of $\underline{E}$ shows that the only possible roots that are greater than unity are in the regimes where $1 < \underline{E}$. Further, for there to be a solution to Eq. 15, it is clear that $|\underline{q}| < |\underline{q}^*|$. This means that off-axis critical points are limited to regime h in Fig. 5.5.2b.



Consider how the critical points evolve for the regimes where $1 < \underline{E}$ as $\underline{q}$ is lowered from a large positive value. First, there is an on-axis critical point in regime c. As $\underline{q}$ is lowered, this point approaches the drop from above. As regime g is entered, a second critical point comes out of the north pole of the drop. As regime h is reached, these points coalesce and split to form a ring in the northern hemisphere. As the charge passes to negative values, this ring moves into the southern hemisphere, where as regime i is reached, the ring collapses into a point, which then splits into two points. As regime ℓ is entered, one of these passes into the south pole while the other moves downward.

Prob. 5.5.2 (cont.)

There are two further clues to the ion trajectories. The part of the particle surface that can possibly accept ions is as in the case considered in the text, and indicated by shading in Fig. 5.5.2b. Over these parts of the surface, there is an inward directed electric field. In addition, if $1 < \underline{E}$, ions must enter the neighborhood of the drop from above, while if $\underline{E} < \underline{1}$ they enter from below.

Finally, the stage is set to sketch the ion trajectories and determine the charging currents. With the singularities already sketched, and with the direction of entry of the characteristic lines from infinity and from the surface of the drop determined, the lines shown in Fig. 5.5.2b follow.

In regions (a), (b) and (c), where there are no lines that reach the drop from the appropriate "infinity", the charging current is zero.

In regions (d) and (e) there are no critical points in the region of interest. The line of demarcation between ions collected by the drop as they come from below and those that pass by is the line reaching the drop where the radial field switches from "out" to "in". Thus, the constant in Eq. 4 is determined by evaluating the expression where $r=R$ and $\cos\theta = -q/q_c = -\underline{q}/\underline{E}$ and hence $\sin^2\theta = 1-\cos^2\theta = 1-(\underline{q}/\underline{E})^2$. Thus, the constant is

$$\text{const.} = \frac{3}{2}\underline{E}\left(1 + \frac{\underline{q}^2}{\underline{E}^2}\right) \tag{16}$$

Now, following this line to $z\to\infty$, where $\cos\theta\to 1$ and $r\sin\theta\to y^*$ gives

$$y^{*2} = \frac{3bER^2}{U}\left(1+\frac{q}{q_c}\right)^2\Big/\left(\frac{bE}{U}-1\right) \tag{17}$$

Thus, the total current being collected is

$$i_1^+ = \pi y^{*2}\rho_+(U-bE) = -3\pi R^2 b|E|\rho_+\left(1-\frac{q}{|q_c|}\right)^2 \tag{18}$$

Prob. 5.5.2 (cont.)

The last form is written by recognizing that in this regime $\underline{E} < 0$, and hence q_c is negative. Note that the charging rate approaches zero as the charge approaches $|q_c|$.

In regime f, the trajectories starting at the lower singularity end at the upper singularity, and hence effectively isolate the drop from trajectories beginning where there is a source of ions. To see this note that the constant for these trajectories, set by evaluating Eq. 4 where $\sin\theta = 0$ and $\cos\theta = 1$ is const. = -3q. So, these lines are

$$-\frac{1}{2}\left(r^2 - \frac{1}{r}\right)\sin^2\theta + E\left(\frac{1}{r} + \frac{1}{2}r^2\right)\sin^2\theta - 3\,q\cos\theta = -3q \qquad (19)$$

Under what conditions do these lines reach the drop surface? To see, evaluate this expression at the particle surface and obtain an expression for the angle at which the trajectory meets the particle surface.

$$\frac{3E}{2}\sin^2\theta = 3q\,(\cos\theta - 1) \qquad (20)$$

Graphical solution of this expression shows that there are no solutions if $\underline{E} > 0$ and $q > 0$. Thus, in regime f, the drop surface does not collect ions.

In regime i, the collection is determined by first evaluating the constant in Eq. 4 for the line passing through the critical point at $\theta = \pi$. It follows that const. = $3q$ and that

$$\underline{y}^{*2} = \frac{-12q}{1-\underline{E}} \Rightarrow y^{*2} = \frac{-12\,R^2\left(\frac{bE}{U}\right)q}{\left(1 - \frac{bE}{U}\right)\;q_c} \qquad (21)$$

Thus, the current is

$$i_z^{+} = \pi y^{*2}\,(U - bE)\rho_+ = -12\pi R^2 \rho_+ b|E|\frac{q}{|q_c|} \qquad (22)$$

Note that this is also the current in regimes k, l and m.

In regime g, the drop surface is shielded from trajectories coming

Prob. 5.5.2 (cont.)

from above. In regime h the critical trajectories pass through the critical points represented by Eqs. 14 and 15. Evaluation of the constant in Eq. 4 then gives

$$\text{const.} = \frac{3}{2}\left(\frac{bE}{U}\right)\frac{\vartheta^*}{\vartheta_c}\left(1 + \frac{\vartheta^2}{\vartheta^{*2}}\right) \tag{23}$$

and it follows that

$$y^{*2} = \frac{2R^2}{\left(\frac{bE}{U} - 1\right)}\left(\frac{bE}{U}\right)\left[-3\frac{\vartheta}{\vartheta_c} + \frac{3}{2}\frac{\vartheta^*}{\vartheta_c}\left(1 + \frac{\vartheta^2}{\vartheta^{*2}}\right)\right] \tag{24}$$

Thus, the current is evaluated as

$$i_3^+ = 2\pi R^2 bE\left[-3\frac{\vartheta}{\vartheta_c} + \frac{3}{2}\frac{\vartheta^*}{\vartheta_c}\left(1 + \frac{\vartheta^2}{\vartheta^{*2}}\right)\right] \tag{25}$$

Note that at the boundary between regimes g and h, where $\vartheta = \vartheta^*$, this expression goes to zero, as it should to match the null current for regime g.

As the charge approaches the boundary between regimes h and j, $\vartheta = -\vartheta^*$ and the current becomes $i_3^+ \rightarrow 12\pi R^2 bE\,\vartheta^*/\vartheta$. This suggests that the current of regime m extends into regime j. That this is the case can be seen by considering that the same critical trajectory determines the current in these latter regimes.

To determine the collection laws for the negative ions, the arguments parallel those given, with the lower signs used in going beyond Eq. 10.

Prob. 5.6.1 A statement that the initial total charge is equal to that at a later time is made by multiplying the initial volume by the initial charge density and setting it equal to the charge density at time t multiplied by the volume at that time. Here, the fact that the cloud remains uniform in its charge density is exploited.

$$\frac{4}{3}\pi\left(R_o^3 - R_i^3\right)\rho_u = \frac{4}{3}\pi R_o^3\left\{1 + \left[\frac{3\Phi_v\tau_e}{4\pi R_o^2} + 1 - \left(\frac{R_i}{R_o}\right)^3\right]\frac{t}{\tau_e} + \left(\frac{R_i}{R_o}\right)^3 - \left(\frac{3\Phi_v\tau_e}{4\pi R_o^2}\right)\frac{t}{\tau_e}\right\}\frac{\rho_u}{1+\frac{t}{\tau_e}}$$

$$= \frac{4}{3}\pi R_o^3\left[1 - \left(\frac{R_i}{R_o}\right)^2\right]\left[1 + \frac{t}{\tau_e}\right]\frac{\rho_u}{1+\frac{t}{\tau_e}}$$

Prob. 5.6.2 a) From Sec. 5.6, the rate of change of charge density for an observer moving along the characteristic line

$$\frac{d\bar{r}}{dt} = \bar{v} + b\bar{E} \tag{1}$$

is given by

$$\frac{d\rho}{dt} = -\frac{\rho^2 b}{\epsilon} \tag{2}$$

Thus, along these characteristics,

$$\rho = \frac{\rho_o}{1 + t/\tau} \quad ; \quad \tau \equiv \frac{\epsilon}{\rho_o b} \tag{3}$$

where throughout this discussion the charge density is presumed positive. The charge density at any given time depends only on the original density (where the characteristic originated) and the elapsed time. So, at any time, points from characteristic lines originating where the charge is uniform have the same charge density. Therefore, the charge-density in the cloud is uniform.

b) The integral form of Gauss' law requires that

$$\oint_S \epsilon \bar{E} \cdot \bar{n}\, da = \int_V \rho\, dV \tag{4}$$

and because the charge density is uniform in the layer, this becomes

$$E_f - E_b = \frac{\rho_o}{\epsilon} \frac{1}{(1 + t/\tau)} (z_f - z_b) \tag{5}$$

The characteristic lines for particles at the front and back of the layer are represented by

$$\frac{dz_f}{dt} = U + bE_f \quad ; \quad \frac{dz_b}{dt} = U + bE_b \tag{6}$$

These expressions combine with Eq. 5 to show that

$$\frac{d}{dt}(z_f - z_b) = \frac{1}{\tau} \frac{1}{1 + t/\tau} (z_f - z_b) \tag{7}$$

Integration gives

$$\int_{z_F - z_B}^{z_f - z_b} \frac{d(z_f - z_b)}{(z_f - z_b)} = \int_0^t \frac{d(t/\tau)}{1 + t/\tau} \tag{8}$$

and hence it follows that

$$z_f - z_b = (1 + t/\tau)(z_F - z_B) \tag{9}$$

Prob. 5.6.2(cont.)

Given the uniform charge distribution in the layer, it follows from Gauss' law that the distribution of electric field intensity is

$$E = \begin{cases} E_b & 0 < z < z_b \\ E_b + (E_f - E_b)\left[\dfrac{z - z_b}{z_f - z_b}\right] & z_b < z < z_f \\ E_f & z_f < z < \ell \end{cases} \tag{10}$$

From this it follows that the voltage, V, is related to E_f and E_b by

$$V = \int_0^\ell E\,dz = E_b z_b + E_b(z_f - z_b) + \frac{1}{2}(E_f - E_b)\frac{(z_f - z_b)^2}{z_f - z_b} + E_f(\ell - z_f) \tag{11}$$

From Eqs. 5 and 9,

$$E_f - E_b = \frac{\rho_o}{\epsilon}(z_F - z_B) \tag{12}$$

$$z_f - z_b = (1 + t/\tau)(z_F - z_B) \tag{13}$$

Substitution for E_b and $z_f - z_b$ as determined by these relations into Eq. 11 then gives an expression that can be solved for E_f.

$$E_f = \frac{V}{\ell} - \frac{1}{2\ell}\frac{\rho_o}{\epsilon}(z_F - z_B)^2\left(1 + \frac{t}{\tau}\right) + \frac{\rho_o}{\ell\epsilon}(z_F - z_B)z_f \tag{14}$$

d) In view of Eq. 6a, this expression makes it possible to write

$$\frac{dz_f}{dt} - \frac{(z_F - z_B)}{\ell}\frac{z_f}{\tau} = \left[U + \frac{bV}{\ell} - \frac{1}{2\ell\tau}(z_F - z_B)^2\right] - \frac{1}{2\ell\tau}(z_F - z_B)^2\frac{t}{\tau} \tag{15}$$

Solutions to this differential equation take the form

$$z_f = A\,e^{\frac{(z_F - z_B)}{\ell}\frac{t}{\tau}} + Bt + C \tag{16}$$

The coefficients of the particular solution, B and C, are found by substituting Eq. 16 into Eq. 15 to obtain

$$B = \frac{z_F - z_B}{2\tau} \tag{17}$$

$$C = \left[1 - K\frac{2\tau}{z_F - z_B}\right]\frac{\ell}{2}\;;\quad K \equiv U + \frac{bV}{\ell} - \frac{1}{2\ell\tau}(z_F - z_B)^2 \tag{18}$$

The coefficient of the homogeneous solution follows from the initial condition that when t=0, $z_f = z_F$.

$$A = z_F - \left(1 - K\frac{2\tau}{z_F - z_B}\right)\frac{\ell}{2} \tag{19}$$

Prob. 5.6.2(cont.)

The position of the back edge of the charge layer follows from this expression and Eq. 9.

$$z_b = z_f - (z_F - z_B)(1 + t/\tau) \tag{20}$$

Normalization of these last two expressions in accordance with

$$\underline{t} \equiv t/\tau \;,\quad \underline{V} \equiv \tau b V/\ell^2 \;,\quad \underline{U} = U/(bV/\ell) \tag{21}$$

$$(\underline{z}_f, \underline{z}_F, \underline{z}_b, \underline{z}_B) = (z_f, z_F, z_b, z_B)/\ell$$

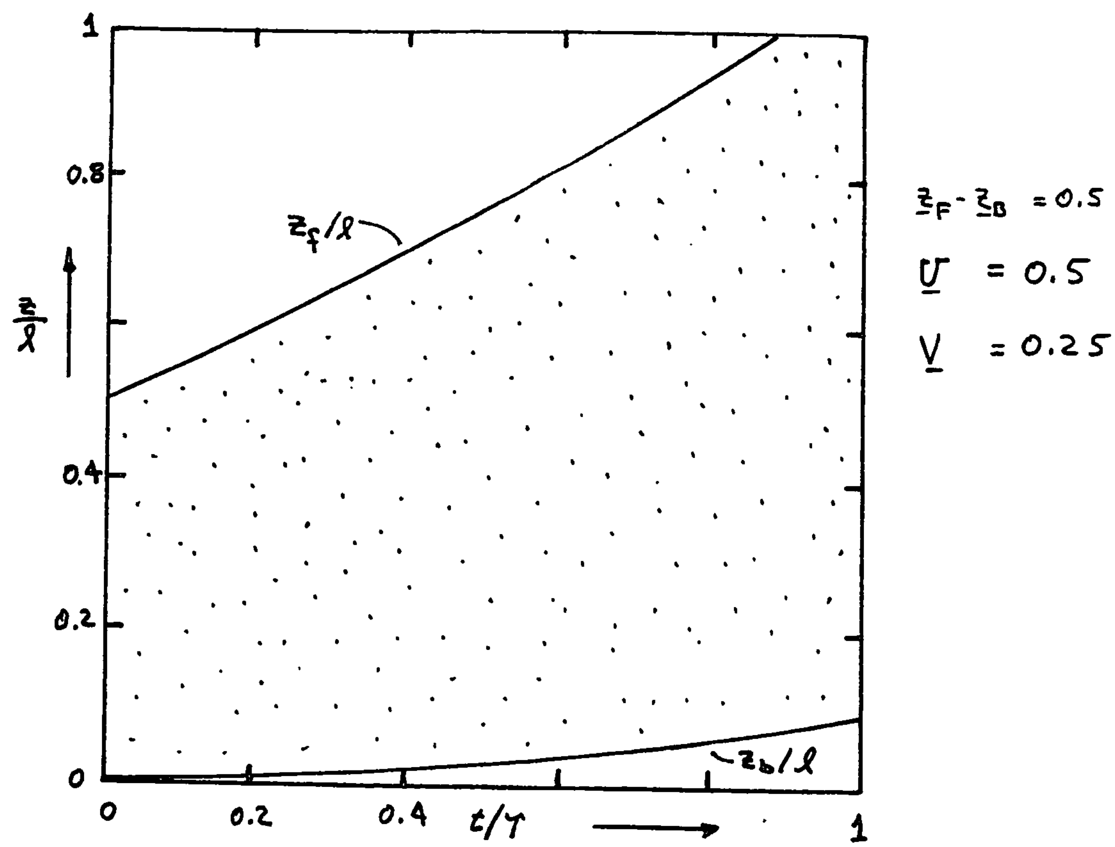
results in

$$z_f = \left\{ z_F - \frac{1}{2} + \frac{V}{z_F - z_B}\left[U + 1 - \frac{(z_F - z_B)^2}{2V}\right]\right\} e^{(z_F - z_B)t} + \frac{1}{2}(z_F - z_B)t + \left\{\frac{1}{2} - \frac{V}{z_F - z_B}\left[U + 1 - \frac{(z_F - z_B)^2}{2V}\right]\right\} \tag{22}$$

and

$$z_b = z_f - (z_F - z_B)(1 + t) \tag{23}$$

The evolution of the charge layer is illustrated in the figure.



Prob. 5.7.1 The characteristic equations are Eqs. 5.6.2 and 5.6.3, written as

$$\frac{d\rho}{dt} = -\frac{\rho^2 b}{\epsilon} \tag{1}$$

$$\frac{dz}{dt} = U + bE \tag{2}$$

It follows from Eq. 1 that

$$\int_1^{\rho/\rho_o} \frac{d(\rho/\rho_o)}{(\rho/\rho_o)^2} = \int_0^t \frac{\rho_o b}{\epsilon}\,dt \Rightarrow \frac{\rho}{\rho_o} = \frac{1}{1 + t/\tau}\;;\; \tau \equiv \frac{\epsilon}{\rho_o b} \tag{3}$$

Charge conservation requires that

$$J = \rho(bE + U) = \frac{i}{A} \tag{4}$$

where i/A is a constant. This is used to evaluate the right hand side of Eq. 2, which then becomes

$$\frac{dz}{dt} = \frac{J}{\rho} = \frac{i}{A\rho_o}\left(1 + \frac{t}{\tau}\right) \tag{5}$$

where Eq. 3 has been used. Integration then gives

$$\int_0^z dz = \int_0^{t/\tau} \frac{i\tau}{A\rho_o}\left(1 + \frac{t}{\tau}\right) d\left(\frac{t}{\tau}\right) = \frac{i\tau}{2A\rho_o}\left[\left(1 + \frac{t}{\tau}\right)^2 - 1\right] \tag{6}$$

Thus,

$$\left(1 + \frac{t}{\tau}\right)^2 = 2\frac{z}{l}\,\frac{i_o}{iR_e} + 1 \tag{7}$$

Finally, substitution into Eq. 3 gives the desired dependence on z.

$$\frac{\rho}{\rho_o} = \left[1 + \left(\frac{2z}{l}\right)\left(\frac{i_o}{iR_e}\right)\right]^{-1/2} \tag{8}$$

Prob. 5.9.1 For uniform distributions, Eqs. 9 and 10 become

$$\frac{d\rho_+}{dt} = \beta n - \frac{\alpha\rho_+\rho_-}{q} \tag{1}$$

$$\frac{d\rho_-}{dt} = \beta n - \frac{\alpha\rho_+\rho_-}{q} \tag{2}$$

$$\frac{dn}{dt} = -\frac{\beta n}{q} + \frac{\alpha\rho_+\rho_-}{q^2} \tag{3}$$

Subtraction of Eqs. 1 and 2 shows that

$$\frac{d}{dt}(\rho_+ - \rho_-) = 0 \tag{4}$$

and given the initial conditions it follows that

$$\rho_+ = \rho_- \tag{5}$$

Note that there being no net charge is consistent with $\bar{E}=0$ in Gauss' law.

(b) Multiplication of Eq. 3 by q and addition to Eq. 1, incorporating Eq. 5, then gives

$$\frac{d}{dt}(\rho_+ + qn) = 0 \tag{6}$$

The constant of integration follows from the initial conditions.

$$\rho_+ + qn = qn_o \tag{7}$$

Introduced into Eq. 3, this expression results in the desired equation for n(t).

$$\frac{dn}{dt} = -\frac{\beta}{q}n + \alpha(n_o - n)^2 \tag{8}$$

Introduced into Eq. 1 it gives an expression for $\rho_+(t)$.

$$\frac{d\rho_+}{dt} = -\frac{\beta}{q}\rho_+ - \frac{\alpha}{q}\rho_+^2 + \beta n_o \tag{9}$$

(c) The stationary state follows from Eq. 8 .

$$n = \left(n_o + \frac{\beta}{2\alpha q}\right) - \sqrt{\left(n_o + \frac{\beta}{2\alpha q}\right)^2 - n_o^2} \tag{10}$$

(d) The first terms on the right in Eqs. 8 and 9 dominate at early times making it clear that the characteristic time for the transients is $\tau_{th} = q/\beta$.

<u>Prob. 5.10.1</u> With $\rho_F(x_0, z_0, 0)$ defined as the charge distribution when t=0, the general solution is

$$\rho_f = \rho_F(x_0, z_0, 0)\, e^{-t/\tau} \quad ; \quad \tau \equiv \epsilon/\sigma \tag{1}$$

on the lines

$$x = x_0$$
$$z = \frac{Ux}{d}t + z_0 \tag{2}$$

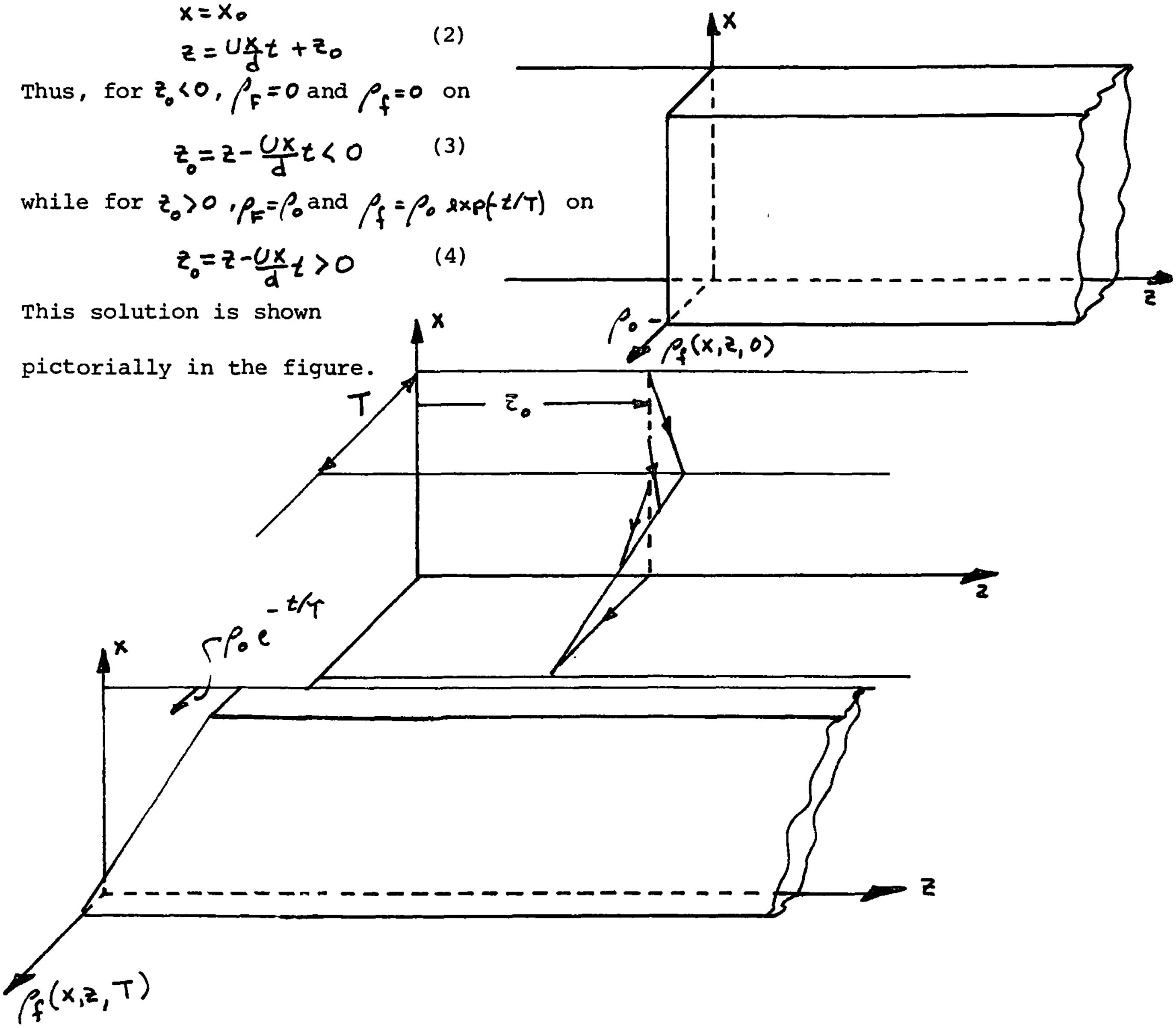
Thus, for $z_0 < 0$, $\rho_F = 0$ and $\rho_f = 0$ on

$$z_0 = z - \frac{Ux}{d}t < 0 \tag{3}$$

while for $z_0 > 0$, $\rho_F = \rho_0$ and $\rho_f = \rho_0 \exp(-t/\tau)$ on

$$z_0 = z - \frac{Ux}{d}t > 0 \tag{4}$$

This solution is shown pictorially in the figure.



<u>Prob. 5.10.2</u> With the understanding that time is measured along a characteristic line, the charge density is

$$\rho = \rho(t = t_a, z = 0)\, e^{-(t-t_a)/\tau} \quad ; \quad \tau \equiv \epsilon/\sigma \tag{1}$$

where t_a is the time when the characteristic passed through the plane z=0, as shown in the figure. The solution to the characteristic equations is

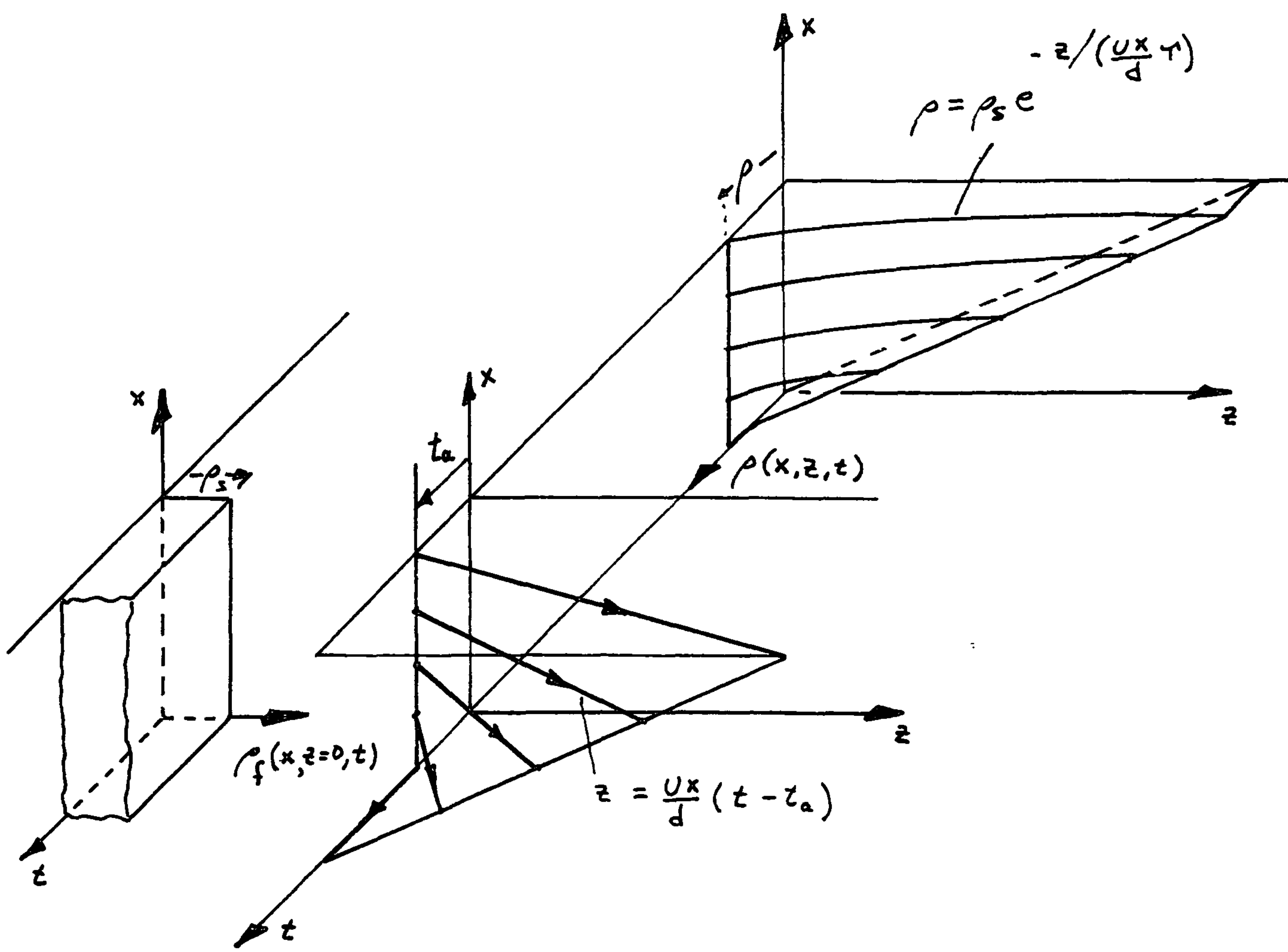
$$x = \text{constant} \tag{2}$$

$$z = \frac{Ux}{d}(t - t_a) \tag{3}$$

Thus, substitution for $t - t_a$ in Eq. 1 gives the charge density as

$$\rho = \begin{cases} \rho_s\, e^{-z/(Ux\tau/d)} & ; 0 < z < Uxt/d \\ 0 & ; Uxt/d < z \end{cases} \tag{4}$$

The time varying boundary condition at z=0, the characteristic lines and the charge distribution are illustrated in the figure. Note that once the wave-front has passed, the charge density remains constant in time.



Prob. 5.10.3 With it understood that

$$q = \int_V \rho_f \, dV \tag{1}$$

the integral form of Gauss' law is

$$\oint_S \epsilon \bar{E} \cdot \bar{n} \, da = q \tag{2}$$

and conservation of charge in integral form is

$$\oint_S \sigma \bar{E} \cdot \bar{n} \, da + \frac{dq}{dt} = 0 \tag{3}$$

Because ϵ and σ are uniform over the enclosing surface, S, these combine to eliminate $\bar{E}$ and require

$$\frac{dq}{dt} + \frac{q}{\tau} = 0 \quad ; \quad \tau \equiv \epsilon/\sigma \tag{4}$$

Thus, the charge decays with the relaxation time.

Prob. 5.12.1 (a) Basic laws are

$$\nabla \times \bar{E} = 0 \Rightarrow \bar{E} = -\nabla \Phi \tag{1}$$

$$\nabla \cdot \epsilon \bar{E} = \rho_f \tag{2}$$

$$\nabla \cdot \bar{J}_f + \frac{\partial \rho_f}{\partial t} = 0 \tag{3}$$

The first and second are substituted into the last with the conduction current as given to obtain an expression for the potential

$$\sigma_x \frac{\partial^2 \Phi}{\partial x^2} + \sigma_y \frac{\partial^2 \Phi}{\partial y^2} + \sigma_z \frac{\partial^2 \Phi}{\partial z^2} + \epsilon \frac{\partial}{\partial t}\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}\right) = 0 \tag{4}$$

With the substitution of the complex amplitude form, this requires of the potential that

$$\frac{d^2 \hat{\Phi}}{dx^2} - \gamma^2 \hat{\Phi} = 0 \tag{5}$$

where

$$\gamma^2 \equiv \left[k_y^2(\sigma_y + j\omega\epsilon) + k_z^2(\sigma_z + j\omega\epsilon)\right] / (\sigma_x + j\omega\epsilon)$$

Although γ is now complex, solution of Eq. 5 is the same as in Sec. 2.16, except that the time dependance has been assumed.

$$\hat{\Phi} = \hat{\Phi}^\alpha \frac{\sinh \gamma x}{\sinh \gamma \Delta} - \hat{\Phi}^\beta \frac{\sinh \gamma (x - \Delta)}{\sinh \gamma \Delta} \tag{6}$$

from which it follows that

$$\hat{J}_x = -(j\omega\epsilon + \sigma_x)\gamma \left[\hat{\Phi}^\alpha \frac{\cosh \gamma x}{\sinh \gamma \Delta} - \hat{\Phi}^\beta \frac{\cosh \gamma (x - \Delta)}{\sinh \gamma \Delta}\right] \tag{7}$$

Evaluation at the (α, β) surfaces, where $x = \Delta$ and $x = 0$, respectively, then gives the required transfer relations

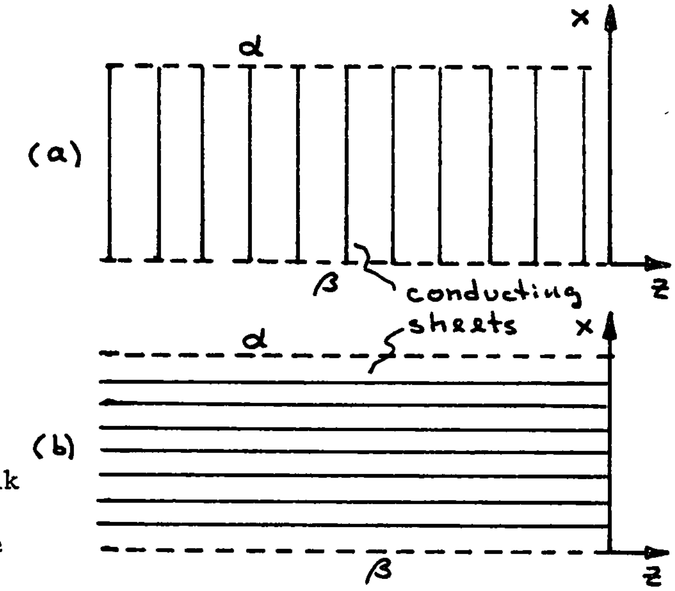
$$\begin{bmatrix} \hat{J}_x^\alpha \\ \hat{J}_x^\beta \end{bmatrix} = (j\omega\epsilon + \sigma_x)\gamma \begin{bmatrix} -\coth \gamma \Delta & \frac{1}{\sinh \gamma \Delta} \\ \frac{-1}{\sinh \gamma \Delta} & \coth \gamma \Delta \end{bmatrix} \begin{bmatrix} \hat{\Phi}^\alpha \\ \hat{\Phi}^\beta \end{bmatrix} \tag{8}$$

Prob. 5.12.1(cont.)

(b) In this limit, the medium might be composed of finely dispersed wires extending in the x direction and insulated from each other, as shown in the figure. With σ_y and $\sigma_z \rightarrow 0$,

$$\gamma^2 = j\omega\epsilon k^2/(\sigma_x + j\omega\epsilon) \rightarrow j\omega\epsilon k^2/\sigma_x$$

as $\omega \rightarrow 0$.

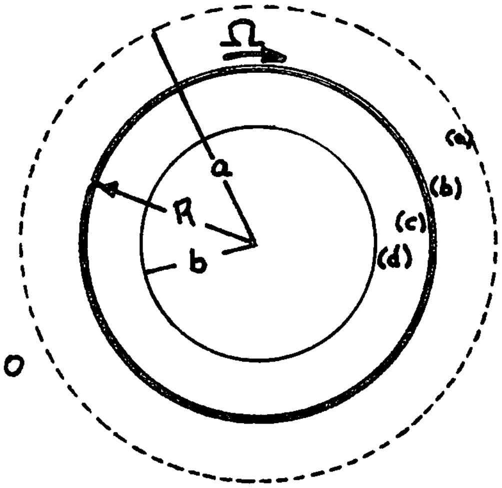


That this factor is complex means that the entries in Eq. 8 are complex. Thus, there is a phase shift (in space and/or in time depending on the nature of the excitations) of the potential in the bulk relative to that on the boundaries. The amplitude of γ gives an indication of the extent to which the potential penetrates into the volume. As $\omega \rightarrow 0, \gamma \rightarrow 0$, which points to an "infinite" penetration at zero frequency. That is, regardless of the spatial distribution of the potential at one surface, at zero frequency it will be reproduced at the other surface regardless of wavelength in the directions y and z. Regardless of k, the transfer relations reduce to

$$\begin{bmatrix} \hat{J}_x^\alpha \\ \hat{J}_x^\beta \end{bmatrix} \rightarrow \frac{\sigma_x}{\Delta} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \hat{\Phi}^\alpha \\ \hat{\Phi}^\beta \end{bmatrix} \tag{9}$$

The "wires" carry the potential in the x direction without loss of spatial resolution.

(c) With no conduction in the x direction but finely dispersed conducting sheets in y-z planes, $\gamma^2 \rightarrow k^2(1 + \sigma_o/j\omega\epsilon)$. Thus, the fields do not penetrate in the x direction at all in the limit $\omega \rightarrow 0$. In the absence of time varying excitations, the y-z planes relax to become equipotentials and effectively shield the surface potentials from the material volume.



Prob. 5.13.1 a) Boundary conditions are

$$\hat{\Phi}^a = \hat{V}_o \tag{1}$$

$$\hat{\Phi}^b = \hat{\Phi}^c \tag{2}$$

Charge conservation for the sheet requires that

$$\frac{1}{R}\frac{\partial}{\partial\theta}(\sigma_s E_\theta) + \left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\theta}\right)(D_r^b - D_r^c) = 0$$

where

$$\hat{E}_\theta = j m \hat{\Phi}$$

In terms of complex amplitudes,

$$\frac{\sigma_s m^2}{R^2}\hat{\Phi}^b + j(\omega - m\Omega)(\hat{D}_r^b - \hat{D}_r^c) = 0 \tag{3}$$

Finally, there is the boundary condition

$$\hat{\Phi}^d = 0 \tag{4}$$

Transfer relations for the two regions follow from Table 2.16.2. They are written with Eqs. 1,2,and 4 taken into account.

$$\begin{bmatrix}\hat{D}_r^a \\ \hat{D}_r^b\end{bmatrix} = \epsilon_o \begin{bmatrix} f_m(R,a) & g_m(a,R) \\ g_m(R,a) & f_m(a,R)\end{bmatrix}\begin{bmatrix}\hat{V}_o \\ \hat{\Phi}^b\end{bmatrix} \tag{5}$$

$$\begin{bmatrix}\hat{D}_r^c \\ \hat{D}_r^d\end{bmatrix} = \epsilon_o \begin{bmatrix} f_m(b,R) & g_m(R,b) \\ g_m(b,R) & f_m(R,b)\end{bmatrix}\begin{bmatrix}\hat{\Phi}^b \\ 0\end{bmatrix} \tag{6}$$

Substitution of Eqs. 5b and 6a into Eq. 3 gives

$$\frac{\sigma_s m^2}{R^2}\hat{\Phi}^b + j(\omega - m\Omega)\epsilon_o\left\{g_m(R,a)\hat{V}_o + \hat{\Phi}^b\left[f_m(a,R) - f_m(b,R)\right]\right\} = 0 \tag{7}$$

or

$$\hat{\Phi}^b = \frac{-j S_e \hat{V}_o g_m(R,a) R}{m^2 + j S_e[f_m(a,R) - f_m(b,R)]R} \tag{8}$$

where $S_e \equiv \epsilon_o(\omega - m\Omega)R/\sigma_s$.

Prob. 5.13.1 (cont.)

b) The torque is

$$\Upsilon_z = (2\pi R^2 \ell)\,\tfrac{1}{2}\,\mathrm{Re}\,\hat{D}_r^b\,\hat{E}_\theta^{b*} \tag{9}$$

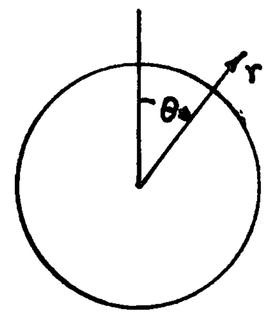
Because $\hat{E}_\theta = jm\hat{\Phi}/R$ and because of Eq. 5b, this expression becomes

$$\Upsilon_z = \pi R^2 \ell\, \mathrm{Re}\left[\epsilon_o g_m(R,a)\hat{V}_o \frac{(-jm)}{R}\hat{\Phi}^{b*}\right] \tag{10}$$

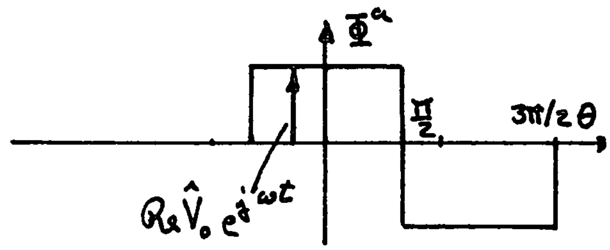
Substitution from Eq. 8 then gives the desired expression

$$\Upsilon_z = \frac{\pi R^2 \ell\, \epsilon_o |\hat{V}_o|^2 g_m^2(R,a)\, S_e\, m^3}{m^4 + S_e^2\left[f_m(a,R) - f_m(b,R)\right]^2 R^2} \tag{11}$$

Prob. 5.13.2 With the (θ,r) coordinates defined as shown, the potential is the function of θ shown to the right. This function is represented by



$$\Phi^a = \mathrm{Re}\left[\sum_{m=-\infty}^{+\infty} \hat{V}_m e^{-jm\theta}\right] e^{j\omega t} \tag{1}$$



The multiplication of both sides by $e^{jn\theta}$ and integration over one period then gives

$$2\pi \hat{V}_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \hat{V}_o e^{jn\theta} d\theta - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \hat{V}_o e^{jn\theta} d\theta \tag{2}$$

which gives $(n \to m)$

$$\hat{V}_m = \frac{2\hat{V}_o}{\pi}\,\frac{\sin\left(\frac{m\pi}{2}\right)}{m} \tag{3}$$

Looking ahead, the current to the upper center electrode is

$$\hat{i} = j\omega\hat{q} = j\omega W \int_0^{\pi} \sum_{-\infty}^{+\infty} (\hat{D}_x^d)_m e^{-jm\theta} d\theta = 2j\omega W \sum_{-\infty}^{+\infty} \frac{\hat{D}_{xm}^b}{m} \sin\left(\frac{m\pi}{2}\right) \tag{4}$$

It then follows from Eqs. 6b and 8 that

$$\hat{i} = \frac{4\omega W \epsilon_o}{\pi} \sum_{m=-\infty}^{+\infty} j^m \frac{\sin^2\left(\frac{m\pi}{2}\right)}{m^2}\; \frac{g_m(b,R)\, S_{em} \hat{V}_o\, g_m(R,a)}{m^2 + j S_{em}\left[f_m(a,R) - f_m(b,R)\right] R} \tag{5}$$

where $S_{em} \equiv (\omega - m\Omega) R\epsilon_o/\sigma_s$.

Prob. 5.13.2 (cont.)

If the series is truncated at m=±1, this expression becomes one analogous to the one in the text.

$$\hat{i} = j\frac{4}{\pi}\omega w \epsilon_o\, g_1(b,R)\, g_1(R,a)\left\{\frac{S_{e1}}{1 + jS_{e1}[f_1(a,R) - f_1(b,R)]R} - \frac{S_{e-1}}{1 + jS_{e-1}[f_1(a,R) - f_1(b,R)]R}\right\} \tag{6}$$

or

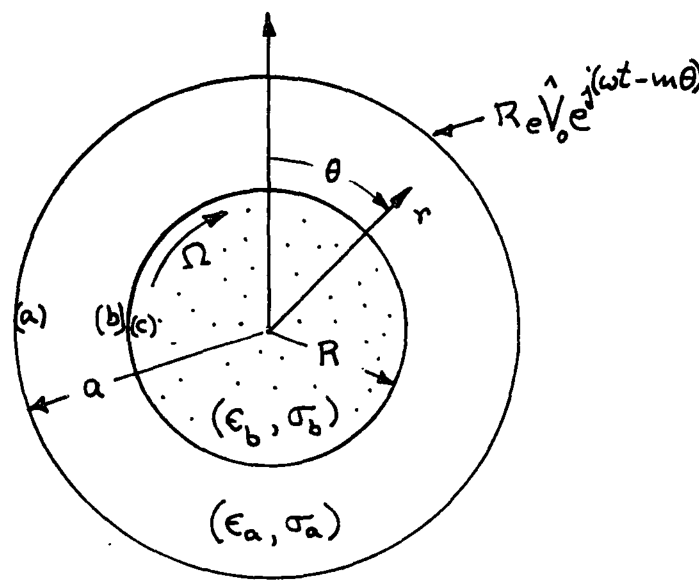
$$|\hat{i}| = \frac{\frac{4}{\pi}\omega w \epsilon_o |\hat{V}_o|\, g_1(b,R)\, g_1(R,a)\left(\frac{2R\epsilon_o}{\sigma_s}\Omega\right)}{\sqrt{\{1 + S_{e1}^2[f_1(a,R) - f_1(b,R)]^2R^2\}\{1 + S_{e-1}^2[f_1(a,R) - f_1(b,R)]^2R^2\}}} \tag{7}$$

Prob. 5.14.1 Bulk relations for the two regions, with surfaces designated as in the figure, are

$$\begin{bmatrix} \hat{D}_r^a \\ \hat{D}_r^b \end{bmatrix} = \epsilon_a \begin{bmatrix} f_m(R,a) & g_m(a,R) \\ g_m(R,a) & f_m(a,R) \end{bmatrix} \begin{bmatrix} \hat{\Phi}^a \\ \hat{\Phi}^b \end{bmatrix} \tag{1}$$

and

$$\hat{D}_r^c = \epsilon_b f_m(0,R)\, \hat{\Phi}^b \tag{2}$$



Integration of the Maxwell stress over a surface enclosing the rotor amounts to a multiplication of the average traction in the θ direction by the surface area, and then to obtain a torque, by the lever arm, R.

$$\tau_z = \frac{1}{2} R_e\left[2\pi \ell R^2\, \hat{E}_\theta^{b*}\, \hat{D}_r^b\right] \tag{3}$$

Because $\hat{E}_\theta^b = +j\frac{m}{R}\hat{\Phi}^b$, introduction of Eq. 1b into Eq. 3 makes it possible to write this torque in terms of the driving potential $\hat{\Phi}^a = \hat{V}_o$ and the potential on the surface of the rotor.

Prob. 5.14.1(cont.)

$$T_z = \pi R^2 \ell \epsilon_a g_m(R,a) \mathrm{Re}\left(j\frac{m}{R}\hat{V}_o^* \hat{\Phi}^b\right) \tag{4}$$

There are two boundary conditions at the surface of the rotor. The potential must be continuous, so

$$\hat{\Phi}^b = \hat{\Phi}^c \tag{5}$$

and charge must be conserved.

$$j(\omega - \Omega m)(\hat{D}_r^b - \hat{D}_r^c) + \left(\frac{\sigma_a}{\epsilon_a}\hat{D}_r^b - \frac{\sigma_b}{\epsilon_b}\hat{D}_r^c\right) = 0 \tag{6}$$

Substitution of Eqs. 1b and 2, again using the boundary condition $\hat{\Phi}^a = \hat{V}_o$ and Eq. 5, then gives an expression that can be solved for the rotor surface potential.

$$\hat{\Phi}^b = \frac{-\hat{V}_o g_m(R,a)[\epsilon_a j(\omega - \Omega m) + \sigma_a]}{\sigma_a f_m(a,R) - \sigma_b f_m(0,R) + j(\omega - \Omega m)(\epsilon_a f_m(a,R) - \epsilon_b f_m(0,R))} \tag{7}$$

Substitution of Eq. 7 into Eq. 4 shows that the torque is

$$T_z = \frac{\pi R^2 \ell m \epsilon_a g_m(R,a) \mathrm{Re}[\epsilon_a(\omega - \Omega m) - j\sigma_a]|\hat{V}_o|^2}{R[\sigma_a f_m(a,R) - \sigma_b f_m(0,R)][1 + jS_e]} \tag{8}$$

where

$$S_e \equiv \frac{(\omega - \Omega m)(\epsilon_a f_m(a,R) - \epsilon_b f_m(0,R))}{\sigma_a f_m(a,R) - \sigma_b f_m(0,R)}$$

Rationalization of Eq. 8 show that the real part is

$$T_z = \frac{-\pi R \ell \epsilon_a |\hat{V}_o|^2 (\epsilon_a \sigma_b - \sigma_a \epsilon_b) g_m^2(R,a) f_m(0,R) m}{[\sigma_a f_m(a,R) - \sigma_b f_m(0,R)][\epsilon_a f_m(a,R) - \epsilon_b f_m(0,R)]} \frac{S_e}{1 + S_e^2} \tag{9}$$

Note that $f_m(0,R)$ is negative, so this expression takes the same form as Eq. 5.14.11.

Prob. 5.14.2 (a) Boundary conditions at the rotor surface require continuity of potential and conservation of charge.

$$[\![\Phi]\!]=0 \tag{1}$$

$$\frac{\partial \sigma_f}{\partial t}+\Omega\frac{\partial \sigma_f}{\partial \theta}=\sigma\frac{\partial \Phi^a}{\partial r} \tag{2}$$

where Gauss' law gives $\sigma_f=\epsilon_a E_r^a-\epsilon_b E_r^b$

Potentials in the fluid and within the rotor are respectively

$$\Phi=E(t)r\cos\theta+P_x(t)\frac{\cos\theta}{r}+P_y(t)\frac{\sin\theta}{r}\;;\quad r>b \tag{3}$$

$$\Phi=Q_x(t)r\cos\theta+Q_y(t)r\sin\theta \tag{4}$$

These are substituted into Eqs. 1 and 2, which are factored according to whether terms have a sin θ or cos θ dependence. Thus, each expression gives rise to two equations in P_x, P_y, Q_x and Q_y. Elimination of Q_x and Q_y reduces the four expressions to two.

$$(\epsilon_a+\epsilon_b)\frac{dP_x}{dt}+(\epsilon_a+\epsilon_b)\Omega P_y+\sigma P_x=-b^2(\epsilon_b-\epsilon_a)\frac{dE}{dt}+\sigma b^2 E \tag{5}$$

$$(\epsilon_a+\epsilon_b)\frac{dP_y}{dt}-(\epsilon_a+\epsilon_b)\Omega P_x-(\epsilon_b-\epsilon_a)E\Omega b^2+\sigma P_y=0 \tag{6}$$

To write the mechanical equation of motion, the electric torque per unit length is computed.

$$T=b\int_0^{2\pi}\frac{\epsilon_a}{b}\frac{\partial \Phi^a}{\partial r}\frac{\partial \Phi^a}{\partial \theta}b\,d\theta \tag{7}$$

Substitution from Eq. 3 and integration gives

$$T=2\epsilon_a\pi E P_y \tag{8}$$

Thus, the torque equation is

$$I\frac{d\Omega}{dt}+B\Omega=-2\epsilon_a\pi E P_y \tag{9}$$

The first of the given equations of motion is obtained from this one by using the normalization that is also given. The second and third relations follow by similarly normalizing Eqs. 5 and 6.

Prob. 5.14.2(cont.)

(b) Steady rotation with $\underline{E}=1$ reduces the equations of motion to

$$\Omega = P_y \tag{10}$$

$$\Omega P_y + P_x = H_e^2 \tag{11}$$

$$-\Omega P_x + P_y = f H_e^2 \Omega \tag{12}$$

Elimination among these for Ω results in the expression

$$H_e^2 (1+f)\Omega = (1+\Omega^2)\Omega \tag{13}$$

One solution to this expression is the static equilibrium $\Omega = 0$.

Another is possible if H_e^2 exceeds the critical value

$$H_e^2 = 1/(1+f) \Rightarrow \frac{\epsilon_a \epsilon_b \mathcal{E}^2}{\sigma \eta} = 1 \tag{14}$$

in which case Ω is given by

$$\Omega = \sqrt{(1+f)H_e^2 - 1} \tag{15}$$

Prob. 5.15.1 From Eq. 8 of the solution to Prob. 5.13.8, the temporal modes are found by setting the denominator equal to zero. Thus,

$$m^2 + j(\omega - m\Omega)\frac{\epsilon_o R^2}{\sigma_s}\left[f_m(a,R) - f_m(b,R)\right] = 0 \tag{1}$$

Solution for ω then gives

$$\omega = m\Omega + \frac{j\sigma_s}{\epsilon_o R^2 m^2 [f_m(a,R) - f_m(b,R)]} \tag{2}$$

where $f_m(a,R) > 0$ and $f_m(b,R) < 0$ so that the imaginary part of ω represents decay.

Prob. 5.15.2 The temporal modes follow from the equation obtained by setting the denominator of Eq. 7 from the solution to Prob. 5.14.1 equal to zero.

$$\sigma_a f_m(a,R) - \sigma_b f_m(0,R) + j(\omega - \Omega m)\left[\epsilon_a f_m(a,R) - \epsilon_b f_m(0,R)\right] = 0 \tag{1}$$

Solved for ω, this gives the desired eigenfrequencies.

$$\omega = \Omega m + j\frac{\sigma_a f_m(a,R) - \sigma_b f_m(0,R)}{\epsilon_a f_m(a,R) - \epsilon_b f_m(0,R)} \tag{2}$$

Note that $f_m(a,R) > 0$ while $f_m(0,R) < 0$, so the frequencies represent decay.

Prob. 5.15.3 The conservation of charge boundary condition takes the form

$$\nabla_\Sigma \cdot \bar{K} + \frac{\partial \sigma_f}{\partial t} = 0 \tag{1}$$

where the surface current density is

$$\bar{K} = \bar{i}_\theta(\sigma_s E_\theta^a) + \bar{i}_\phi(\sigma_s E_\phi + \sigma_f \Omega R \sin\theta) \tag{2}$$

Using Eq. (2) to evaluate Eq. (1) and writing $\bar{E}$ in terms of the potential, Φ, the conservation of charge boundary condition becomes

$$\frac{1}{R}\frac{\partial}{\partial\theta}(\sigma_s E_\theta^a \sin\theta) + \frac{\sigma_s}{R}\frac{\partial E_\phi}{\partial\phi} + \left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\phi}\right)(\sigma_f \sin\theta) = 0 \tag{3}$$

With the substitution of the solutions to Laplace's equation in spherical coordinates

$$\Phi = Re\, \hat{\Phi}(r) P_n^m(\cos\theta)\, e^{-jm\phi} e^{j\omega t} \tag{4}$$

the boundary condition stipulates that

$$\frac{-\sigma_s \sin\theta}{R^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\hat{\Phi}^a P_n^m\right) - \frac{m^2\hat{\Phi}^a P_n^m}{\sin^2\theta}\right] + j(\omega - m\Omega)\sin\theta\, \hat{\sigma}_f^a P_n^m = 0 \tag{5}$$

By definition, the operator in square brackets is

$$-n(n+1)\hat{\Phi}^a P_n^m \tag{6}$$

and so the boundary condition becomes simply

$$\frac{\sigma_s}{R^2}\hat{\Phi}^a n(n+1) + j(\omega - m\Omega)\hat{\sigma}_f^a = 0 \tag{7}$$

In addition, the potential is continuous at the boundary r = R.

$$\hat{\Phi}^a = \hat{\Phi}^b \tag{8}$$

Transfer relations representing the fields in the volume regions are Eqs. 4.8.18 and 4.8.19. For the outside region $\beta \rightarrow$(a) while for the inside region, $\alpha \rightarrow$ (b). Thus, Eq. (7), which can also be written as

Prob. 5.15.3 (cont.)

$$\frac{\sigma_s}{R^2} n(n+1)\hat{\Phi}^a + j(\omega - m\Omega)(\hat{D}_r^a - \hat{D}_r^b) = 0 \tag{9}$$

becomes, with substitution for $\hat{D}_r^a$ and $\hat{D}_r^b$, and use of Eq. (8),

$$\frac{\sigma_s}{R^2} n(n+1)\hat{\Phi}^a + j(\omega - m\Omega)\left[\frac{\epsilon_a(n+1)}{R}\hat{\Phi}^a + \frac{\epsilon_b n}{R}\hat{\Phi}^a\right] = 0 \tag{10}$$

This expression is homogeneous in the amplitude $\hat{\Phi}^a$, (there is no drive) and it follows that the natural modes satisfy the dispersion equation

$$\omega = m\Omega + j\frac{\sigma_s n(n+1)}{R[\epsilon_b n + \epsilon_a(n+1)]} \tag{11}$$

where (n,m) are the integer mode numbers in spherical coordinates.

In a uniform electric field, surface charge on the spherical surface would assume the same distribution as on a perfectly conducting sphere.... a cos θ distribution. Hence, the associated mode which describes the build up or decay of this distribution is n = 1, m = 0. The time constant for charging or discharging a particle where the conduction is primarily on the surface is therefore

$$\tau = R(2\epsilon_a + \epsilon_b)/2\sigma_s \tag{12}$$

Prob. 5.15.4 The desired modes of charge relaxation are the homogeneous response. This can be found by considering the system without excitations. Thus, for the exterior region,

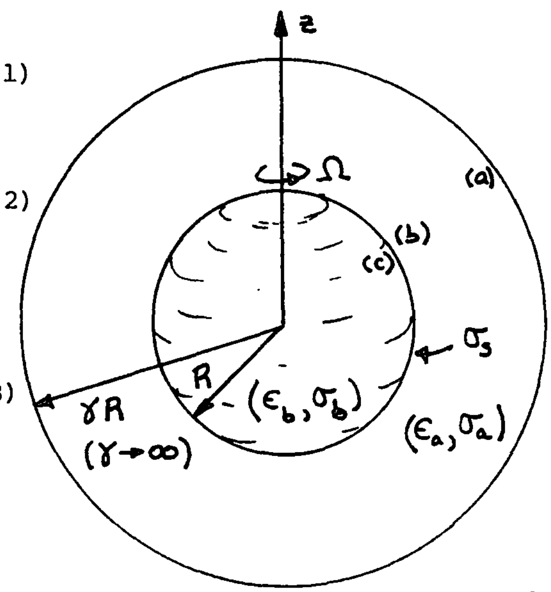
$$\hat{D}_r^b = \epsilon_a f_n(\infty, R)\hat{\Phi}^b = \epsilon_a \frac{(n+1)}{R}\hat{\Phi}^b \quad (1)$$

while for the interior region,

$$\hat{D}_r^c = \epsilon_b f_n(0,R)\hat{\Phi}^c = -\frac{\epsilon_b n}{R}\hat{\Phi}^c \quad (2)$$

At the interface, the potential must be continuous, so

$$\hat{\Phi}^c = \hat{\Phi}^b \quad (3)$$



The second boundary condition combines conservation of charge and Gauss' law. To express this in terms of complex amplitudes, first observe that charge conservation requires that the accumulation of surface charge either is the result of a net divergence of surface current in the region of surface conduction, or results from a difference of conduction current from the volume regions.

$$\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\phi}\right)\sigma_f = -\nabla_\Sigma \cdot \bar{K}_f' - \bar{n}\cdot[\![\bar{J}_f']\!] = \sigma_s \nabla_\Sigma^2 \Phi^b - \left(\frac{\sigma_a}{\epsilon_a}D_r^b - \frac{\sigma_b}{\epsilon_b}D_r^c\right) \quad (4)$$

where

$$\nabla_\Sigma^2 = \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

For solutions having the complex amplitude form in spherical coordinates,

$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) = -n(n+1) \quad (5)$$

so, with the use of Gauss' law, Eq. 4 becomes

$$j(\omega - m\Omega)(\hat{D}_r^b - \hat{D}_r^c) = \frac{-n(n+1)\sigma_s}{R^2}\hat{\Phi}^b + \left(\frac{\sigma_b}{\epsilon_b}\hat{D}_r^c - \frac{\sigma_a}{\epsilon_a}\hat{D}_r^b\right) \quad (6)$$

Substitution of Eqs. 1-3 into this expression gives an equation that is homogeneous in $\hat{\Phi}^b$. The coefficient of $\hat{\Phi}^b$ must therefore vanish. Solved for $j\omega$, the resulting expression is

$$j\omega = j\Omega m - \left[\frac{n(n+1)\sigma_s}{R} + \sigma_a(n+1) + \sigma_b n\right] \Big/ \left[\epsilon_a(n+1) + \epsilon_b n\right] \quad (7)$$

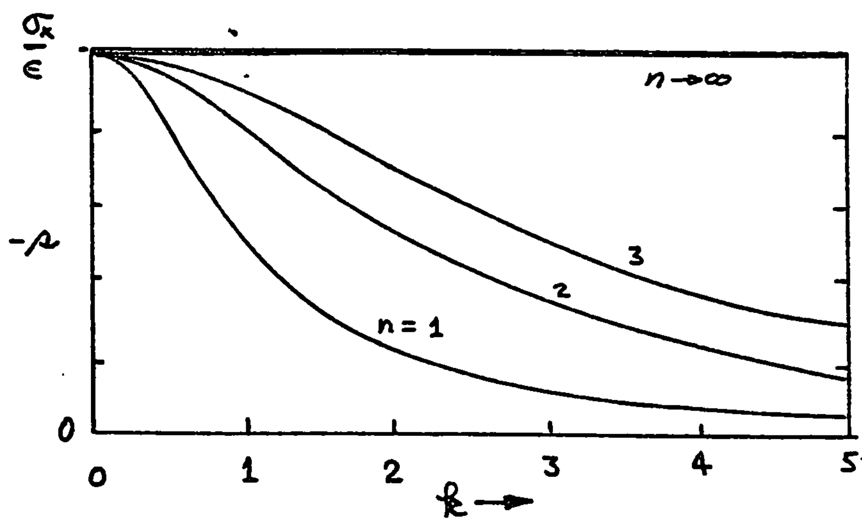
Prob. 5.15.5 (a) With the potentials in the transfer relations of Prob. 5.12.1 constrained to zero, the response cannot be finite unless the determinant of the coefficients is infinite. This condition is met if $\sinh \gamma\Delta = 0$. Roots to this expression are $\gamma\Delta = jn\pi$, n = 1, 2, ... and it follows that the required eigenfrequency equation is the expression for γ^2 with $\gamma^2 = -(n\pi/\Delta)^2$.

$$s \equiv j\omega = -\frac{\left[\sigma_x\left(\frac{n\pi}{\Delta}\right)^2 + \sigma_y k_y^2 + \sigma_z k_z^2\right]}{\epsilon\left[k^2 + \left(\frac{n\pi}{\Delta}\right)^2\right]}\ ; \quad k^2 = k_y^2 + k_z^2 \tag{1}$$

(b) Note that if $\sigma_x = \sigma_y = \sigma_z \equiv \sigma$, this expression reduces to $-\sigma/\epsilon$ regardless of n. The discrete modes degenerate into a continuum of modes representing the charge relaxation process in a uniform conductor. (c) For $\sigma_y \to 0$ and $\sigma_z \to 0$, Eq. 1 reduces to

$$s = -\frac{\sigma_x}{\epsilon}\left(\frac{n\pi}{\Delta}\right)^2 \Big/ \left[k^2 + \left(\frac{n\pi}{\Delta}\right)^2\right] \tag{2}$$

Thus, the eigenfrequencies as shown in Fig. P5.15.5a depend on k with the mode number as a parameter.

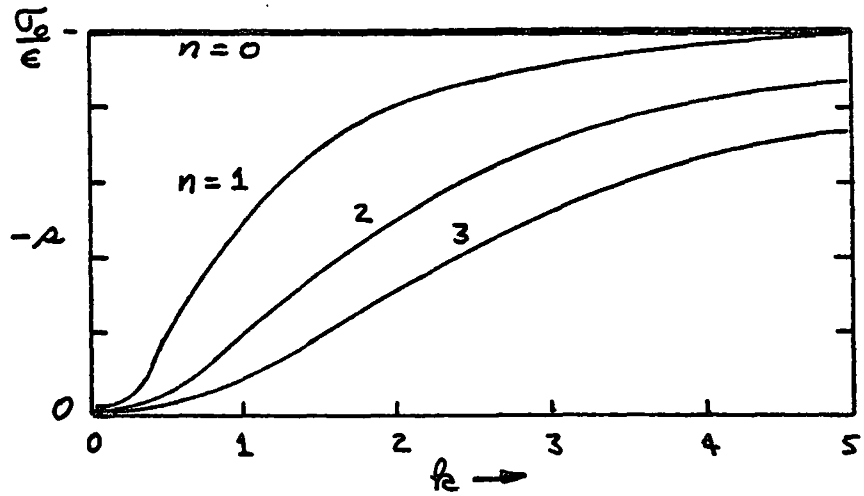


Prob. 5.15.5(cont.)

(d) With $\sigma_x=0, \sigma_y=\sigma_z=\sigma_o$, Eq. 1 reduces to

$$s=-\frac{\sigma_o}{\epsilon}k^2\Big/\left[k^2+\left(\frac{n\pi}{\Delta}\right)^2\right] \qquad (3)$$

and the eigenfrequencies depend on k as shown in Fig. P5.15.5b.



Prob. 5.17.1 In the upper region, solutions to Laplace's equation take the form

$$\hat{\Phi}_n = \hat{\Phi}^a \frac{\sinh k_n x}{\sinh k_n d} - \hat{\Phi}^b \frac{\sinh k_n(x-d)}{\sinh k_n d} \tag{1}$$

It follows from this fact alone and Eqs. 5.17.17-5.17.19 that in region I, where $\hat{\Phi}^a = 0$

$$\Phi = -Re\, \frac{j\hat{V}_o d\epsilon}{\sigma_s} \sum_{n=-1}^{-\infty} \frac{(\omega - k_n U)[e^{j(k_n-\beta)\ell} - 1]e^{j(\omega t - k_n z)}}{(k_n-\beta)D'(\omega,k_n)} \frac{\sinh k_n(x-d)}{\sinh k_n d} \tag{2}$$

Similarly, in region II, where $\hat{\Phi}^a = \hat{V}_o$

$$\begin{aligned} \Phi = -Re\, \frac{j\hat{V}_o d\epsilon}{\sigma_s} \Bigg\{ &\sum_{n=-1}^{-\infty} \frac{(\omega - k_n U) e^{j(k_n-\beta)\ell - jk_n z} \sinh k_n(x-d)}{(k_n-\beta)D'(\omega,k_n) \sinh k_n d} \\ &+ \frac{(\omega - \beta U) e^{-j\beta z} \sinh \beta(x-d)}{D(\omega,\beta) \sinh \beta d} \\ &+ \sum_{n=1}^{\infty} \frac{(\omega - k_n U) e^{-jk_n z} \sinh k_n(x-d)}{(k_n-\beta)D'(\omega,k_n)} \Bigg\} e^{j\omega t} + Re\, \hat{V}_o e^{j(\omega t - \beta z)} \frac{\sinh \beta x}{\sinh \beta d} \end{aligned} \tag{3}$$

and in region III, where $\hat{\Phi}^a = 0$

$$\Phi = Re\, \frac{j\hat{V}_o d\epsilon}{\sigma_s} \Bigg\{ \sum_{n=1}^{\infty} \frac{(\omega - k_n U)[e^{j(k_n-\beta)\ell} - 1] e^{j(\omega t - k_n z)}}{(k_n-\beta)D'(\omega,k_n)} \frac{\sinh k_n(x-d)}{\sinh k_n d} \tag{4}$$

In the lower region, $\hat{\Phi}^d = 0$ throughout, so

$$\hat{\Phi}_n = \hat{\Phi}^b \frac{\sinh k_n(x+d)}{\sinh k_n d} \tag{5}$$

and in region I

$$\Phi = Re\, \frac{j\hat{V}_o d\epsilon}{\sigma_s} \sum_{n=-1}^{-\infty} \frac{(\omega - k_n U)[e^{j(k_n-\beta)\ell} - 1] e^{j(\omega t - k_n z)}}{(k_n-\beta)D'(\omega,k_n)} \frac{\sinh k_n(x+d)}{\sinh k_n d} \tag{6}$$

in region II

Prob. 5.17.1 (cont.)

$$\Phi = Re\, j\frac{\hat{V}_o d\epsilon}{\sigma_s}\Bigg\{\sum_{n=-1}^{-\infty}\frac{(\omega-k_n U)[e^{j(k_n-\beta)\ell}]e^{-jk_n z}}{(k_n-\beta)D'(\omega,k_n)}\frac{\sinh k_n(x+d)}{\sinh k_n d}$$

$$+\frac{(\omega-\beta U)e^{-j\beta z}}{D(\omega,\beta)}\frac{\sinh\beta(x+d)}{\sinh\beta d} \tag{7}$$

$$+\sum_{n=1}^{\infty}\frac{(\omega-k_n U)e^{-jk_n z}}{(k_n-\beta)D'(\omega,k_n)}\frac{\sinh k_n(x+d)}{\sinh k_n d}\Bigg\}e^{j\omega t}$$

and in region III

$$\Phi = -Re\, j\frac{\hat{V}_o d\epsilon}{\sigma_s}\Bigg\{\sum_{n=1}^{\infty}\frac{(\omega-k_n U)[e^{j(k_n-\beta)\ell}-1]e^{-jk_n z}}{(k_n-\beta)D'(\omega,k_n)}\Bigg\}e^{j\omega t} \tag{8}$$

Prob. 5.17.2 The relation between Fourier transforms has already been determined in Sec. 5.14, where the response to a single complex amplitude was found. Here, the single traveling wave on the (a) surface is replaced by

$$\Phi^a(z,t) = Re\left\{\hat{V}_o[u_{-1}(z)-u_{-1}(z-\ell)]e^{j(\omega t-\beta z)}\right\} = Re\,\hat{\Phi}(z)e^{j\omega t} \tag{1}$$

where

$$\hat{\Phi}^a = \hat{V}_o[u_{-1}(z)-u_{-1}(z-\ell)]e^{-j\beta z} \tag{2}$$

Thus, the Fourier transform of the driving potential is

$$\hat{\tilde{\Phi}}^a = \int_{-\infty}^{+\infty}\hat{\Phi}e^{jkz}dz = \int_0^{\ell}\hat{V}_o e^{-j(\beta-k)z}dz = \frac{\hat{V}_o[e^{j(k-\beta)\ell}-1]}{j(k-\beta)} \tag{3}$$

It follows that the transform of the potential in the (b) surface is given by Eq. 5.14.8 with $\hat{V}_o \to \hat{\tilde{\Phi}}$, and a=b=d.

$$\hat{\tilde{\Phi}}^b = \frac{1}{\cosh kd}\frac{\sigma_a}{\sigma_a+\sigma_b}\left[\frac{1+j(\omega-kU)\frac{\epsilon_a}{\sigma_a}}{1+j(\omega-kU)\frac{(\epsilon_a+\epsilon_b)}{\sigma_a+\sigma_b}}\right]\hat{\tilde{\Phi}}^a \tag{4}$$

where $\hat{\tilde{\Phi}}^a$ is given by Eqs. 1 and 2. The spatial distribution follows by taking the inverse Fourier transform.

Prob. 5.17.2(cont.)

$$\hat{\Phi}^b = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{\hat{\Phi}}^b e^{-jkz}\, dk \tag{5}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\hat{V}_o\left[\epsilon_a(\omega - kU) - j\sigma_a\right]\left[e^{j(\ell - z)k} e^{-j\beta\ell} - e^{-jkz}\right] dk}{(k-\beta)\, D(\omega,k)}$$

where

$$D(\omega,k) \equiv \cosh kd\left[(\sigma_a+\sigma_b) + j(\omega - kU)(\epsilon_a+\epsilon_b)\right]$$

Singularities of the integrand given by $D(\omega,k)=0$ are either

$$\cosh(kd) = 0 \Rightarrow jkd = \pm(|2n|-1)\pi/2 \Rightarrow k_n = \mp\frac{(|2n|-1)}{d}\frac{\pi}{2}j,\; n=\pm 1\cdots\pm\infty \tag{6}$$

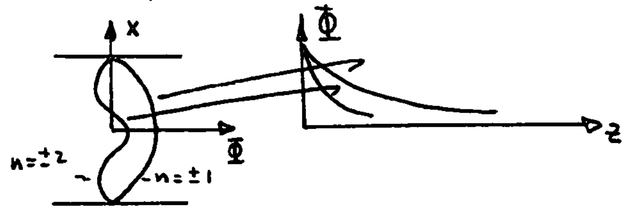
or

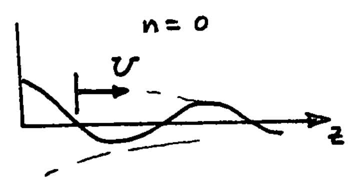
$$(\sigma_a+\sigma_b) + j(\omega - kU)(\epsilon_a+\epsilon_b) = 0 \Rightarrow k = \frac{\omega}{U} - j\frac{(\sigma_a+\sigma_b)}{(\epsilon_a+\epsilon_b)}\frac{1}{U} \equiv k_0 \tag{7}$$

With the transverse coordinate, x, taken as having its origin on the moving sheet, the distribution of potential is in general given by $(\hat{\hat{\Phi}}^a = 0)$

$$\hat{\hat{\Phi}} = \begin{cases} -\hat{\hat{\Phi}}^b \dfrac{\sinh k(x-d)}{\sinh kd} & ; x > 0 \\ \hat{\hat{\Phi}}^b \dfrac{\sinh k(x+d)}{\sinh kd} & ; x < 0 \end{cases} \tag{8}$$

Thus, the $n \neq 0$ modes, which are either purelly growing or decaying with an exponential dependence in the longitudinal direction, have the sinusoidal transverse dependence sketched. Note that these are the modes expected from Laplace's equation in the absence of a sheet. They have no derivative in the x direction at the sheet surface, and therefore represent modes with no net surface charge on the sheet.





These modes, which are uncoupled from the sheet, are possible because of the symmetry of the configuration obtained by making a=b. The n=0 mode is the only one involving the charge relaxation on the sheet. Because the wavenumber is complex, the transverse dependence is neither purelly exponential or sinusoidal. In fact, the transverse dependence can no longer be represented by a single amplitude, since all positions in a given z plane do not have the

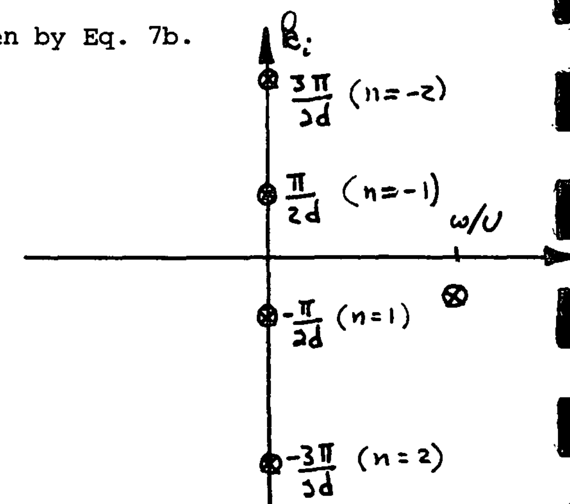
Prob. 5.17.2(cont.)

same phase. By using the identity $\sinh(u+jv) = \sinh u \cos v + j\cosh u \sin v$, the magnitude of the transverse dependence in the upper region given by Eq. 8 can be shown to be

$$\left|\frac{\sinh k(x-d)}{\sinh kd}\right| = \sqrt{\frac{\sinh^2 k_r(x-d)\cos^2 k_i(x-d) + \cosh^2 k_r(x-d)\sin^2 k_i(x-d)}{\sinh^2 k_r d \cos^2 k_i d + \cosh^2 k_r d \sin^2 k_i d}} \quad (9)$$

where the real and imaginary parts of k are given by Eq. 7b.

In the complex k plane, the poles of Eq. 5 are as shown in the sketch. Note that $k=\beta$ is not a singular point because the numerator contains a zero also at $k=\beta$. In using the Residue theorem, the contour is closed in the upper half plane for $z < 0$ and in the lower half for $z > \ell$. For the intermediate region, II, the term multiplying $\exp jk(\ell - z)$ must be closed from above while that multiplying $\exp -jkz$ is closed from below. Thus, in region I, $z < 0$,



$$\Phi^b = \mathrm{Re}\, e^{j\omega t}\hat{V}_o \sum_{n=-1}^{-\infty} \frac{[j(\omega - k_n U)\epsilon_a + \sigma_a][e^{j(\ell - z)k_n} e^{-j\beta\ell} - e^{-jk_n z}]}{-j(k_n - \beta)(-1)^n d[(\sigma_a+\sigma_b) + j(\omega - k_n U)(\epsilon_a + \epsilon_b)]} \quad (10)$$

in region II, the integral is split as described and the "pole" at $k=\beta$ is now actually a singularity, and hence makes a contribution. $0 < z < \ell$

$$\Phi^b = \mathrm{Re}\,\hat{V}_o e^{j\omega t}\left\{ \sum_{n=-1}^{-\infty} \frac{[j(\omega - k_n U)\epsilon_a + \sigma_a]\, e^{j(\ell - z)k_n} e^{-j\beta\ell}}{j(-1)^{n+1} d(k_n - \beta)[(\sigma_a+\sigma_b) + j(\omega - k_n U)(\epsilon_a+\epsilon_b)]} \right.$$

$$+ \frac{[\sigma_b\epsilon_a - \sigma_a\epsilon_b]\, e^{-jk_o z}}{\cosh k_o d\,[U(\epsilon_a+\epsilon_b)^2](k_o - \beta)} \quad (11)$$

$$\left. + \sum_{n=1}^{\infty} \frac{[j(\omega - k_n U)\epsilon_a + \sigma_a]\, e^{-jk_n z}}{j(k_n - \beta)(-1)^n d[(\sigma_a+\sigma_b) + j(\omega - k_n U)(\epsilon_a+\epsilon_b)]} + \frac{[j\epsilon_a(\omega - \beta U) + \sigma_a]\, e^{-j\beta z}}{D(\omega, \beta)} \right\}$$

Finally in region III, $z > \ell$,

Prob. 5.17.2(cont.)

$$\hat{\Phi}^b = -\mathcal{R}_e \hat{V}_o e^{j\omega t}\left\{\sum_{n=1}^{\infty} \frac{[j(\omega - k_n U)\epsilon_a + \sigma_a][e^{j(\ell - z)k_n} e^{-j\beta\ell} - e^{-jk_n z}]}{j(k_n - \beta)(-1)^n d[(\sigma_a + \sigma_b) + j(\omega - k_n U)(\epsilon_a + \epsilon_b)]}\right. \tag{12}$$

$$\left. - j\left[\frac{\epsilon_a \sigma_b - \epsilon_b \sigma_a}{\epsilon_a + \epsilon_b}\right] \frac{e^{j(\ell - z)k_o} e^{-j\beta\ell} - e^{-jk_o z}}{\cosh k_o d \; U(\epsilon_a + \epsilon_b)(k_o - \beta)}\right\}$$

The total force follows from an evaluation of

$$f = \frac{1}{4\pi}\mathcal{R}_e \int_{-\infty}^{+\infty} \hat{E}_z^b [\hat{D}_x^b - \hat{D}_x^c]^* dk = \frac{\mathcal{R}_e}{4\pi}\int_{-\infty}^{+\infty} jk\hat{\Phi}^b[\hat{D}_x^b - \hat{D}_x^c]^* dk \tag{13}$$

Use of Eqs. 5.14.8 and 5.14.9 for $\hat{\Phi}^b$ and $[\![\hat{D}_x]\!]$ results in

$$f = -\frac{\mathcal{R}_e}{4\pi}\int_{-\infty}^{+\infty} \frac{jk^2[j(\omega - kU)\epsilon_a + \sigma_a]\hat{\Phi}^a \hat{\Phi}^{a*}(\epsilon_a\sigma_b - \epsilon_b\sigma_a)\,dk}{\sinh kd \cosh kd[(\sigma_a + \sigma_b)^2 + (\omega - kU)^2(\epsilon_a + \epsilon_b)^2]} \tag{14}$$

The real part is therefore simply

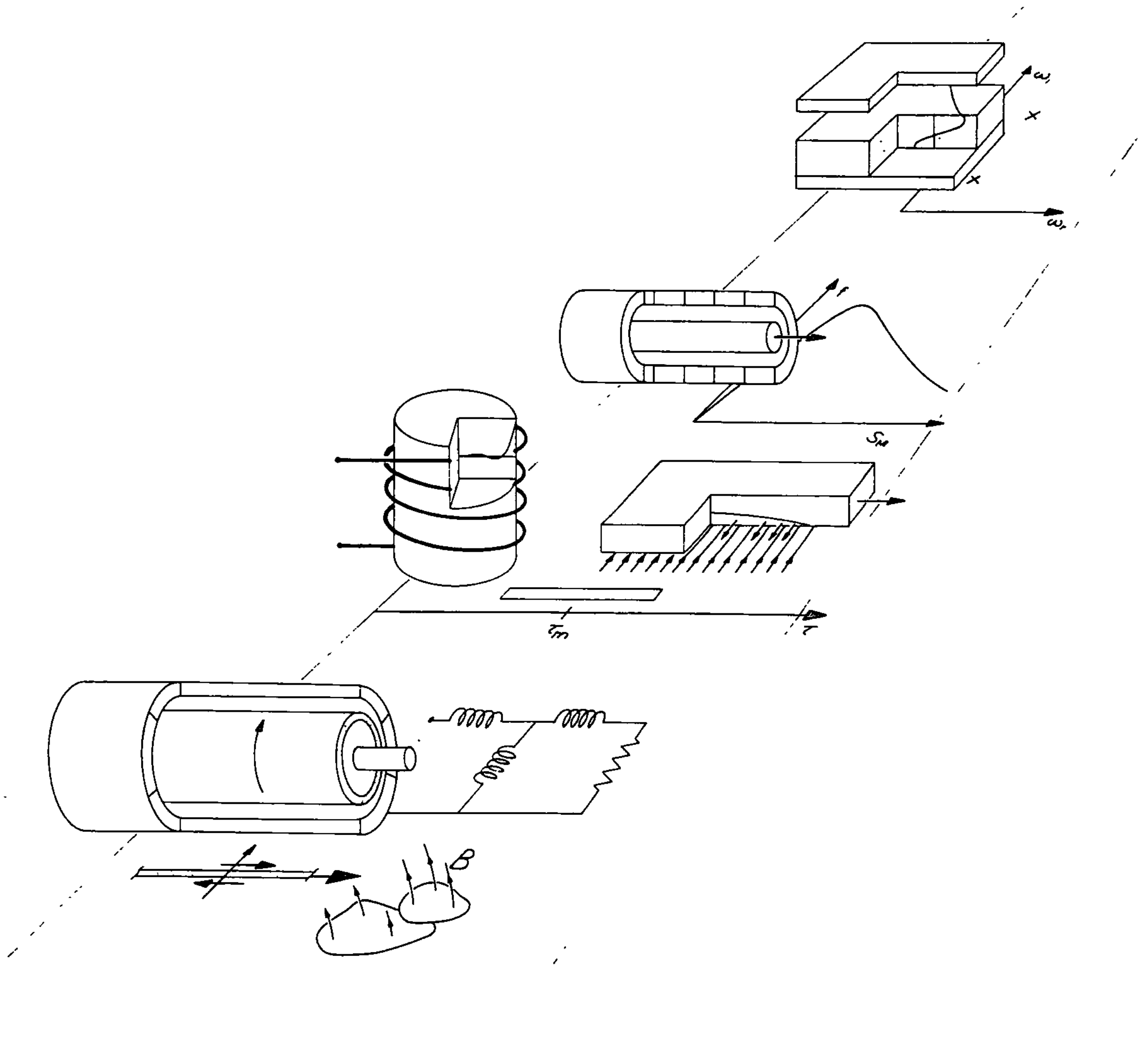
$$f = \frac{\epsilon_a\sigma_b - \epsilon_b\sigma_a}{4\pi}\int_{-\infty}^{+\infty} \frac{k^2(\omega - kU)\epsilon_a \hat{\Phi}^a \hat{\Phi}^{a*}\,dk}{\sinh kd \cosh kd[(\sigma_a + \sigma_b)^2 + (\omega - kU)^2(\epsilon_a + \epsilon_b)^2]} \tag{15}$$

where the square of the driving amplitudes follows from Eq. 3.

$$\hat{\Phi}^a \hat{\Phi}^{a*} = \frac{4|\hat{V}_o|^2 \sin^2\left[\frac{(k - \beta)\ell}{2}\right]}{(k - \beta)^2} \tag{16}$$

6

Magnetic Diffusion and Induction Interactions



Prob. 6.2.1 (a) The zero order fields follow from current continuity and Ampere's law,

$$\bar{J} = J_o \bar{i}_x \ ; \ \bar{E} = (J_o/\sigma)\bar{i}_x \tag{1}$$

$$-\frac{\partial H_y}{\partial z} = J_o \Rightarrow H_y = -\frac{i}{d\ell} z \tag{2}$$

where d is the length in the y direction.

Thus, the magnetic energy storage is

$$\frac{1}{2} L i^2 = \frac{da}{2}\mu \int_{-\ell}^{0} H_y^2 dz = \frac{1}{2}\left(\frac{\mu a \ell}{3d}\right) i^2 \tag{3}$$

from which it follows that the inductance is $L = \mu a \ell/3d$.

With this zero order H_y substituted on the right in Eq. 7, it follows that

$$\frac{\partial^2 H_{y1}}{\partial z^2} = -\frac{\mu\sigma}{d\ell} z \frac{di}{dt} \tag{4}$$

Two integrations bring in two integration functions, the second of which is zero because H_y=0 at z=0.

$$H_{y1} = \frac{-\mu\sigma z^3}{6d\ell}\frac{di}{dt} + f(t) z \tag{5}$$

So that the current at $z=-\ell$ on the plate at x=0 is i(t), the function f(t) is evaluated by making H_{y1}=0 there

$$f = \frac{\mu\sigma\ell}{6d}\frac{di}{dt} \tag{6}$$

Thus, the zero plus first order fields are

$$H_y = -\frac{i}{d\ell} z + \frac{di}{dt}\frac{\mu\sigma\ell}{6d}\left(z - \frac{z^3}{\ell^2}\right) \tag{7}$$

The current density implied by this follows from Ampere's law

$$J_x = -\frac{\partial H_y}{\partial z} = \frac{i}{d\ell} - \frac{\mu\sigma\ell}{6d}\left(1 - \frac{3z^2}{\ell^2}\right)\frac{di}{dt}$$

Finally, the voltage at the terminals is evaluated by recognizing from Ohm's law that $v = aE_z = aJ_x(-\ell)/\sigma$. Thus, Eq. 8 gives

$$v = Ri + L\frac{di}{dt} \tag{9}$$

Prob. 6.2.1 (cont.)

where $L = \mu l a/3d$ and $R = \sigma/\sigma d l$.

Prob. 6.3.1 For the cylindrical rotating shell, Eq. 6.3.2 becomes

$$\frac{1}{r}\left[\frac{\partial K_z}{\partial \theta} - \frac{\partial (K_\theta r)}{\partial z}\right] = -\sigma_s\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial \theta}\right)B_r \tag{1}$$

and Eq. 6.3.3 becomes

$$\frac{1}{r}\frac{\partial K_\theta}{\partial \theta} + \frac{\partial K_z}{\partial z} = 0 \tag{2}$$

The desired result involves $[\![H_\theta]\!]$, which in view of Ampere's law is K_z. So, between these two equations, K_θ is eliminated by operating on Eq. 1 with $r\partial(\;)/\partial\theta$ and adding to Eq. 2 operated on by $r^2\partial(\;)/\partial z$.

$$\left(r^2\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \theta^2}\right)K_z = -r\sigma_s\frac{\partial}{\partial \theta}\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial \theta}\right)B_r \tag{3}$$

Then, because $K_z = [\![H_\theta]\!]$, the desired result, Eq. (b) of Table 6.3.1, is obtained.

Prob. 6.3.2 Equation 6.3.2 becomes

$$(\nabla\times\bar{K}_f)_r = -\sigma_s\frac{\partial B_r}{\partial t} + \sigma_s[\nabla\times(\bar{v}\times\bar{B})]_r \tag{1}$$

or, in cylindrical coordinates

$$\frac{1}{a}\frac{\partial K_z}{\partial \theta} - \frac{\partial K_\theta}{\partial z} = -\sigma_s\left(\frac{\partial B_r}{\partial t} + U\frac{\partial B_r}{\partial z}\right) \tag{2}$$

Equation 6.3.3 is

$$\nabla_\Sigma\cdot\bar{K}_f = \frac{1}{a}\frac{\partial K_\theta}{\partial \theta} + \frac{\partial K_z}{\partial z} = 0 \tag{3}$$

while Eq. 6.3.4 requires that

$$[\![H_\theta]\!] = K_z\;;\; -[\![H_z]\!] = K_\theta \tag{4}$$

The $\partial/\partial z$ of Eq. 2 and $\partial/\partial\theta$ of Eq. 3 then combine (to eliminate $\partial^2 K_z/\partial\theta\partial z$) to give

$$-\left(\frac{1}{a^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right)K_\theta = \sigma_s\frac{\partial}{\partial z}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial z}\right)B_r \tag{5}$$

Substitution for K_θ from Eq. 4b then gives Eq. c of Table 6.3.1.

Prob. 6.3.3 Interest is in the radial component of Eq. (2) evaluated at r= d .

$$\frac{1}{d^2 \sin\theta}\left[\frac{\partial(K_\phi d \sin\theta)}{\partial\theta} - \frac{\partial(K_\theta d)}{\partial\phi}\right] = -\sigma_s\left(\frac{\partial B_r}{\partial t} + \Omega \frac{d \sin\theta}{d \sin\theta}\frac{\partial B_r}{\partial\phi}\right) \quad (1)$$

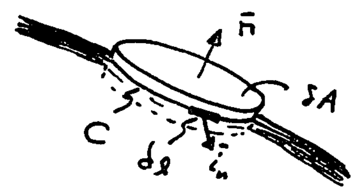
In spherical coordinates, Eq. 3 becomes

$$\frac{1}{d^2 \sin\theta}\left[\frac{\partial}{\partial\theta}(K_\theta d \sin\theta) + \frac{\partial}{\partial\phi}(K_\phi d)\right] = 0$$

To eliminate K_ϕ , multiply Eq. 2 by $\frac{\partial}{\partial\theta}\sin\theta$ and subtract Eq. 1 operated on by $\partial/\partial\phi$. Because Eq. 4 shows that $[\![H_\phi]\!] = -K_\theta$, Eq. (d) of Table 6.3.1 follows. To obtain Eq. (e) of Table 6.3.1, operate on Eq. (1) with $d\frac{\partial}{\partial\theta}(\sin^2\theta)$, on Eq. (2) with $\frac{\partial}{\partial\phi}(d \sin\theta)$ and add the latter to the former. Then use Eq. (4) to replace K_ϕ with $[\![H_\theta]\!]$.

Prob. 6.3.4 Gauss' law for $\bar{B}$ in integral form is applied to a pill-box enclosing a section of the sheet. The box has the thickness Δ of the sheet and an incremental area δA in the plane of the sheet. With C defined as a contour following the intersection of the sheet and the box, the integral law requires that

$$\Delta\mu \oint_C \bar{H}\cdot\bar{i}_n dl + \delta A [\![B_n]\!] = 0 \quad (1)$$



The surface divergence is defined as

$$\nabla_\Sigma \cdot H \equiv \lim_{\delta A \to 0} \frac{1}{\delta A}\oint_C \bar{H}\cdot\bar{i}_n dl \quad (2)$$

Under the assumption that the tangential field intensity is continuous through the sheet, Eq. 1 therefore becomes the required boundary condition.

$$\Delta\mu \nabla_\Sigma \cdot \bar{H} + [\![B_n]\!] = 0 \quad (3)$$

In cartesian coordinates and for a planar sheet, $\bar{H} = -\nabla\Psi$ and Eq. 3 becomes

$$-\Delta\mu\left[\frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2}\right] + [\![B_x]\!] = 0 \quad (4)$$

In terms of complex amplitudes, this is equivalent to

$$\Delta\mu k^2 \tilde{\Psi} + [\tilde{B}_x^a - \tilde{B}_x^b] = 0 \quad (5)$$

Prob. 6.3.4 (cont.)

From Table 2.16.1, the transfer relations for a layer of arbitrary thickness are

$$\begin{bmatrix} \tilde{B}_x^a \\ \tilde{B}_x^b \end{bmatrix} = \mu k \begin{bmatrix} -\coth k\Delta & \frac{1}{\sinh k\Delta} \\ -\frac{1}{\sinh k\Delta} & \coth k\Delta \end{bmatrix} \begin{bmatrix} \tilde{\psi}^a \\ \tilde{\psi}^b \end{bmatrix} \tag{6}$$

Subtraction of the second expression from the first gives

$$\tilde{B}_x^a - \tilde{B}_x^b = \mu k \left[\frac{1 - \cosh k\Delta}{\sinh k\Delta}\right](\tilde{\psi}^a + \tilde{\psi}^b) \tag{7}$$

In the long-wave limit, $\cosh k\Delta \to 1+(k\Delta)^2/2$ and $\sinh k\Delta \to k\Delta$ so this expression becomes

$$\tilde{B}_x^a - \tilde{B}_x^b = -\mu\Delta k^2 \frac{(\tilde{\psi}^a + \tilde{\psi}^b)}{2} \tag{8}$$

continuity of tangential $\bar{H}$ requires that $\tilde{\psi}^a \to \tilde{\psi}^b$, so that this expression agrees with Eq. 5.

<u>Prob. 6.3.5</u> The boundary condition reflecting the solenoidal nature of the flux density is determined as in Prob. 6.3.4 except that the integral over the sheet cross-section is not simply a multiplication by the thickness. Thus,

$$\mu \oint_C \left[\int_0^\Delta \bar{H}\cdot\bar{i}_n\,dx\right] d\ell + \delta A [\![B_n]\!] = 0 \tag{1}$$

is evaluated using $\bar{H}_t = \bar{H}_t^b + \frac{x}{\Delta}(\bar{H}_t^a - \bar{H}_t^b)$. To that end, observe that

$$\int_0^\Delta \bar{H}\cdot\bar{i}_n\,dx = \bar{H}_t^b\cdot\bar{i}_n\Delta + \frac{1}{2}\Delta(\bar{H}_t^a - \bar{H}_t^b)\cdot\bar{i}_n = \Delta\langle\bar{H}_t\rangle\cdot\bar{i}_n \tag{2}$$

so that Eq. 1 becomes

$$\frac{1}{\delta A}\oint_C \langle H_t\rangle\cdot\bar{i}_n\,d\ell + [\![B_n]\!] = 0 \tag{3}$$

In the limit this becomes the required boundary condition.

$$\mu\Delta\nabla_\Sigma\cdot\langle\bar{H}\rangle + [\![B_n]\!] = 0 \tag{4}$$

With the definition

$$\bar{K}_f \equiv \int_0^\Delta \bar{J}\,dx \tag{5}$$

and the assumption that contributions to the line integration of $\bar{H}$ through the sheet are negligible compared to those tangential, Ampere's law still requires

Prob. 6.3.5(cont.)

that

$$\bar{n} \times [\![\bar{H}]\!] = \bar{K}_f \tag{6}$$

The combination of Faraday's and Ohm's laws, Eq. 6.2.3, is integrated over the sheet cross-section.

$$\int_0^\Delta (\nabla \times \bar{J}_f)_n \, dx = \sigma \int_0^\Delta \left\{ -\frac{\partial B_n}{\partial t} + [\nabla \times (\bar{v} \times \bar{B})]_n \right\} dx \tag{7}$$

This reduces to

$$(\nabla \times \bar{K}_f)_n = -\sigma\Delta \left[\frac{\partial}{\partial t} + U \frac{\partial}{\partial y} \right] \langle B_n \rangle \tag{8}$$

where evaluation using the presumed constant plus linear dependence for B_n shows that

$$\int_0^\Delta B_n \, dx = \Delta \langle B_n \rangle \tag{9}$$

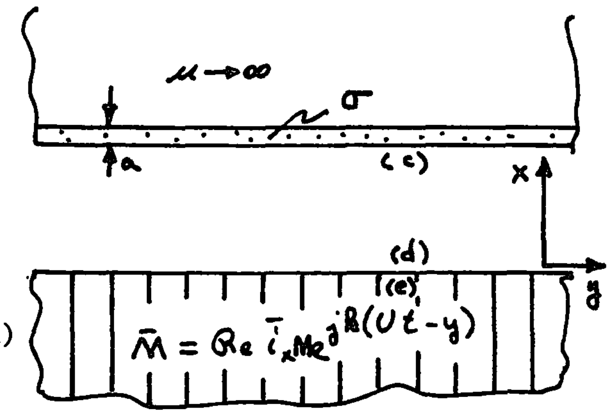
It is still true that

$$\nabla_\Sigma \cdot \bar{K}_f = 0 \tag{10}$$

To eliminate K_y, the y derivative of Eq. 9 is added to the z derivative of Eq. 10 and the z component of Eq. 6 is in turn used to replace K_z. Thus, the second boundary condition becomes

$$\left[\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] [\![H_y]\!] = -\sigma\Delta \frac{\partial}{\partial y} \left[\frac{\partial}{\partial t} + U \frac{\partial}{\partial y} \right] \langle B_x \rangle \tag{11}$$

Note that this is the same as given in Table 6.3.1 provided B_n is taken as the average.



Prob. 6.4.1 For solutions of the form $\exp j(\omega t - ky)$ where $\omega = kU$, let $\bar H = -\nabla\psi$. Then, boundary conditions begin with the conducting sheet

$$-\frac{\partial^2 H_y^c}{\partial y^2} = -\sigma_s \frac{\partial}{\partial y}\left(\frac{\partial}{\partial t}\right) B_x^c \tag{1}$$

or, in terms of complex amplitudes,

$$k^2 \hat H_y^c = -\sigma_s \omega k \hat H_x^c \Rightarrow \mu_o \hat H_x^c = -\frac{jk^2}{\sigma_s \omega} \hat\psi^c. \tag{2}$$

At this same boundary the normal flux density is continuous, but because the region above is infinitely permeable, this condition is implicite to Eq. 1. At the interface of the moving magnetized member,

$$n \times [\![\hat H]\!] = 0 \Rightarrow \hat\psi^d = \hat\psi^e \tag{3}$$

and

$$\bar n \cdot [\![\mu_o \bar H]\!] = -\bar n \cdot [\![\mu_o \bar M]\!] = Re\, \mu_o M e^{j(\omega t - ky)} \Rightarrow \hat H_x^d - \hat H_x^e = M \tag{4}$$

and because the lower region is an infinite half space, $\psi \to 0$ as $x \to -\infty$.

Bulk relations reflecting Laplace's equation in the air-gap are (from Table 2.16.1 with $B_x \to \mu_o H_x$)

$$\begin{bmatrix} \hat H_x^c \\ \hat H_x^d \end{bmatrix} = k \begin{bmatrix} -\coth kd & \dfrac{1}{\sinh kd} \\ \dfrac{-1}{\sinh kd} & \coth kd \end{bmatrix} \begin{bmatrix} \hat\psi^c \\ \hat\psi^d \end{bmatrix} \tag{5}$$

In the lower region, $\nabla \cdot \mu_o \bar M = 0$, so again $\nabla^2 \psi = 0$ and the transfer relation (which represents a solution of $\bar H = -\nabla\psi$ where $\nabla^2\psi = 0$ with $\mu \to \mu_o$ and hence $B_x \to \mu_o H_x$. Of course, in the actual problem, $B_x = \mu_o(H_x + M_x)$)is

$$\mu_o \hat H_x^e = -\mu_o k \hat\psi^e \tag{6}$$

Looking ahead, what is desired is

$$\langle T_y \rangle_t = -\frac{\mu_o}{2} Re\, \hat H_y^c \hat H_x^{c*} = -\frac{\mu_o k}{2} Re\, j \hat\psi^c \hat H_x^{c*} \tag{7}$$

From Eq. 2 (again with $\hat H_y^c = jk\hat\psi^c$)

$$\langle T_y \rangle_t = \frac{k}{2} Re\, \hat\psi^c \left(\frac{k^2}{\sigma_s \omega}\right) \hat\psi^{e*} \tag{8}$$

Prob. 6.4.1 (cont.)

To solve for $\hat{\psi}^c$, plug Eqs. 2 and 3 into Eq. 5a

$$\begin{bmatrix} \frac{jk^2}{\mu_o \sigma_s \omega} - k\coth kd & \frac{k}{\sinh kd} \\ \frac{-k}{\sinh kd} & k(1+\coth kd) \end{bmatrix} \begin{bmatrix} \hat{\psi}^c \\ \hat{\psi}^e \end{bmatrix} = \begin{bmatrix} 0 \\ M \end{bmatrix} \tag{9}$$

The second of these follows by using Eqs. 3,4 and 6 in Eq. 5b. Thus,

$$\hat{\psi}^c = \frac{Mk}{\sinh kd\left\{\left[\frac{k^2}{\sinh^2 kd} - k^2\coth kd(1+\coth kd)\right] + \frac{jk^2}{\mu_o\sigma_s\omega}(1+\coth kd)k\right\}} \tag{10}$$

Thus with $\underline{U} \equiv \mu_o \sigma_s U$, Eq. 8 becomes

$$\langle T_y \rangle_t = \frac{\mu_o M^2}{2\sinh^2 kd} \frac{\underline{U}}{\sqrt{\underline{U}^2\left[\frac{1}{\sinh^2 kd} - \coth kd(1+\coth kd)\right]^2 + (1+\coth kd)^2}} \tag{11}$$

To make $\langle T_y \rangle_t$ proportional to U, design the device to have

$$\underline{U}^2\left[\frac{1}{\sinh^2 kd} - \coth kd(1+\coth kd)\right]^2 << (1+\coth kd)^2 \tag{12}$$

In which case

$$\langle T_y \rangle_t = \frac{\mu_o M^2 (\mu_o \sigma_s U)}{2\sinh^2 kd(1+\coth kd)} \tag{13}$$

so that the force per unit area is proportional to the velocity of the rotor.

Prob. 6.4.2 For the circuit, loop equations are

$$\begin{bmatrix} j\omega(L_1+M) & -j\omega M \\ -j\omega M & j\omega(L_2+M)+\frac{R}{s_m} \end{bmatrix}\begin{bmatrix} \hat{i}_a \\ \hat{i}_b \end{bmatrix} = \begin{bmatrix} \hat{v}_a \\ 0 \end{bmatrix} \tag{1}$$

Thus,

$$\hat{i}_a = \frac{\hat{v}_a\left[j\omega\,(L_2+M)+\frac{R}{s_m}\right]}{j\omega\,(L_1+M)\left[j\omega\,(L_2+M)+\frac{R}{s_m}\right]+\omega^2M^2} \tag{2}$$

and written in the form of Eq. 6.4.17, this becomes

$$\hat{v}_a = \left\{j\omega\,(L_1+M) - j\omega s_m \frac{\left[j\omega M^2 R + \omega^2 M^2 (L_2+M) s_m\right]}{R^2\left[1+\frac{\omega^2(L_2+M)^2 s_m^2}{R^2}\right]}\right. \tag{3}$$

where comparison with Eq. 6.4.17 shows that

$$\frac{s_m}{R}\,\omega\,(L_2+M) = S_m \coth kd \tag{4}$$

$$L_1+M = \frac{w\ell N_a^2\mu_o}{2k}\coth kd \tag{5}$$

$$s_m\omega M^2/R = S_m\, w\ell N_a^2\mu_o/2k\sinh^2 kd \tag{6}$$

These three conditions do not uniquely specify the unknowns. But, add to them the condition that $L_1=L_2$ and it follows from Eq. 6 that

$$\frac{s_m}{R} = \frac{S_m}{\sinh^2 kd}\,\frac{w\ell^2N_a^2\mu_o}{4\pi\omega M^2} \tag{7}$$

so that Eq. 4 becomes an expression that can be solved for M

$$M = \frac{wN_a^2\mu_o\ell^2}{4\pi\sinh kd} \tag{8}$$

and Eq. 5 then gives

$$L_1 = L_2 = \frac{w\ell^2N_a^2\mu_o}{4\pi}\left[\coth kd - \frac{1}{\sinh kd}\right] = \frac{w\ell^2N_a^2\mu_o}{4\pi}\tanh\left(\frac{kd}{2}\right) \tag{9}$$

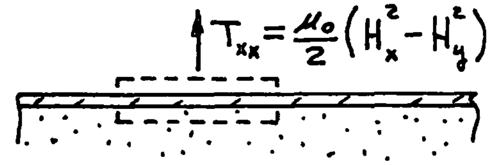
Finally, a return to Eq. 7 gives

$$\frac{s_m}{R} = \frac{S_m}{\omega}\,\frac{4\pi}{w\ell^2N_a^2\mu_o} \tag{10}$$

These parameters check with those from the figure.

Prob. 6.4.3 The force on the "stator" is the negative of that on the "rotor".

$$\langle f_x\rangle_t = -\frac{p\ell w}{2}\frac{\mu_o}{4}\,Re\Big\{\hat{H}^r_{x+}\hat{H}^{r*}_{x+} - \hat{H}^r_{y+}\hat{H}^{r*}_{y+} + \hat{H}^r_{x-}\hat{H}^{r*}_{x-} - \hat{H}^r_{y-}\hat{H}^{r*}_{y-}\Big\} \qquad (1)$$



In the following, the response is found for the ± waves separately, and then these are combined to evaluate Eq. 1. From Eq. 6.4.9,

$$\hat{H}^r_{x\pm} = \mp j\Big[\frac{\hat{K}^A_\pm}{\sinh kd} + \coth kd\,\hat{H}^r_{y\pm}\Big] \qquad (2)$$

So that

$$|\hat{H}^r_{x\pm}|^2 - |\hat{H}^r_{y\pm}|^2 = \Big\{\frac{|\hat{K}^A_\pm|}{\sinh^2 kd} + \frac{\coth kd}{\sinh kd}\big[\hat{K}^A_\pm\hat{H}^{r*}_{y\pm} + \hat{K}^{A*}_\pm\hat{H}^r_{y\pm}\big] + (\coth^2 kd - 1)|\hat{H}^r_{y\pm}|^2\Big\} \qquad (3)$$

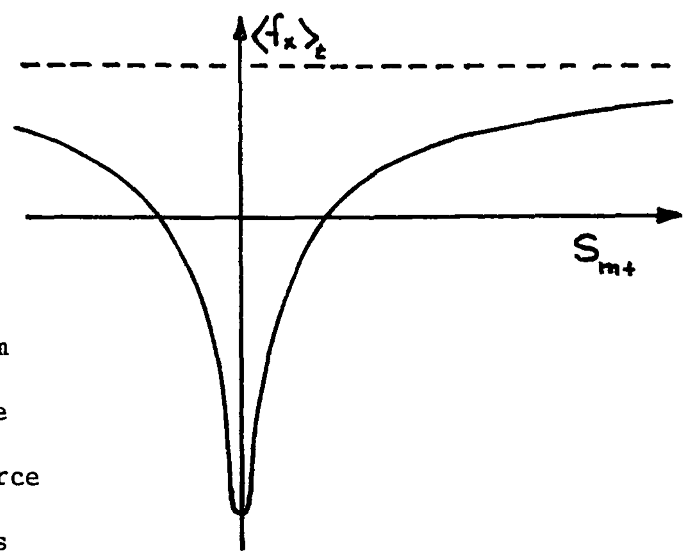
Now, use is made of Eq. 6.4.6 to write Eq. 3 as

$$|\hat{H}^r_{x\pm}|^2 - |\hat{H}^r_{y\pm}|^2 = \frac{|\hat{K}^A_\pm|^2}{\sinh^2 kd}\Big\{1 - \frac{S^2_{m\pm}}{1 + S^2_{m\pm}\coth^2 kd}\Big\} \qquad (4)$$

So, in general

$$\langle f_x\rangle_t = -\frac{p\ell w}{8}\frac{\mu_o}{\sinh^2 kd}\Big\{\frac{|\hat{K}^A_+|^2(1 - S^2_{m+})}{1 + S^2_{m+}\coth^2 kd} + \frac{|\hat{K}^A_-|^2(1 - S^2_{m-})}{1 + S^2_{m-}\coth^2 kd}\Big\} \qquad (5)$$

With two-phase excitation (a pure traveling wave) the second term does not contribute and the dependence of the normal force on S_m is as shown to the right. At low frequency (from the conductor frame of reference) the magnetization force prevails (the force is attractive). For high frequencies



Prob. 6.4.3 (cont.)

($|S_m| > 1$) the force is one of repulsion, as would be expected for a force associated with the induced currents.

With single phase excitation, the currents are as given by Eq. 6.4.18

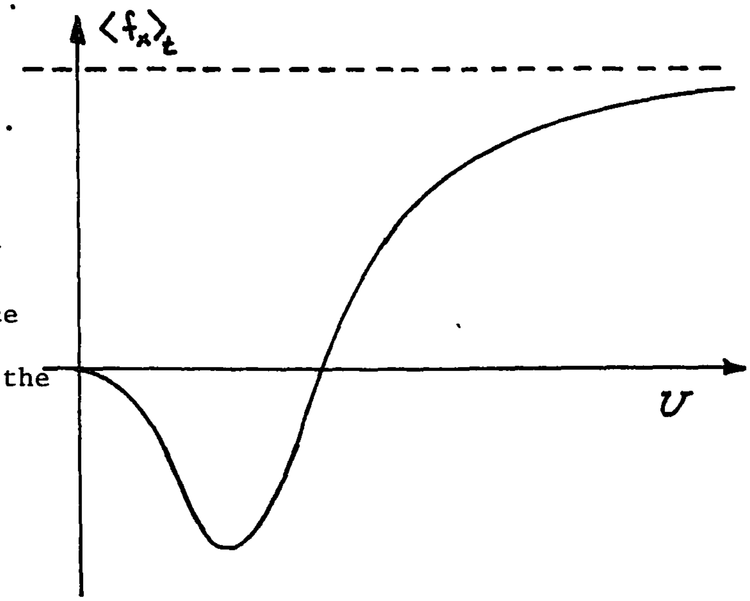
$$\hat{K}_+^a = \hat{K}_-^a = \frac{1}{2} N_a \hat{i}_a \tag{6}$$

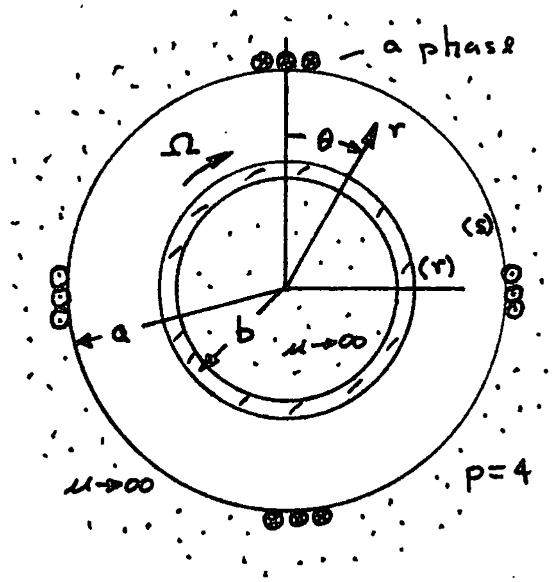
and Eq. 5 becomes

$$\langle f_x \rangle_t = \frac{p l w}{32} \frac{\mu_o N_a^2 |\hat{i}_a|}{\sinh^2 kd} \left\{ \frac{S_{m+}^2 - 1}{1 + S_{m+}^2 \coth^2 kd} + \frac{S_{m-}^2 - 1}{1 + S_{m-}^2 \coth^2 kd} \right\} \tag{7}$$

where $S_{m\pm} = \mu_o \sigma_s (\omega \mp kU)/k$.

The dependence of the force on the speed is illustrated by the figure. Making the velocity large is equivalent to making the frequency high, so at high velocity the force tends to be one of repulsion. In the neighborhood of the synchronous condition there is little induced current and the force is one of attraction.





Prob. 6.4.4 Two-phase stator currents are represented by

$$K_z^s = \mathrm{Re}\,\hat{i}_a e^{j\omega t} N_a \cos\left(\frac{\theta p}{2}\right) + \hat{i}_b e^{j\omega t} N_b \cos\left[\left(\frac{p\theta}{2}\right) - \frac{\pi}{2}\right] \tag{1}$$

and this expression can be written in terms of complex amplitudes as

$$K_z^s = \mathrm{Re}\left[\hat{K}_+^s e^{j(\omega t - m\theta)} + \hat{K}_-^s e^{j(\omega t + m\theta)}\right] \tag{2}$$

where

$$\hat{K}_\pm^s = \frac{1}{2}\left(\hat{i}_a N_a + \hat{i}_b N_b e^{\pm j\frac{\pi}{2}}\right)$$

Boundary conditions are written using designations shown in the figure. At the stator surface,

$$\hat{H}_\theta^s = -\hat{K}^s \tag{3}$$

while at the rotor surface (Eq. b, Table 6.3.1)

$$\frac{m^2}{b^2}\hat{H}_\theta^r = \frac{\sigma_s m}{b}(\omega - m\Omega)\hat{B}_r^r \Rightarrow \hat{H}_\theta^r = \sigma_s(\omega - m\Omega)(-j\hat{A}^r) \tag{4}$$

In the gap, the vector potential is used to make calculation of the terminal relations more convenient. Thus, Eq. d of Table 2.19.1 is

$$\begin{bmatrix}\hat{A}^s\\ \hat{A}^r\end{bmatrix} = \mu_o \begin{bmatrix} F_m(b,a) & G_m(a,b)\\ G_m(b,a) & F_m(a,b)\end{bmatrix}\begin{bmatrix}\hat{H}_\theta^s\\ \hat{H}_\theta^r\end{bmatrix} \tag{5}$$

To determine $\hat{H}_\theta^r$, write Eq. 5b using Eq. 3 for $\hat{H}_\theta^s$ and Eq. 4 for $\hat{A}^r$.

$$\frac{\hat{H}_\theta^r}{-j\sigma_s(\omega - m\Omega)} = -\mu_o G_m(b,a)\hat{K}^s + \mu_o F_m(a,b)\hat{H}_\theta^r \tag{6}$$

This expression is solved and rationalized to give

$$\hat{H}_{\theta\pm}^r = \frac{\hat{K}_\pm^s G_m(b,a)\mu_o\sigma_s(\omega \mp m\Omega)[j + F_m(a,b)\mu_o\sigma_s(\omega \mp m\Omega)]}{1 + F_m^2(a,b)[\mu_o\sigma_s(\omega \mp m\Omega)]^2} \tag{7}$$

Here, $\hat{H}_{\theta r}$ is written by replacing $m \to -m$ and recognizing that F_m and G_m are even in m.

Prob. 6.4.4 (cont.)

The torque is

$$\langle \tau \rangle_t = 2\pi b^2 w \frac{1}{2} \mathcal{R}e\left[\hat{B}^r_{r+} (\hat{H}^r_{\theta+})^* + \hat{B}^r_{r-} (\hat{H}^r_{\theta-})^* \right] \tag{8}$$

which in view of Eq. 5b and $\hat{B}_r = -jm\hat{A}/r$ becomes

$$\langle \tau \rangle_t = \pi b^2 w \mathcal{R}e\left[\frac{-jm}{b} \hat{A}^r_+ (\hat{H}^r_{\theta+})^* + \frac{jm}{b} \hat{A}^r_- (\hat{H}^r_{\theta-})^* \right] \tag{9}$$

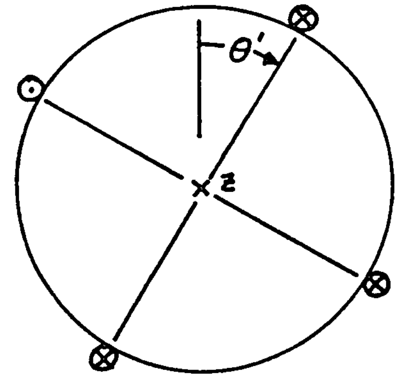
$$= \pi b^2 w \mathcal{R}e\left[\frac{jm\mu_o}{b} \hat{K}^a_+ G_m(b,a) \hat{H}^{r*}_{\theta+} - \frac{jm}{b} \mu_o \hat{K}^a_- G_m(b,a) (\hat{H}^r_{\theta-})^* \right]$$

Finally, with the use of Eq. 7,

$$\langle \tau \rangle_t = \pi b w m \mu_o G_m^2(b,a) \left\{ \frac{|\hat{K}^a_+|^2 \mu_o \sigma_s (\omega - m\Omega)}{1 + F_m^2(a,b)[\mu_o \sigma_s (\omega - m\Omega)]^2} - \frac{|\hat{K}^a_-|^2 \mu_o \sigma_s (\omega + m\Omega)}{1 + F_m^2(a,b)[\mu_o \sigma_s (\omega + m\Omega)]^2} \right\} \tag{10}$$

where m=p/2. This expression is similar in form to Eq. 6.4.11.

Prob. 6.4.5 From Eq. (f), Table 2.18.1



$$\Phi_\lambda = \frac{pw}{2}\left[A^a(\theta') - A^a\left(\theta' + \frac{2\pi}{p}\right)\right] \quad (1)$$

Because $A^a\left(\theta' + \frac{4\pi}{p}\right) = A^a(\theta')$, the flux linked by the total coil is just $p/2$ times that linked by the turns having the positive current in the z direction at θ' and returned at $\theta' + \pi/p$.

In terms of the complex amplitudes

$$\Phi_\lambda = \frac{pw}{2}\mathrm{Re}\left[\hat{A}_+^a e^{j(\omega t - m\theta')} + \hat{A}_-^a e^{j(\omega t + m\theta')} - \hat{A}_+^a e^{j(\omega t - m\theta' - \pi)} - \hat{A}_-^a e^{j(\omega t + m\theta' + \pi)}\right] \quad (2)$$

$$= pw\,\mathrm{Re}\left[\hat{A}_+^a e^{-jm\theta'} + \hat{A}_-^a e^{jm\theta'}\right] e^{j\omega t}$$

so

$$\lambda_a = \int_{-\pi/p}^{\pi/p} \Phi_\lambda N_a \cos\left(\frac{\theta' p}{2}\right) a\, d\theta' \quad (3)$$

or

$$\lambda_a = \frac{N_a p w a}{2}\mathrm{Re}\int_{-\frac{\pi}{p}}^{\frac{\pi}{p}}\left[\hat{A}_+^a e^{-j\frac{p\theta'}{2}} + \hat{A}_-^a e^{j\frac{p\theta'}{2}}\right]\left[e^{j\frac{p\theta'}{2}} + e^{-j\frac{p\theta'}{2}}\right]d\theta' e^{j\omega t} \quad (4)$$

The only terms contributing are those independent of θ'

$$\lambda_a = \frac{N_a p w a}{2}\mathrm{Re}\left[\hat{A}_+^a + \hat{A}_-^a\right] e^{j\omega t} \quad (5)$$

Substitution from Eqs. 5a and 7 from Prob. 6.4.4 then gives

Prob. 6.4.5 (cont.)

$$\lambda_a = \frac{N_a p w a}{2} Re\left\{ -\mu_o F_m(b,a)\hat{K}_+^a - \mu_o F_m(b,a)\hat{K}_-^a \right.$$

$$+ \frac{\mu_o G_m(a,b)\hat{K}_+^a G_m(b,a)\mu_o\sigma_s(\omega - m\Omega)[j + F_m(a,b)\mu_o\sigma_s(\omega - m\Omega)]}{1 + F_m^2(a,b)[\mu_o\sigma_s(\omega - m\Omega)]^2} \tag{6}$$

$$\left. - \frac{\mu_o G_m(a,b)\hat{K}_-^a G_m(b,a)\mu_o\sigma_s(\omega + m\Omega)[j + F_m(a,b)\mu_o\sigma_s(\omega + m\Omega)]}{1 + F_m^2(a,b)[\mu_o\sigma_s(\omega + m\Omega)]^2} \right\}$$

For two phase excitation $\hat{K}_+^a = \frac{1}{2} N_a \hat{i}_a$, $\hat{K}_-^a = 0$

this becomes

$$\lambda_a = Re\, \hat{\lambda}_a e^{j\omega t} \tag{7}$$

where

$$\hat{\lambda}_a = \frac{\mu_o N_a^2 p w a b}{4} \hat{i}_a \left\{ -\frac{F_m(b,a)}{b} + \frac{G_m(a,b)G_m(b,a)}{b^2} \cdot \left[\frac{S_m\left[j + \frac{F_m(a,b)}{b} S_m\right]}{1 + \frac{F_m^2(a,b)}{b^2} S_m^2} \right] \right\}$$

$$S_m \equiv \mu_o\sigma_s b(\omega - m\Omega)$$

For the circuit of Fig. 6.4.3,

$$\hat{v}_a = j\omega\hat{\lambda}_a = j\omega\left\{ (L_1 + M) - \frac{[\omega^2 M^2 (L_2 + M) + j\omega M^2 \frac{R}{r}]}{\omega^2 (L_2 + M)^2 + (\frac{R}{r})^2} \right\}$$

$$= j\omega\left\{ (L_1 + M) - \omega M^2 \frac{r}{R} \frac{[j + \omega(L_2 + M)\frac{r}{R}]}{[1 + (\frac{r}{R})^2\omega^2(L_2 + M)^2]} \right\} \tag{8}$$

Prob. 6.4.5 (cont.)

compared to Eq. 7 with $\mathcal{A}_m \equiv S_m/\mu_o\sigma_s b\omega = (1-\frac{m\Omega}{\omega})$ this expression gives

$$L_1 + M = \frac{-\mu_o N_a^2 pwba}{4}\ \frac{F_m(b,a)}{b} \tag{9}$$

$$\omega(L_2+M)\frac{A}{R} = \frac{F_m(a,b)}{b} = \frac{(L_2+M)}{R\mu_o\sigma_s b} \tag{10}$$

$$\frac{-M^2}{R\mu_o\sigma_s b} = \frac{\mu_o N_a^2 pwab}{4}\ \frac{G_m(a,b)\,G_m(b,a)}{b^2} \tag{11}$$

Assume $L_1 = L_2$ and Eqs. ~~2~~ 9 and ~~3~~ 10 then give

$$\frac{F_m(a,b)}{b} = -\frac{N_a^2 pwa}{R\sigma_s 4}\ \frac{F_m(b,a)}{b} \tag{12}$$

from which it follows that

$$R = -\frac{N_a^2 pwa}{4\sigma_s}\ \frac{F_m(b,a)}{F_m(a,b)} \tag{13}$$

Note from Eq. (b) of Table 2.16.2 that $F_m(b,a)/F_m(a,b) = -a/b$ so Eq. 6 becomes

$$R = \frac{N_a^2 pwa^2}{4\sigma_s b} \tag{14}$$

From this and Eq. 4 it follows that

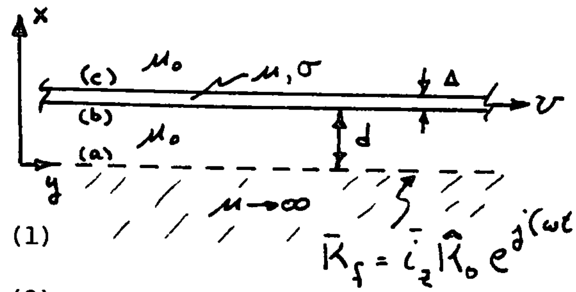
$$M = \frac{\mu_o N_a^2 pwa^2}{4}\sqrt{\frac{-G_m(a,b)G_m(b,a)}{ab}} \tag{15}$$

Note that $G_m(a,b) = -G_m(b,a)b/a$, so this can also be written as

$$M = \frac{\mu_o N_a^2 pwab\ G_m(b,a)}{4b} \tag{16}$$

Finally, from Eqs. 2 and 9

$$L_1 = L_2 = \frac{\mu_o N_a^2 pwba}{4}\left\{-\frac{F_m(b,a)}{b} - \frac{G_m(b,a)}{b}\right\} \tag{17}$$



Prob. 6.4.6 In terms of the cross-section shown, boundary conditions from Prob. 6.3.5 require that

$$-j\frac{k\mu\Delta}{2}(\hat{H}_y^c + \hat{H}_y^b) + (\hat{B}_x^c - \hat{B}_x^b) = 0 \tag{1}$$

$$-k^2(\hat{H}_y^c - \hat{H}_y^b) + \sigma\Delta k(\omega - kU)(\hat{B}_x^a + \hat{B}_x^b) \tag{2}$$

In addition, the fields must vanish as $x \to \infty$ and at the current sheet

$$\hat{H}_y^a = jk\hat{\psi}^a = \hat{K}_o \Rightarrow \hat{\psi}^a = \hat{K}_o/jk \tag{3}$$

Bulk conditions require that

$$\hat{B}_x^c = \mu_o k \hat{\psi}^c \quad (k > 0) \tag{4}$$

$$\begin{bmatrix} \hat{B}_x^b \\ \hat{B}_x^a \end{bmatrix} = \mu_o k \begin{bmatrix} -\coth k\Delta & \dfrac{1}{\sinh k\Delta} \\ \dfrac{-1}{\sinh k\Delta} & \coth k\Delta \end{bmatrix} \begin{bmatrix} \hat{\psi}^b \\ \dfrac{\hat{K}_o}{jk} \end{bmatrix} \tag{5}$$

In terms of the magnetic potential, Eqs. 1 and 2 are

$$\frac{k^2\mu\Delta}{2}(\hat{\psi}^c + \hat{\psi}^b) + (\hat{B}_x^c - \hat{B}_x^b) = 0 \tag{6}$$

$$-jk^2(\hat{\psi}^c - \hat{\psi}^b) + \sigma\Delta(\omega - kU)(\hat{B}_x^c + \hat{B}_x^b) = 0 \tag{7}$$

These two conditions are now written using Eqs. 3 and 5a to eliminate B_x^c and B_x^b.

$$\begin{bmatrix} k\mu_o\left(\dfrac{k\Delta\mu}{2\mu_o} + 1\right) & k\mu_o\left(\dfrac{k\Delta\mu}{2\mu_o} + \coth kd\right) \\ k^2\left[-j + \dfrac{\sigma\Delta(\omega - kU)\mu_o}{k}\right] & k^2\left[j - \dfrac{\sigma\Delta(\omega - kU)\mu_o}{k}\coth kd\right] \end{bmatrix} \begin{bmatrix} \hat{\psi}^c \\ \hat{\psi}^b \end{bmatrix} = \begin{bmatrix} \dfrac{\mu_o\hat{K}_o}{j\sinh kd} \\ \dfrac{j\sigma\Delta(\omega - kU)\mu_o\hat{K}_o}{\sinh kd} \end{bmatrix} \tag{8}$$

From these expressions it follows that

$$\tilde{\psi}^c = \frac{\hat{K}_o}{k\sinh kd} \frac{\left\{1 + j\dfrac{\mu_o\sigma\Delta}{k}(\omega - kU)\left(-\dfrac{\mu\Delta k}{2\mu_o}\right)\right\}}{\left\{j\left[(1 + \coth kd) + \dfrac{k\Delta\mu}{\mu_o}\right] - \dfrac{\sigma\mu_o\Delta(\omega - kU)}{k}\left[2\coth kd + \dfrac{k\Delta\mu}{2\mu_o}(1 + \coth kd)\right]\right\}} \tag{9}$$

Prob. 6.4.6(cont.)

In the limit where $\mu \to \mu_o$, having $\mu_o \sigma \Delta (\omega - kU)/k >> 1$ results in Eq. 9 becoming

$$\hat{\psi}^c \to \frac{-\hat{K}_o}{k \sinh kd} \Big/ \left[\frac{\mu_o \sigma \Delta (\omega - kU)}{k} 2 \coth kd\right] \tag{10}$$

Thus, as $\mu_o \sigma \Delta (\omega - kU)/k$ is raised, the field is shielded out of the region above the sheet by the induced currents.

In the limit where $\sigma \to 0$, for $(k\Delta)\mu/\mu_o >> 1$, Eq. 9 becomes

$$\hat{\psi}^c = \frac{\hat{K}_o}{k \sinh kd} \Big/ j\left(\frac{k\Delta \mu}{\mu_o}\right) \tag{11}$$

and again as $k\Delta \mu/\mu_o$ is made large the field is shielded out. (Note that by the requirements of the thin sheet model, $k\Delta << 1$, so μ/μ_o must be very large to obtain this shielding.)

With $k\Delta \mu/\mu_o$ finite, the numerator as well as the denominator of Eq. 9 becomes large as $\mu_o \sigma \Delta (\omega - kU)/k$ is raised. The conduction current shielding tends to be compromised by having a magnetizable sheet. This conflict should be expected, since the conduction current shields by making the normal flux density vanish. By contrast, the magnetizable sheet shields by virtue of tending to make the tangential field intensity zero. The tendency for the magnetization to duct the flux density through the sheet is in conflict with the effect of the induced current, which is to prevent a normal flux density.

Prob. 6.4.7 For the given distribution of surface current, the Fourier transform of the complex amplitude is

$$\hat{K}^s = \hat{K}_o \int_0^{\ell} e^{j(k-\beta)} dy = \frac{\hat{K}_o [e^{j(k-\beta)\ell} - 1]}{j(k-\beta)} \tag{1}$$

It follows from Eq. 5.16.8 that the desired force is

$$\langle f_y \rangle_t = \frac{w}{2} \mathrm{Re} \int_{-\infty}^{+\infty} \hat{B}_x^r (\hat{H}_y^r)^* dz = \frac{w}{4\pi} \mathrm{Re} \int_{-\infty}^{+\infty} \hat{B}_x^r (\hat{H}_y^r)^* dk \tag{2}$$

In evaluating the integral on k, observe first that Eq. 6.4.9 can be used to evaluate $\hat{B}_x^r$.

$$\langle f_y \rangle_t = \frac{-w}{4\pi} \mathrm{Re} \int_{-\infty}^{+\infty} j\mu_o \left[\frac{\hat{K}^s}{\sinh kd} + \coth kd\, \hat{H}_y^r\right] (\hat{H}_y^r)^* dk \tag{3}$$

Because the integration is over real values of k only, it is clear that the second term of the two in brackets is purely imaginary and hence makes no contribution. With Eq. 6.4.6 used to substitute for $\hat{H}_y^r$, the expression then becomes

$$\langle f_y \rangle_t = \frac{w}{4\pi} \mu_o \int_{-\infty}^{+\infty} \frac{|\hat{K}^s|^2 S_m \, dk}{\sinh^2 kd \,(1 + S_m^2 \coth^2 kd)} \tag{4}$$

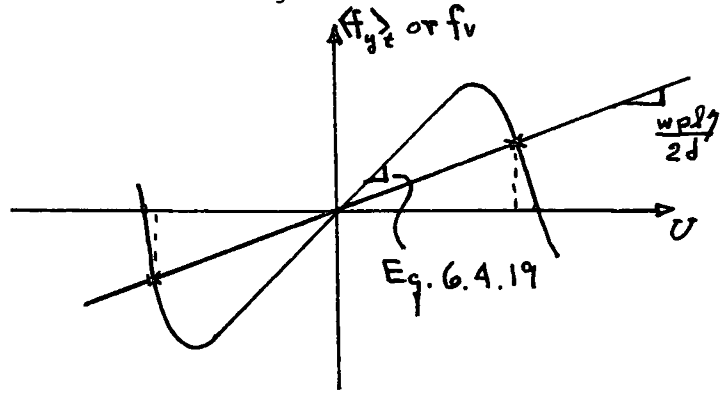
The magnitude $|\hat{K}^s|$ is conveniently found from Eq. 1 by first recognizing that

$$\hat{K}^s = \frac{2j\hat{K}_o}{j(k-\beta)} \left[\frac{e^{j\frac{(k-\beta)\ell}{2}} - e^{-j\frac{(k-\beta)\ell}{2}}}{2j}\right] e^{j\frac{(k-\beta)\ell}{2}} = 2j\hat{K}_o \frac{\sin\left[\frac{(k-\beta)\ell}{2}\right]}{j(k-\beta)} e^{j\frac{(k-\beta)\ell}{2}} \tag{5}$$

Substitution of this expression into Eq. 4 finally results in the integral given in the problem statement.

Prob. 6.4.8 From Eq. 7.13.1, the viscous force retarding the motion of the rotor is

$$f_v = \frac{wp\ell}{2}\left(\frac{\eta U}{d}\right) \tag{1}$$



Thus, the balance of viscous and magnetic forces is represented graphically as shown in the sketch. The slope of the magnetic force curve near the origin is given by Eq. 6.4.19.

As the magnetic field is raised, the static equilibrium at the origin becomes one with U either positive or negative as the slopes of the respective curves are equal at the origin. Thus, instability is incipient as

$$\frac{kd}{\sinh^2 kd} R_m \frac{[R_m^2 \coth^2 kd - 1]}{[R_m^2 \coth^2 kd + 1]^2} > \omega \tau_{mv} \tag{2}$$

where $R_m = \omega\tau_m$, $\tau_m \equiv \mu_o \sigma_s / k$, $\tau_{mv} = \eta / \mu_o H_o^2$.

Prob. 6.5.1 The z component of Eq. 6.5.3 is written with $\bar{v} = \Omega r \bar{i}_\theta$ and $\bar{A} = A(r,\theta,t)\bar{i}_z$ by recognizing that

$$\nabla \times \bar{A} = \frac{1}{r}\frac{\partial A}{\partial \theta}\bar{i}_r - \frac{\partial A}{\partial r}\bar{i}_\theta \tag{1}$$

so that

$$\bar{v} \times \nabla \times \bar{A} = \begin{bmatrix} \bar{i}_r & \bar{i}_\theta & \bar{i}_z \\ 0 & \Omega r & 0 \\ \frac{1}{r}\frac{\partial A}{\partial \theta} & -\frac{\partial A}{\partial r} & 0 \end{bmatrix} = -\bar{i}_z \Omega \frac{\partial A}{\partial \theta} \tag{2}$$

Thus, because the z component of the vector Laplacian in polar coordinates is the same as the scalar Laplacian, Eq. 6.5.8 is obtained from Eq. 6.5.3

$$\frac{1}{\mu\sigma}\nabla^2 A = \frac{\partial A}{\partial t} + \Omega\frac{\partial A}{\partial \theta} \tag{3}$$

Solutions $A = \mathcal{R}e\, \hat{A}(r) \exp j(\omega t - m\theta)$ are introduced into this expression to obtain

$$\frac{1}{\mu\sigma}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\hat{A}}{dr}\right) - \frac{m^2}{r^2}\hat{A}\right] = j(\omega - m\Omega)\hat{A} \tag{4}$$

which becomes Eq. 6.5.9

$$\frac{d^2\hat{A}}{dr^2} + \frac{1}{r}\frac{d\hat{A}}{dr} - \left(\gamma^2 + \frac{m^2}{r^2}\right)\hat{A} = 0 \tag{5}$$

where

$$\gamma^2 \equiv j\mu\sigma(\omega - m\Omega)$$

Compare this to Eq. 2.16.19 and it is clear that the solution is the linear combination of $H_m(j\gamma r)$ and $J_m(j\gamma r)$ that make

Prob. 6.5.1 (cont.)

$$\hat{A}(\alpha) = \hat{A}^{\alpha} \qquad \hat{A}(\beta) = \hat{A}^{\beta} \quad .$$

This can be accomplished by writing two equations in the two uknown coefficients of H_m and J_m or by inspection as follows. The "answer" will look like

$$\hat{A}(r) = \hat{A}^{\alpha}\left[\frac{(\quad)}{(\quad)}H_m(j\gamma r) + \frac{(\quad)}{(\quad)}J_m(j\gamma r)\right] + \hat{A}^{\beta}\left[\frac{(\quad)}{(\quad)}H_m(j\gamma r) + \frac{(\quad)}{(\quad)}J_m(j\gamma r)\right] \tag{6}$$

The coefficients of the first term must be such that the combination multiplying $\hat{A}^{\alpha}$ vanishes where $r=\beta$ (because there, the answer cannot depend on $\hat{A}^{\alpha}$). To this end, make them $J_m(j\gamma\beta)$ and $H_m(j\gamma\beta)$ respectively. The denominator is then set to make the coefficient of $\hat{A}^{\alpha}$ unity where $r = \alpha$. Similar reasoning sets the coefficient of $\hat{A}^{\beta}$. The result is

$$\hat{A}(r) = \hat{A}^{\alpha}\frac{[H_m(j\gamma r)J_m(j\gamma\beta) - J_m(j\gamma r)H_m(j\gamma\beta)]}{[H_m(j\gamma\alpha)J_m(j\gamma\beta) - J_m(j\gamma\alpha)H_m(j\gamma\beta)]} + \hat{A}^{\beta}\frac{[H_m(j\gamma r)J_m(j\gamma\alpha) - J_m(j\gamma r)H_m(j\gamma\alpha)]}{[H_m(j\gamma\beta)J_m(j\gamma\alpha) - J_m(j\gamma\beta)H_m(j\gamma\alpha)]} \tag{7}$$

The tangential $\bar{H}$, $H_\theta = -(\partial A/\partial r)/\mu$ so it follows from Eq. 7 that

$$\hat{H}_\theta = -\frac{j\gamma}{\mu}\left\{\hat{A}^{\alpha}\frac{[H_m'(j\gamma r)J_m(j\gamma\beta) - J_m'(j\gamma r)H_m(j\gamma\beta)]}{[H_m(j\gamma\alpha)J_m(j\gamma\beta) - J_m(j\gamma\alpha)H_m(j\gamma\beta)]} + \hat{A}^{\beta}\frac{[H_m'(j\gamma r)J_m(j\gamma\alpha) - J_m'(j\gamma r)H_m(j\gamma\alpha)]}{[H_m(j\gamma\beta)J_m(j\gamma\alpha) - J_m(j\gamma\beta)H_m(j\gamma\alpha)]}\right\} \tag{8}$$

Prob. 6.5.1 (cont.)

Evaluation of this expression at $r = \alpha$ gives $\hat{H}_\theta^\alpha$

$$\hat{H}_\theta^\alpha = \frac{1}{\mu}\left\{ f_m(\beta,\alpha,\gamma)\hat{A}^\alpha + g_m(\alpha,\beta,\gamma)\hat{A}^\beta \right\} \tag{9}$$

where

$$f_m(\beta,\alpha,\gamma) = j\gamma \frac{\left[J_m'(j\gamma\alpha)H_m(j\gamma\beta) - H_m'(j\gamma\alpha)J_m(j\gamma\beta)\right]}{\left[H_m(j\gamma\alpha)J_m(j\gamma\beta) - J_m(j\gamma\alpha)H_m(j\gamma\beta)\right]}$$

and

$$g_m(\alpha,\beta,\gamma) = \frac{j}{\pi\alpha} \frac{\left[J_m'(j\gamma\alpha)H_m(j\gamma\alpha) - H_m'(j\gamma\alpha)J_m(j\gamma\alpha)\right]}{\left[H_m(j\gamma\beta)J_m(j\gamma\alpha) - J_m(j\gamma\beta)H_m(j\gamma\alpha)\right]}$$

Of course, Eq. 9 is the first of the desired transfer relations, the first of Eqs. (c) of Table 6.5.1. The second follows by evaluating Eq. 9 at $r = \beta$. Note that these definitions are consistent with those given in Table 2.16.2 with $k \rightarrow \gamma$. Because γ generally differs according to the region being described, it is included in the argument of the function.

To determine Eq. (d) of Table 6.5.1, these relations are inverted. For example, by Kramer's rule

$$F_m(\beta,\alpha,\gamma) = \frac{1}{\mu} \frac{f_m(\alpha,\beta,\gamma)}{\frac{1}{\mu^2}\left[f_m(\beta,\alpha,\gamma)f_m(\alpha,\beta,\gamma) - g_m(\beta,\alpha,\gamma)g_m(\alpha,\beta,\gamma)\right]} \tag{10}$$

Prob. 6.5.2 By way of establishing the representation, Eqs. g and h of Table 2.18.1 define the scalar component of the vector potential.

$$\bar{B} = -\frac{1}{r}\frac{\partial \Lambda}{\partial z}\bar{i}_r + \frac{1}{r}\frac{\partial \Lambda}{\partial r}\bar{i}_z \;;\; \Lambda \equiv Ar \tag{1}$$

$$\bar{A} = \bar{i}_\theta A(r,z,t) \tag{2}$$

Thus, the θ component of Eq. 6.5.3 requires that (Appendix A)

$$\frac{1}{\mu\sigma}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rA)\right) + \frac{\partial^2 A}{\partial z^2}\right] = \frac{\partial A}{\partial t} + U\frac{\partial A}{\partial z} \tag{3}$$

In terms of the complex amplitude, this requires that

$$\frac{d^2\hat{A}}{dr^2} + \frac{1}{r}\frac{d\hat{A}}{dr} - \left(\gamma^2 + \frac{1}{r^2}\right)\hat{A} = 0 \tag{4}$$

where $\gamma^2 = k^2 + j(\omega - kU)\mu\sigma$. The solution to this expression satisfying the appropriate boundary conditions is Eq. 156.14.15. In view of Eq.1 ,

$$H_z = \frac{B_z}{\mu} = \frac{1}{\mu r}\frac{\partial \Lambda}{\partial r} \tag{5}$$

Observe from Eq. 2.16.26d (evaluated using m=0) that $uR_1' + R_1 = (uR_1)' = uR_0$ where R_m can be either J_m or H_m and the prime indicates a derivative with respect to the argument. Thus, with Eq. 6.5.15 used to evaluate Eq. 5, it follows that

$$H_z = \frac{j\gamma}{\mu}\left\{\hat{A}^\alpha \frac{[H_1(j\gamma\beta)J_0(j\gamma r) - J_1(j\gamma\beta)H_0(j\gamma r)]}{[H_1(j\gamma\beta)J_1(j\gamma\alpha) - J_1(j\gamma\beta)H_1(j\gamma\alpha)]} + \hat{A}^\beta \frac{[J_1(j\gamma\alpha)H_0(j\gamma r) - H_1(j\gamma\alpha)J_0(j\gamma r)]}{[J_1(j\gamma\alpha)H_1(j\gamma\beta) - H_1(j\gamma\alpha)J_1(j\gamma\beta)]}\right\} \tag{6}$$

Further, observe that(Eq. 2.16.26c) $J_1(j\gamma x) = -J_0'(j\gamma x)$ so, Eq. 6 becomes

$$H_z = -\frac{\gamma^2}{\mu}\left\{\frac{\hat{A}^\alpha}{j\gamma}\frac{[J_0'(j\gamma\beta)H_0(j\gamma r) - H_0'(j\gamma\beta)J_0(j\gamma r)]}{[H_0'(j\gamma\beta)J_0'(j\gamma\alpha) - J_0'(j\gamma\beta)H_0'(j\gamma\alpha)]} \quad \frac{\hat{A}^\beta}{j\gamma}\frac{[H_0'(j\gamma\alpha)J_0(j\gamma r) - J_0'(j\gamma\alpha)H_0(j\gamma r)]}{[J_0'(j\gamma\alpha)H_0'(j\gamma\beta) - H_0'(j\gamma\alpha)J_0'(j\gamma\beta)]}\right\} \tag{7}$$

This expression is evaluated at r= α and r = β respectively to obtain the equations e of Table 6.5.1. Because Eqs. e and f take the same form as Eqs. b and a respectively of Table 2.16.2, the inversion to obtain Eqs. f has already been shown.

Prob. 6.6.1 For the pure traveling wave, Eq. 6.7.7 reduces to

$$\langle S_d\rangle_{yt} = -\frac{1}{2}(\omega - kU)\,\mathrm{Re}\, j\left[\hat{A}^b(\hat{H}_y^b)^* - \hat{A}^c(\hat{H}_y^c)^*\right] \tag{1}$$

The boundary condition represented by Eq. 6.6.3 makes the second term zero while Eq. 6.6.5b shows that the remaining expression can also be written as

$$\langle S_d\rangle_{yt} = -\frac{1}{2}(\omega - kU)\,\mathrm{Re}\left\{ j\frac{\mu_0}{k}\left[\frac{\hat{K}_+^a}{\sinh kd} + \coth kd\,\hat{H}_y^b\right]\right\}(\hat{H}_y^b)^* \tag{2}$$

The "self" term therefore makes no contribution. The remaining term is evaluated by using Eq. 6.6.9.

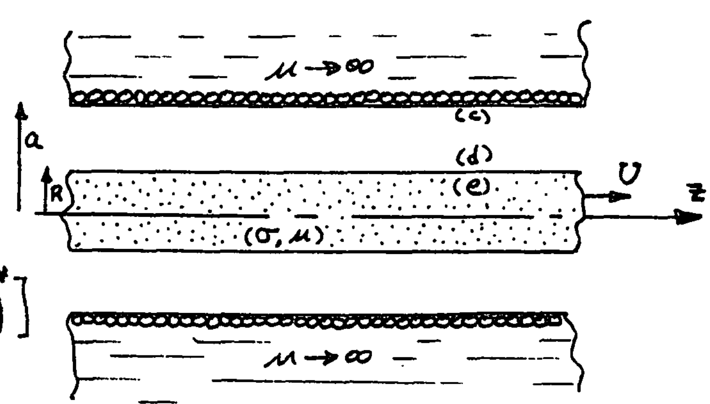
$$\langle S_d\rangle_{yt} = \frac{1}{2}(\omega - kU)\frac{\mu_0}{k}\frac{|\hat{K}_+^a|^2}{\sinh^2 kd}\,\mathrm{Re}\,\frac{j}{\left[\frac{k}{\gamma^*}\frac{\mu}{\mu_0}\coth\gamma^* a + \coth kd\right]} \tag{3}$$

Prob. 6.6.2 (a) To obtain the drive in terms of complex amplitudes, write the cosines in complex form and group terms as forward and backward traveling waves. It follows that

$$\hat{K}_\pm^a = \hat{i}_a\frac{N_a}{2} + \hat{i}_b\frac{N_b}{2}e^{j\frac{2\pi}{3}} + \hat{i}_c\frac{N_c}{2}e^{j\frac{4\pi}{3}} \tag{1}$$

To determine the time average force, the rod is enclosed by a circular cylindrical surface having radius R and axial length ℓ . Boundary locations are as indicated in the diagram. Using the theorem of Eq. 5.16.4, it follows that

$$\langle f_z\rangle_t = 2\pi R\ell\langle B_r^d H_z^d\rangle_{zt} \tag{2}$$



$$= \pi R\ell\,\mathrm{Re}\left[\hat{B}_{r+}^d(\hat{H}_{z+}^d)^* - \hat{B}_{r-}^d(\hat{H}_{z-}^d)^*\right]$$

With the use of Eqs. (e) from Table 2.19.1 to represent the air-gap fields

Prob. 6.6.2 (cont.)

the "self" terms are dropped and Eq. (2) becomes

$$\langle f_z \rangle_t = \mu_o \frac{\pi R \ell}{k} Re\left[j g_o(R,a,k)(\hat{H}^d_{z+})^* - j g_o(R,a,-k)(\hat{H}^d_{z-})^* \right] \tag{3}$$

So, $\hat{H}^d_{z\pm}$ is desired. To this end observe that boundary and jump conditions are

$$\hat{H}^c_z = \hat{K}^a \tag{4}$$

$$\hat{H}^d_z = \hat{H}^e_z \tag{5}$$

$$\hat{A}^d = \hat{A}^e \Rightarrow \hat{\Lambda}^d = \hat{\Lambda}^e \tag{6}$$

It follows from Eqs. (f) of Table 6.5.1 applied to the air-gap and to the rod that

$$\frac{\hat{\Lambda}^e}{R} = -\frac{\mu}{\gamma^2} f_o(0,R,\gamma)\hat{H}^e_z = -\frac{\mu_o}{k^2} g_o(R,a,k)\hat{K}^a - \frac{\mu_o}{k^2} f_o(a,R,k)\hat{H}^d_z \tag{7}$$

Hence,

$$\hat{H}^d_{z\pm} = \hat{H}^e_{z\pm} = \frac{-g_o(R,a,\pm k)\hat{K}^a}{f_o(a,R,\pm k) - \frac{k^2}{\gamma_\pm^2}\frac{\mu}{\mu_o} f_o(0,R,\pm\gamma)} \; ; \gamma \equiv \sqrt{k^2 + j\mu\sigma(\omega \mp kU)} \tag{8}$$

<u>Prob. 6.6.3</u> The Fourier transform of the excitation surface current is

$$\hat{K}^a = \hat{K}_o \frac{e^{j(k-\beta)\ell} - 1}{j(k-\beta)} = \frac{2\hat{K}_o e^{j\frac{(k-\beta)\ell}{2}}}{k-\beta} \sin\left[\frac{(k-\beta)\ell}{2}\right] \tag{1}$$

In terms of the Fourier transforms, Eq. 5.16.8 shows that the total force is

$$\langle f_y \rangle_t = \frac{w}{4\pi} Re \int_{-\infty}^{+\infty} (\hat{B}^b_x)^* \hat{H}^b_y \, dk \tag{2}$$

In view of Eq. 6.6.5b, this expression becomes

$$\langle f_y \rangle_t = \frac{w}{4\pi} Re \int_{-\infty}^{+\infty} \mu_o j \frac{(\hat{K}^a)^*}{\sinh kd} \hat{H}^b_y \, dk \tag{3}$$

where the term in $\hat{H}^b_y(\hat{H}^b_y)^*$ has been eliminated by taking the real part.

With the use of Eq. 6.6.9, this expression becomes

$$\langle f_y \rangle_t = \frac{-w\mu_o}{4\pi} Re \int_{-\infty}^{+\infty} \frac{j\,|\hat{K}^a|^2 dk}{\sinh^2 kd\left[\frac{k}{\gamma}\frac{\mu}{\mu_o}\coth\gamma a + \coth kd\right]} \tag{4}$$

With the further substitution of Eq. 1, the expression stated with the problem is found.

Prob. 6.7.1 It follows from Eq. 6.7.7 that the power dissipation (per unit y-z area) is

$$P_d \equiv \langle S_d \rangle_{yt} = -\frac{1}{2}(\omega - kU)\,\mathrm{Re}\,j\left[\hat{A}^\alpha(\hat{H}_y^\alpha)^* - \hat{A}^\beta(\hat{H}_y^\beta)^*\right] \tag{1}$$

The time average mechanical power output (again per unit y-z area) is the product of the velocity U and the difference in magnetic shear stress acting on the respective surfaces

$$P_m = \frac{1}{2}\,\mathrm{Re}\left[\hat{B}_x^\alpha(\hat{H}_y^\alpha)^* - \hat{B}_x^\beta(\hat{H}_y^\beta)^*\right]U \tag{2}$$

Because $\hat{B}_x = -jk\hat{A}$, this expression can be written in terms of the same combination of amplitudes as appears in Eq. 1

$$P_m = -\frac{k}{2}\,\mathrm{Re}\,j\left[\hat{A}^\alpha(\hat{H}_y^\alpha)^* - \hat{A}^\beta(\hat{H}_y^\beta)^*\right] \tag{3}$$

Thus, it follows from Eqs. 1 and 3 that

$$E_{ff} \equiv \frac{P_m}{P_m + P_d} = \frac{U}{(\omega/k)} \tag{4}$$

From the definition of s,

$$\frac{U}{(\omega/k)} = 1 - s \tag{5}$$

so that

$$E_{ff} = 1 - s \tag{6}$$

Prob. 6.7.2

The time average and space average power dissipation per unit y-z area is given by Eq. 6.7.7. For this example n=1 and

$$\langle S_d \rangle_{yt} = -\mathrm{Re}\, j\frac{(\omega - kU)}{2}\hat{A}^b(\hat{H}_y^b)^* \tag{1}$$
$$= \mathrm{Re}\, j\frac{(\omega - kU)}{2}(\hat{A}^b)^*\hat{H}_y^b$$

because $\hat{H}_y^\beta = \hat{H}_y^d = 0.$

From Eq. 6.5.5b

$$\langle S_d \rangle_{yt} = \mathrm{Re}\, j\frac{(\omega - kU)}{2}\frac{\mu_o}{k}\left[\frac{\hat{K}^{a*}}{\sinh kd}\hat{H}_y^b\right] \tag{2}$$

where, in expressing $\hat{A}^b$, the term in $\hat{H}_y^b$ has been dropped because the real part is taken.

In view of Eq. 6.6.9, this expression becomes

$$\langle S_d \rangle_{yt} = -\mathrm{Re}\, j\frac{(\omega - kU)}{2}\frac{\mu_o}{k}\frac{|\hat{K}_+^a|^2}{\sinh^2 kd\left[\frac{k}{\gamma}\frac{\mu}{\mu_o}\coth\gamma a + \coth kd\right]} \tag{3}$$

Note that it is only because $\gamma \equiv \sqrt{(ka)^2 + jS_m}/a$ is complex that this function has a non-zero value.

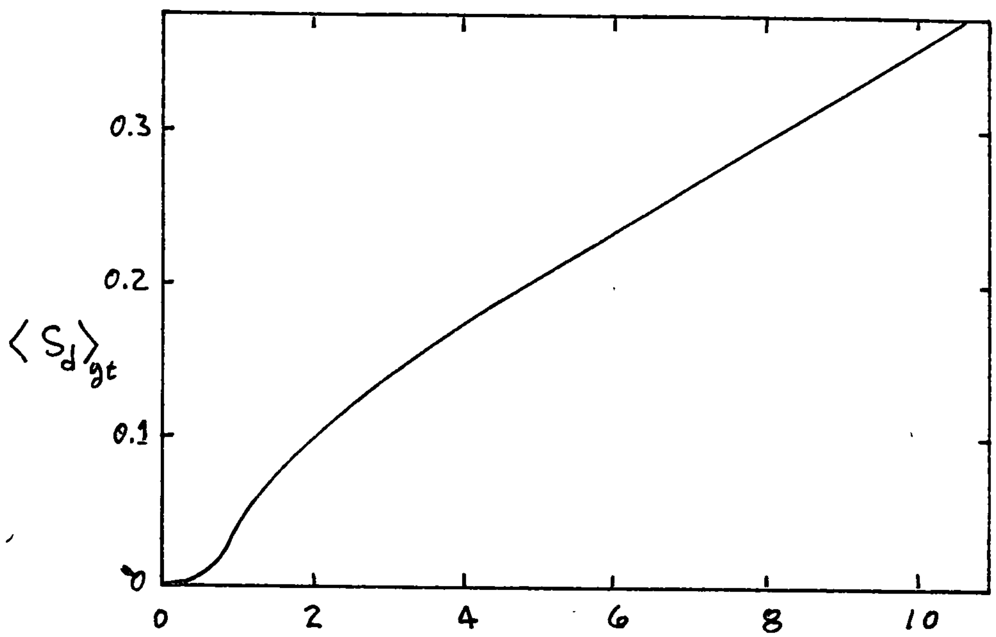
In terms of $S_m \equiv \mu\sigma a^2(\omega - kU)$

$$\langle S_d \rangle_{yt} = -\mathrm{Re}\frac{S_m\mu_o}{2\mu\sigma a^2}\frac{|\hat{K}_+^a|^2}{k\sinh^2 kd}\left\{\frac{j}{\left[\frac{k}{\gamma}\frac{\mu}{\mu_o}\coth\gamma a + \coth kd\right]}\right\} \tag{4}$$

Note that the term in { } is the same function as represents the S_m dependence of the time average force/unit area, Fig. 6.6.2. Thus, the dependence

Prob. 6.7.2 (cont.)

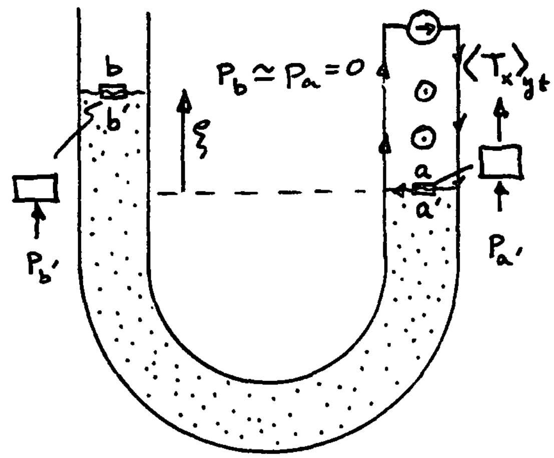
of $\langle S_d \rangle_{yt}$ on S_m is the function shown in that figure multiplied by S_m.



Prob. 6.8.1 Equations 6.8.10 and 6.8.11 are directly applicable. The skin depth is short, so $\hat{H}_y^\beta$ is negligible. Elimination of $\hat{H}_y^\alpha$ between the two expressions gives

$$\langle T_x \rangle_{yt} = -\frac{\mu_o}{4}(2\sigma\delta)\langle S_d \rangle_{yt} = -\sqrt{\frac{\mu_o\sigma}{2\omega}}\langle S_d \rangle_{yt} \tag{1}$$

where $\langle S_d \rangle_{yt}$ is the time average power dissipated per unit area of the interface. Force equilibrium at the interfaces can be pictured from the control volumes shown.



$$P_{b'} = 0 \tag{2}$$

$$\langle T_x \rangle_{yt} + P_{a'} = 0 \tag{3}$$

Bernoulli's equation relates the pressures at the interfaces inside the liquid.

$$P_{a'} = P_{b'} + \rho g \xi \tag{4}$$

Elimination of the p's between these last three expressions then gives

$$\langle T_x \rangle_{yt} = -\rho g \xi \tag{5}$$

So, in terms of the power dissipation as given by Eq. 1, the "head" is

$$\xi = \frac{1}{\rho g}\sqrt{\frac{\mu_o\sigma}{2\omega}}\langle S_d \rangle_{yt} \tag{6}$$

Prob. 6.9.1

With

$$\xi = \frac{x}{2}\sqrt{\frac{\mu\sigma}{t'}} \tag{1}$$

$$\frac{\partial}{\partial t'} f(\xi) = \frac{df}{d\xi}\frac{\partial \xi}{\partial t'} = -\frac{x}{4}\sqrt{\mu\sigma}\, t^{-\frac{3}{2}} \frac{df}{d\xi} \tag{2}$$

and

$$\frac{\partial f}{\partial x} = \frac{df}{d\xi}\frac{\partial \xi}{\partial x} = \frac{1}{2}\sqrt{\mu\sigma}\, t'^{-\frac{1}{2}} \frac{df}{d\xi} \tag{3}$$

Taking this latter derivative again gives

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{2}\sqrt{\mu\sigma}\, t'^{-\frac{1}{2}} \frac{d^2 f}{d\xi^2}\frac{\partial \xi}{\partial x} = \frac{1}{4}\mu\sigma\, t^{-1} \frac{d^2 f}{d\xi^2} \tag{4}$$

Thus, Eq. 6.9.3 becomes

$$\frac{1}{\mu\sigma}\, t'^{-1} \frac{d^2 H_y}{d\xi^2} = -x\sqrt{\mu\sigma}\, t'^{-\frac{3}{2}} \frac{dH_y}{d\xi} \tag{5}$$

or,

$$\frac{d^2 H_y}{d\xi^2} + 2\frac{x}{2}\sqrt{\frac{\mu\sigma}{t'}}\frac{dH_y}{d\xi} = 0 \tag{6}$$

In view of the definition of ξ , Eq. 1, this expression is the same as Eq. 6.9.7.

Prob. 6.9.2 (a) The field in the liquid metal is approximated by Eq. 6.9.1 with U=0. Thus, the field is computed as though it had no y dependence and is simply

$$H_y = \mathcal{R}e\, \hat{H}_y\, e^{\frac{x}{\delta}}\, e^{j(\omega t + \frac{x}{\delta})} \tag{1}$$

The amplitude of this field is a slowly varying function of y, however, given by the fact that the flux is essentially trapped in the air-gap. Thus, $\hat{H}_y = a\hat{H}_o/h$ and Eq. (1) becomes

$$H_y = \mathcal{R}e\, \frac{a\hat{H}_o}{h}\, e^{\frac{x}{\delta}}\, e^{j(\omega t + \frac{x}{\delta})} \tag{2}$$

(b) Gauss' Law can now be used to find H_x. First, observe from Eq. (2) that

$$\frac{\partial H_x}{\partial x} = -\frac{\partial H_y}{\partial y} = \mathcal{R}e\, \frac{a\hat{H}_o}{h^2}\, \frac{dh}{dy}\, e^{\frac{x}{\delta}}\, e^{j(\omega t + \frac{x}{\delta})} \tag{3}$$

Then, integration gives H_x

$$H_x = \mathcal{R}e\, \frac{a\hat{H}_o\delta}{1+j}\, \frac{1}{h^2}\frac{dh}{dy}\, e^{\frac{x}{\delta}}\, e^{j(\omega t - \frac{x}{\delta})} \tag{4}$$

The integration constant is zero because the field must vanish as $x \to -\infty$.

(c) The time-average shearing surface force density is found by integrating the Maxwell stress tensor over a pill box enclosing the complete skin region.

$$\langle T_y\rangle_t = \frac{1}{2}\mathcal{R}e\,\mu_o \hat{H}_x \hat{H}_y^*\Big|_{x=0} = \frac{\mu_o}{4} a^2 |\hat{H}_o|^2 \delta \frac{dh}{dy} \tag{5}$$

As would be expected, this surface force density goes to zero as either the skin depth or the slope of the electrode vanish.

(d) If Eq. 5 is to be independent of y,

$$\frac{1}{h^3}\frac{dh}{dy} = \text{constant} = \frac{S}{a^3} \tag{6}$$

Integration follows by multiplying by dy

$$\int_a^h \frac{dh}{h^3} = \int_0^y \frac{S}{a^3}\, dy$$

and the given distribution h(y) follows.

Prob. 6.9.2(cont.)

(e) Evaluated using h(y), Eq. 6 becomes

$$\langle T_y \rangle_t = \frac{\mu_o}{4} |\hat{H}_o|^2 \frac{\delta}{a} S \tag{8}$$

Prob. 6.9.3 From Eq. 6.8.11, the power dissipated per unit area is (there is no $\mathcal{B}$ surface)

$$\langle S_d \rangle_{yt} = \frac{1}{2\sigma\delta'} |\hat{H}_y^d|^2 \tag{1}$$

where

$$\delta' \rightarrow \sqrt{\frac{2}{|\omega|\mu\sigma}}$$

Thus, Eq. 2 of Prob. 6.9.2 can be exploited to write $H_y(x=0)$ in Eq. 1 as

$$\langle S_d \rangle_{yt} = \frac{1}{2\sigma\delta} |\hat{H}_o|^2 [1 + 2 S(\tfrac{y}{a})] \tag{2}$$

The total power dissipation per unit depth in the z direction is

$$\int_0^{\ell} \langle S_d \rangle_{yt} dy = \frac{|\hat{H}_o|^2}{2\sigma\delta} \int_0^{\ell} [1 - 2 S(\tfrac{y}{a})] dy = \frac{|\hat{H}_o|^2 \ell}{2\sigma\delta}\left(1 - \frac{S\ell}{a}\right) \tag{3}$$

Prob. 6.9.4 Because $\bar{J}_f' = \bar{J}_f$ and $\bar{J}_f' = \sigma \bar{E}'$, the power dissipation per unit y-z area is

$$S_d = \int_{-\infty}^{0} \bar{E}' \cdot \bar{J}_f' \, dx = \int_{-\infty}^{0} \frac{\bar{J}_f \cdot \bar{J}_f}{\sigma} \, dx \tag{1}$$

In the "boundary-layer" approximation, the z component of Ampere's law becomes

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \simeq \frac{\partial H_y}{\partial x} = J_z \tag{2}$$

So that the dissipation density is

$$\frac{J_z^2}{\sigma} \simeq \frac{1}{\sigma} \left(\frac{\partial H_y}{\partial x} \right)^2 \tag{3}$$

In view of Eq. 6.9.8,

$$\begin{aligned} \frac{J_z^2}{\sigma} &= \frac{H_o^2}{\sigma} \left[\frac{\partial}{\partial x} \operatorname{erf}(\xi) \right]^2 = \frac{H_o^2}{\sigma} \left(\frac{2 e^{-\xi^2}}{\sqrt{\pi}} \right)^2 \left(\frac{\partial \xi}{\partial x} \right)^2 \\ &= H_o^2 \frac{\mu}{t' \pi} e^{-2\xi^2} \end{aligned} \tag{4}$$

Note that the only x dependence is now through ξ. Thus,

$$S_d = \frac{\mu H_o^2}{\pi t'} \int_{-\infty}^{0} e^{-2\xi^2} dx = \frac{2\sqrt{2}}{\pi} \frac{\mu H_o^2}{t'} \sqrt{\frac{t'}{\mu\sigma}} \int_{-\infty}^{0} e^{-2\xi^2} d(\xi\sqrt{2}) \tag{5}$$

$$= \sqrt{\frac{2}{\pi}} \frac{\mu H_o^2}{\sqrt{t'\mu\sigma}} \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-(\sqrt{2}\xi)^2} d(\sqrt{2}\xi) = \frac{\mu H_o^2 \sqrt{2}}{\sqrt{\pi \mu \sigma t'}}$$

Prob. 6.9.4 (cont.)

So, for $y > Ut$ where $t' = y/U$

$$S_d = \begin{cases} \dfrac{\mu H_o^2 \sqrt{2}}{\sqrt{\pi\mu\sigma t}} & ; \quad y > Ut \\[2ex] \dfrac{\mu H_o^2 \sqrt{2}}{\sqrt{\dfrac{\pi\mu\sigma y}{U}}} & ; \quad 0 < y < Ut \end{cases} \tag{6}$$

For $Ut < L$ the total power per unit length in the z direction is

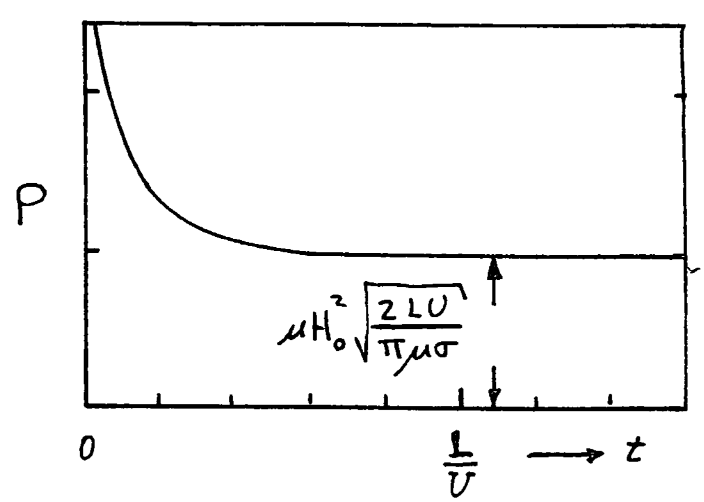
$$P = \int_0^{Ut} \frac{\sqrt{2}\,\mu H_o^2}{\sqrt{\dfrac{\pi\mu\sigma y}{U}}} + \int_{Ut}^{L} \frac{\sqrt{2}\,\mu H_o^2}{\sqrt{\pi\mu\sigma t}}\, dy \tag{7}$$

and this becomes

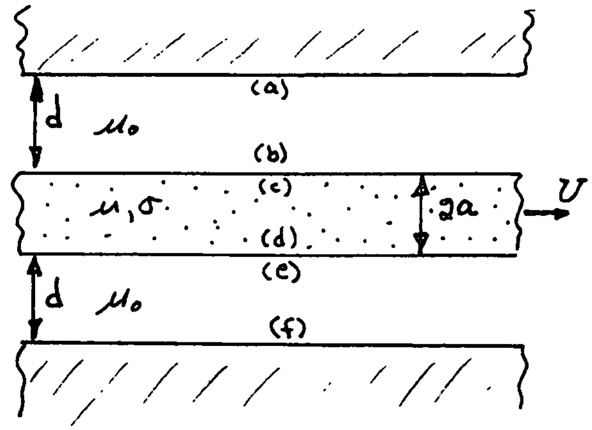
$$P = \frac{\sqrt{2}\,\mu H_o^2}{\sqrt{\pi\mu\sigma}}\left[2\sqrt{U}\sqrt{Ut} + \frac{1}{\sqrt{t}}(L - Ut)\right] \tag{8}$$

$$= \frac{\sqrt{2}\,\mu H_o^2}{\sqrt{\pi\mu\sigma}}\left[U\sqrt{t} + L/\sqrt{t}\right]$$

The time dependence of the total force is therefore as shown in the sketch.



<u>Prob. 6.10.1</u> Boundary conditions for the eigenmodes are homogeneous. In terms of the designations shown in the sketch,



$$\hat{H}_y^a = 0 \tag{1}$$

$$\hat{H}_y^b = \hat{H}_y^c \tag{2}$$

$$\hat{A}^b = \hat{A}^c \tag{3}$$

$$\hat{H}_y^d = \hat{H}_y^e \tag{4}$$

$$\hat{A}^d = \hat{A}^e \tag{5}$$

(6)

The bulk conditions are conveniently written with these conditions incorporated from the outset. In all three regions they are as given by Eq. (b) of Table 6.5.1 with suitable identification of properties and dimensions. In the upper air gap, it is the second equation that is required.

$$\hat{A}^b = \frac{\mu_o}{k} \coth kd \; \hat{H}_y^b \tag{7}$$

For the slab

$$\begin{bmatrix} \hat{A}^b \\ \hat{A}^e \end{bmatrix} = \frac{\mu}{\gamma} \begin{bmatrix} -\coth 2\gamma a & \dfrac{1}{\sinh 2\gamma a} \\ \dfrac{-1}{\sinh 2\gamma a} & \coth 2\gamma a \end{bmatrix} \begin{bmatrix} \hat{H}_y^b \\ \hat{H}_y^e \end{bmatrix} \tag{8}$$

while for the lower gap it is the first equation that applies

$$\hat{A}^e = -\frac{\mu_o}{k} \coth kd \; \hat{H}_y^e \tag{9}$$

Now, with Eqs. 7 and 9 used to evaluate Eq. 8, it follows that

$$\begin{bmatrix} -\dfrac{\mu_o}{k} \coth kd - \dfrac{\mu}{\gamma} \coth 2\gamma a & \dfrac{\mu}{\gamma} \dfrac{1}{\sinh 2\gamma a} \\ -\dfrac{\mu}{\gamma} \dfrac{1}{\sinh 2\gamma a} & \dfrac{\mu_o}{\gamma} \coth kd + \dfrac{\mu}{\gamma} \coth 2\gamma a \end{bmatrix} \begin{bmatrix} \hat{H}_y^b \\ \hat{H}_y^e \end{bmatrix} = 0 \tag{10}$$

Note that <u>both</u> of these equations are satisfied if $H_y^b = H_y^e$ so that

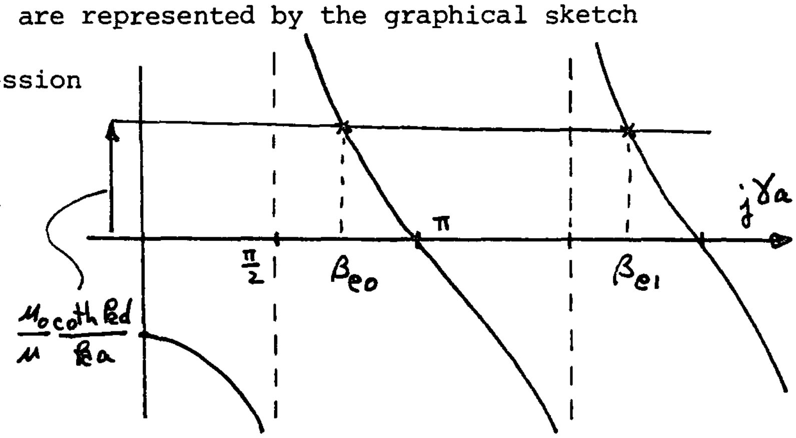
Prob. 6.10.1(cont.)

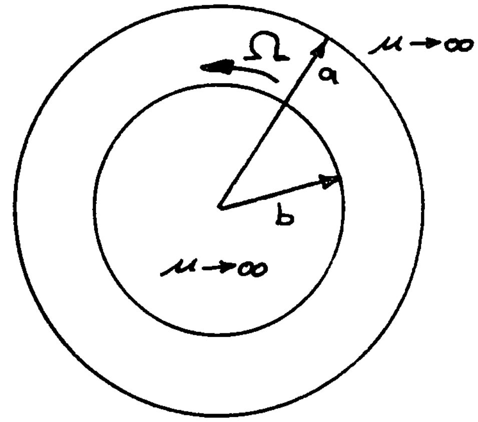
$$-\frac{\mu_o}{k}\coth kd-\frac{\mu}{\gamma}\left(\coth 2\gamma a \mp \frac{1}{\sinh 2\gamma a}\right)=0 \tag{11}$$

with the upper sign applying. Similarly, if $H_y^b = -H_y^e$, both expressions are satisfied and Eq. 11 is found with the lower sign applying. In this way, it has been shown that the eigenvalue equation that would be obtained by setting the determinant of the coefficients in Eq. 10 equal to zero can be factored into expressions that are given by Eq. 11. Further, it is seen that the roots given by these factors can respectively be identified with the even and odd modes. By using the identity $(\cosh x-1)/\sinh x = \tanh(x/2)$ and $(\cosh x+1)/\sinh x = \coth(x/2)$ it follows that the eigenvalue equations can be written as

$$\frac{\mu_o}{\mu}\frac{\coth kd}{ka} = \begin{cases} -\dfrac{\tan j\gamma a}{j\gamma a}\,; & \text{even} \\ \dfrac{\cot j\gamma a}{j\gamma a}\,; & \text{odd}\end{cases} \tag{12}$$

so that the expression for the odd solutions is the same as Eq. 6.10.1 with roots given by the graphical solution of Fig. 6.10.2 and eigenfrequencies given by Eq. 6.10.7. The even solutions are represented by the graphical sketch shown. The roots of this expression can be used in Eq. 6.10.7 to obtain the eigenfrequencies for these modes. Note that the dominant mode is odd, as would be expected for the tangential magnetic field associated with a current tending to be uniform over the sheet cross-section.





Prob. 6.10.2 (a) In Eq. (d) of Table 6.5.1, $\hat{H}_\theta^\alpha$ and $\hat{H}_\theta^\beta$ are zero so the determinant of the coefficients is zero. But, the resulting expression can be written out and then factored using the identity footnote to Table 2.16.2. This is the common denominator of the coefficients in the inverse matrix, Eq. (c) of that table. Thus, the required equation is (see Table 2.16.2 for denominators of f_m and g_m to which the determinant is proportional).

$$J_m(j\gamma a)H_m(j\gamma b) - J_m(j\gamma b)H_m(j\gamma a) = 0 \tag{1}$$

This can be written, using the recommended dimensionless parameters, and the definition of H_m in terms of N_m (Eq. 2.16.29) as

$$J_m[j(\gamma a)]\, N_m[j(\gamma a)\lambda] - J_m[j(\gamma a)\lambda]\, N_m[j(\gamma a)] = 0 \tag{2}$$

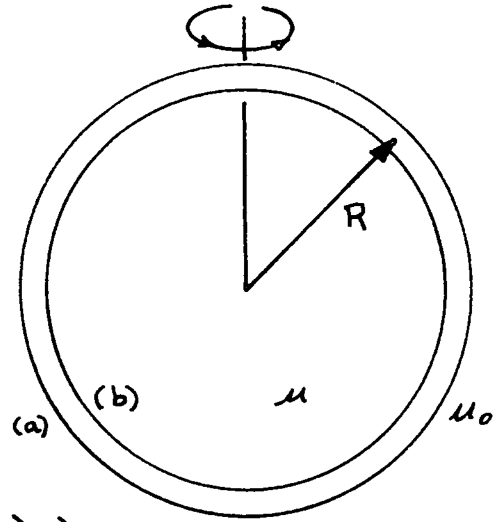
where $\lambda \equiv b/a$ ranges from 0 to 1 and $\gamma a \equiv \sqrt{j\mu\sigma a^2(\omega - m\Omega)}$.

(b) Given $\lambda \equiv b/a$ and the azimuthal wavenumber, m, Eq. 2 is a transcendental equation for the eigenvalues $\gamma a \equiv (\gamma a)_{mn}$ (which turn out to be real). The eigenfrequencies then follow. an

$$\omega_{mn} = m\Omega - j\frac{(\gamma a)_{mn}^2}{\mu\sigma a^2} \tag{3}$$

For example, for m=0 and 1, the roots to Eq. 2 are tabulated (Abramowitz, M. and Stegun, I.A., Handbook of Mathematical Functions, (National Bureau of Standards Applied Math Series, 1964) p. 415.) However, to make use of their tabulation, the eigenvalue should be made γb and the expression written as

$$J_m(j\gamma b)\, N_m\!\left[(j\gamma b)\tfrac{a}{b}\right] - J_m\!\left[(j\gamma b)\tfrac{a}{b}\right] N_m(j\gamma b) = 0 \tag{4}$$



Prob. 6.10.3 Solutions are of form

$$\psi = \mathrm{Re}\,\hat{\psi}(r)\,P_n^m \exp j(\omega t - m\phi)$$

(a) The first boundary condition is Eq. d, Table 6.3.1

$$\left(\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\sin\theta + \frac{\partial^2}{\partial\phi^2}\right)[\![H_\phi]\!]$$

$$= -\sigma_s R\sin\theta\frac{\partial}{\partial\phi}\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\phi}\right)B_r^a \tag{2}$$

With the substitution of the assumed form and $\hat{H}_\phi = jm\hat{\psi}/r\sin\theta$

$$\frac{jm}{R}(\hat{\psi}^a - \hat{\psi}^b)\left[\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}\sin\theta - m^2\right]\frac{P_n^m(\cos\theta)}{\sin\theta} \tag{3}$$

$$= -\sigma_s R\sin\theta\, m(\omega - m\Omega)\hat{B}_r^a P_n^m(\cos\theta)$$

In view of Eq. 2.16.31a, this becomes

$$-\frac{jm}{R}(\hat{\psi}^a - \hat{\psi}^b)n(n+1) = -\sigma_s R\hat{B}_r^a \tag{4}$$

The second boundary condition is

$$\hat{B}_r^a = \hat{B}_r^b \tag{5}$$

Prob. 6.10.3 (cont.)

Bulk relations are (Eq. (d) of Table 2.16.3)

$$\hat{B}_r^a = \frac{\mu_o(n+1)}{R}\hat{\psi}^b \tag{6}$$

for the exterior region and (Eq. (c) of Table 2.16.3)

$$\hat{B}_r^b = -\frac{\mu n}{R}\hat{\psi}^b \tag{7}$$

for the interior region.

These last three expressions, substituted into Eq. 4, then give

$$-\frac{jm}{R}n(n+1)\left[\frac{R}{\mu_o(n+1)} + \frac{R}{\mu n}\right]\hat{B}_r^a = -\sigma_s R m(\omega - m\Omega)\hat{B}_r^a \tag{8}$$

Thus, the desired eigenfrequency expression requires that the coefficients of $\hat{B}_r^a$ be zero. Solved for ω, this gives.

$$\omega = m\Omega + \frac{j}{\sigma_s R\mu_o}\left[n + \frac{(n+1)}{\mu/\mu_o}\right] \tag{9}$$

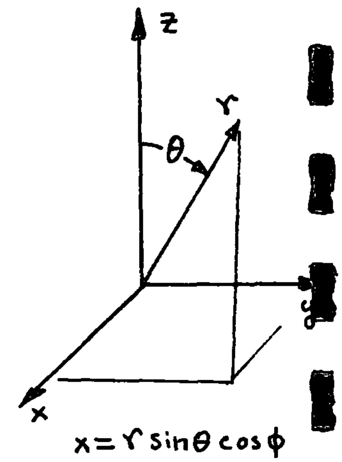
(b) A uniform field in the z direction superimposes on the homogeneous solution a field $\psi = -H_o z = -H_o r\cos\theta$. This has the same θ dependence as the mode m=0, n=1. Thus the mode necessary to satisfy the initial condition is (m,n) = (0,1) (Table 2.16.2) and the eigenfrequency is

$$\omega_{01} = \frac{j}{\sigma_s R\mu_o}\left(1 + \frac{2\mu_o}{\mu}\right) \tag{10}$$

Prob. 6.10.3 (cont.)

The response is a pure decay because there is no dependence of the excitation on the direction of rotation.

(c) With the initial field uniform perpendicular to the z axis there is a ϕ dependence.



$$\psi = -H_o x = -H_o r \sin\theta \cos\phi$$

This is the θ-ϕ dependence of the n=1, m=1 mode (Table 2.16.2).

So

$$\omega_{11} = \Omega + \frac{j}{\sigma_s R \mu_o}\left(1 + \frac{2\mu_o}{\mu}\right) \tag{11}$$

The decay rate is the same as before, but because the dipole field is now rotating, there is a real part.

Prob. 6.10.4 (a) The temporal modes exist even if the excitation is turned off. Hence, the denominator of Eq. 8 from Prob. 6.6.2 must vanish.

$$\frac{\mu_o}{\mu}\frac{f_o(a,R,k)}{k^2} = \frac{f_o(0,R,\gamma)}{\gamma^2} \tag{1}$$

(b) It is convenient to group

$$j\mu\sigma(\omega - kU) = S_n \tag{2}$$

Finding the roots S_n to Eq. 1 is tantamount to finding the desired eigen-frequencies because it then follows from Eq. 2 that

$$\omega_n = \frac{S_n}{j\mu\sigma} + kU \tag{3}$$

Note that for S_n real both sides of Eq. 1 are real. Thus, a graphical procedure can be used to find these roots.

Prob. 6.10.5 Even with nonuniform conductivity and velocity, Eq. 6.5.3 describes the vector potential. For the z component it follows that

$$\frac{1}{\mu\sigma}\nabla^2 A = \frac{\partial A}{\partial t} + U\frac{\partial A}{\partial y} \tag{1}$$

Thus, the complex amplitude satisfies the equation

$$\frac{d^2 A}{dx^2} - \gamma^2 A = 0\ ;\ \gamma^2(x) \equiv k^2 + j\mu\sigma(x)\left[\omega - kU(x)\right] \tag{2}$$

On the infinitely permeable walls, $H_y=0$ and so

$$\frac{dA}{dx}(\ell) = 0\ ;\ \frac{dA}{dx}(0) = 0 \tag{3}$$

Because Eq. 1 applies over the entire interval $0 < x < a+d \equiv \ell$, there is no need to use a piece-wise continuous representation. Multiply Eq. 2 by another eigenmode, $\hat{A}_m$, and integrate by parts to obtain

$$\hat{A}_m \frac{d\hat{A}_n}{dx}\Big|_0^{\ell} - \int_0^{\ell}\left(\frac{d\hat{A}_m}{dx}\frac{d\hat{A}_n}{dx} + \gamma_n^2 \hat{A}_m \hat{A}_n\right)dx = 0 \tag{4}$$

With the roles of m and n reversed, these same steps are carried out and the result subtracted from Eq. 4.

$$\left[\hat{A}_m \frac{d\hat{A}_n}{dx} - \hat{A}_n \frac{d\hat{A}_m}{dx}\right]_0^{\ell} - \int_0^{\ell}(\gamma_n^2 - \gamma_m^2)\hat{A}_m \hat{A}_n\, dx = 0 \tag{5}$$

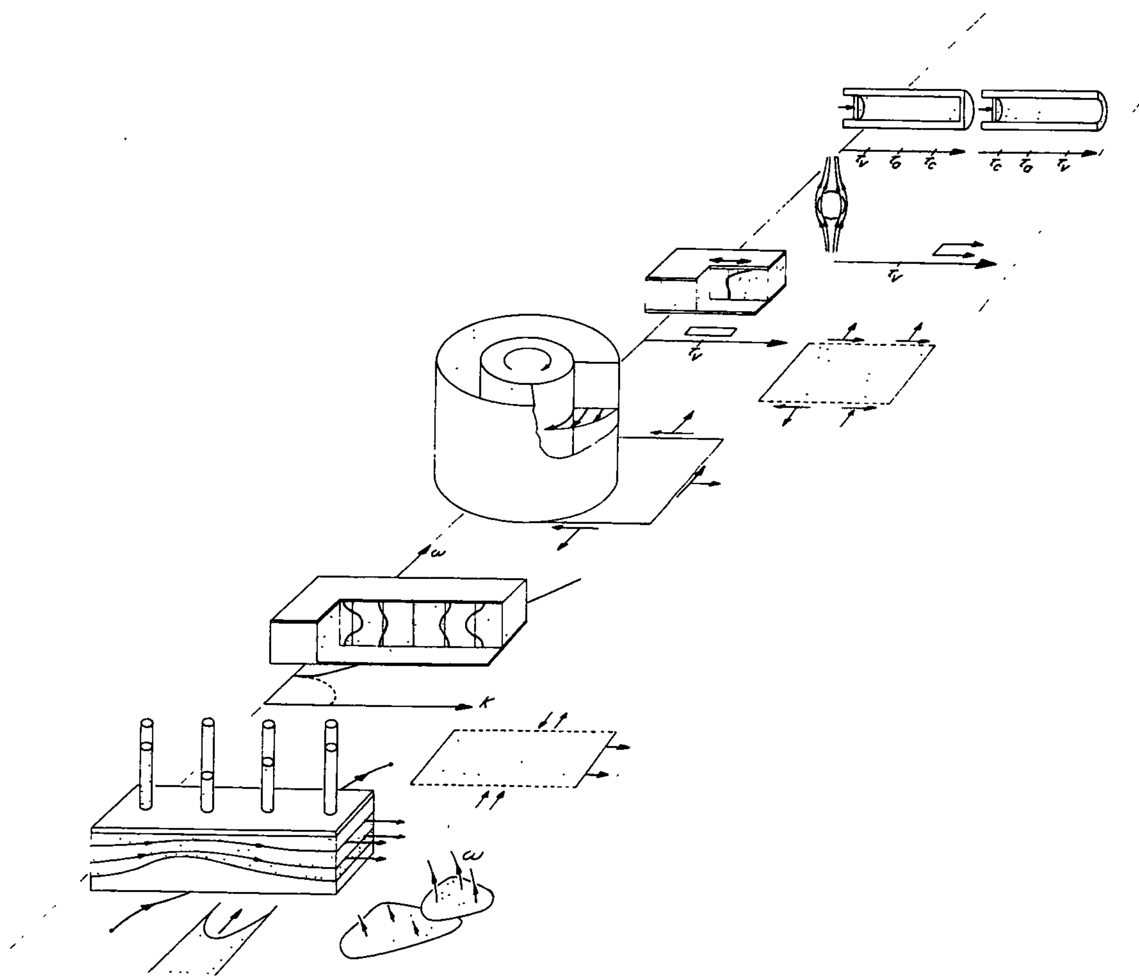
Note that by definition, $\gamma_n^2 - \gamma_m^2 = j\mu\sigma\,(\omega_n - \omega_m)$

In view of the boundary conditions applying at x=0 and x= ℓ , Eq. , the required orthogonality condition follows.

$$(\omega_n - \omega_m)\int_0^{\ell}\sigma(x)\hat{A}_m \hat{A}_n\, dx \tag{6}$$

7

Laws, Approximations and Relations of Fluid Mechanics



Prob. 7.2.1 If for a volume of fixed identity (Eq. 3.7.3)

$$\int_V \alpha_i \, dV = \text{constant} \tag{1}$$

then

$$\frac{d}{dt}\int \alpha_i \, dV = 0 \tag{2}$$

From the scalar form of the Leibnitz rule (Eq. 2.6.5 with $\varsigma \to \alpha_i$)

$$\int_V \frac{\partial \alpha_i}{\partial t} dV + \oint_S \alpha_i \bar{v}\cdot\bar{n}\, da = 0 \tag{3}$$

where $\bar{v}$ is the velocity of the material supporting the property α_i. With the use of the Gauss theorem on the surface integral

$$\int_V \left[\frac{\partial \alpha_i}{\partial t} + \nabla\cdot(\alpha_i \bar{v})\right] dV = 0 \tag{4}$$

Because the volume of fixed identity is arbitrary

$$\frac{\partial \alpha_i}{\partial t} + \nabla\cdot \alpha_i \bar{v} = 0 \tag{5}$$

Now, if $\alpha_i = \rho\beta_i$, then Eq. (5) becomes

$$\rho\frac{\partial \beta_i}{\partial t} + \beta_i\frac{\partial \rho}{\partial t} + \beta_i \nabla\cdot\rho\bar{v} + \rho\bar{v}\cdot\nabla\beta_i = 0 \tag{6}$$

The second and third terms cancel by virtue of mass conservation, Eq. 7.2.3, leaving

$$\frac{\partial \beta_i}{\partial t} + \bar{v}\cdot\nabla\beta_i = 0 \tag{7}$$

Prob. 7.6.1 To linear terms, the normal vector is

$$\bar{n} = \bar{i}_x - \frac{\partial \xi}{\partial y}\bar{i}_y - \frac{\partial \xi}{\partial z}\bar{i}_z \tag{1}$$

and inserted into Eq. 7.6.12, this gives the surface force density to linear terms

$$(\bar{T}_s)_x = -\gamma\left(-\frac{\partial^2 \xi}{\partial y^2} - \frac{\partial^2 \xi}{\partial z^2}\right) \tag{2}$$

<u>Prob. 7.6.2</u> The initially spherical surface has a position represented by

$$F = r - (R + \xi(\theta,\phi,t)) = 0 \tag{1}$$

Thus, to linear terms in the amplitude, ξ , the normal vector is

$$\bar{n} = -\frac{\nabla F}{|\nabla F|} \approx \bar{i}_r - \frac{1}{R}\frac{\partial \xi}{\partial \theta}\bar{i}_\theta - \frac{1}{R\sin\theta}\frac{\partial \xi}{\partial \phi}\bar{i}_\phi \tag{2}$$

It follows from the divergence operator in spherical coordinates that

$$\nabla\cdot\bar{n} = \frac{1}{r^2\sin\theta}\left[\frac{\partial r^2\sin\theta}{\partial r} - \frac{\partial}{\partial\theta}\left(\frac{r\sin\theta}{R}\frac{\partial\xi}{\partial\theta}\right) - \frac{\partial}{\partial\phi}\frac{r}{R\sin\theta}\frac{\partial\xi}{\partial\phi}\right] \tag{3}$$

Evaluation of Eq. 3 using the approximation that

$$\frac{1}{r} \approx \frac{1}{R} - \frac{\xi}{R^2} \tag{4}$$

therefore gives

$$(\tilde{T}_s)_r = \gamma\left[-\frac{2}{R} + \frac{2\xi}{R^2} + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\xi}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2\xi}{\partial\phi^2}\right] \tag{5}$$

where terms that are quadradic in ξ have been dropped.

To obtain a convenient complex amplitude representation, where $\xi = Re\,\tilde{\xi}\,P_n^m(\cos\theta)\exp(-jm\phi)$, use is made of the relation, Eq. 2.16.31,

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}P_n^m(\cos\theta)\right) - \frac{m^2}{\sin^2\theta}P_n^m(\cos\theta) = -n(n+1) \tag{6}$$

Thus, the complex amplitude of the surface force density due to surface tension is

$$(\tilde{T}_s)_r = -\frac{\gamma}{R^2}\left[(n-1)(n+2)\right]\tilde{\xi} \tag{7}$$

Actually, Eqs. 2 and 3 show that $\bar{T}_s$ also has θ and ϕ components (to linear terms in ξ). Because the surface force density is always normal to the interface, these components are balanced by pressure forces from the fluid to either side of the interface. To linear terms, the radial force balance represents the balance in the normal direction while the θ and ϕ components represent the shear balance. For an inviscid fluid it is not appropriate to include any shearing surface force density, so the stress equilibrium equations written to linear terms in the θ and ϕ directions must automatically balance.

Prob. 7.6.3 Mass conservation requires that

$$\frac{4}{3}\pi\xi_1^3 + \frac{4}{3}\pi\xi_2^3 = 2\left(\frac{4}{3}\pi\xi_o^3\right) \Rightarrow \xi_1^3 + \xi_2^3 = 2\xi_o^3 \tag{1}$$

With the pressure outside the bubbles defined as p_o, the pressures inside the respective bubbles are

$$p_a - p_o = \frac{2\gamma}{\xi_1} \quad ; \quad p_b - p_o = \frac{2\gamma}{\xi_2} \tag{2}$$

so that the pressure difference driving fluid between the bubbles once the valve is opened is

$$p_a - p_b = 2\gamma\left[\frac{1}{\xi_1} - \frac{1}{\xi_2}\right] \tag{3}$$

The flow rate between bubbles given by differentiating Eq. 1 is then equal to Q_v and hence to the given expression for the pressure drop through the connecting tubing.

$$Q_v = -\frac{4}{3}\pi\, 3\xi_1^2\frac{d\xi_1}{dt} = \frac{\pi R^4}{8\eta\ell}(p_a - p_b) = \frac{\pi R^4}{8\eta\ell}2\gamma\left[\frac{1}{\xi_1} - \frac{1}{\xi_2}\right] \tag{4}$$

Thus, the combination of Eqs. 1 and 4 give a first order differential equation describing the evolution of ξ_1 or ξ_2. In normalized terms, that expression is

$$\frac{d\xi_1}{dt} = \frac{1}{\xi_1^2}\left[\frac{1}{(2-\xi_1^3)^{1/3}} - \frac{1}{\xi_1}\right] \tag{5}$$

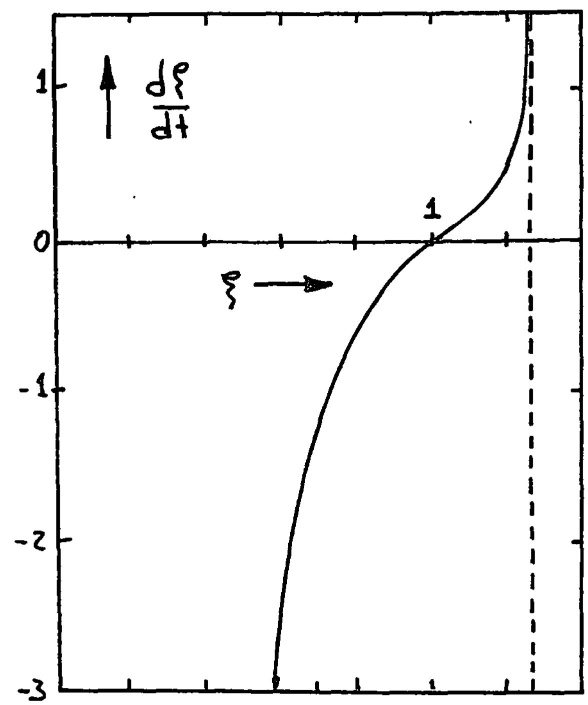
where

$$\xi_1 = \|\xi_1\|\,\xi_o \quad , \quad t = \underline{t}\left[\frac{16\eta\ell\xi_o^4}{R^4\gamma}\right]$$

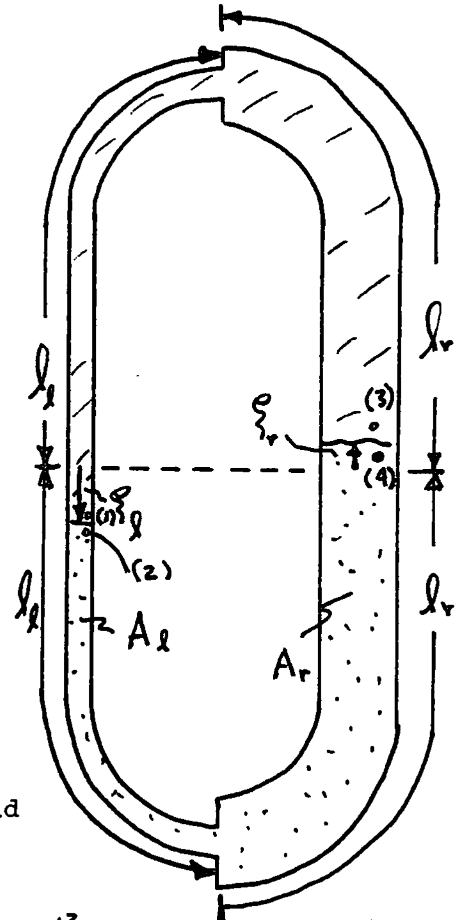
Thus, the velocity is a function of ξ_1, and can be pictured as shown in the figure. It is therefore evident that if ξ_1 increases slightly, it will tend to further increase. The static equilibrium at $\xi_1 = \xi_o$ is unstable. Physically this results from the fact that γ is constant. As the radius of curvature of a bubble decreases, the pressure increases and forces the air into the other bubble. Note that this is not what would be found if the bubbles were replaced by most elastic membranes. The example is useful for giving a reminder of what is implied by the concept of a surface tension. Of course, if the bubble

Prob. 7.6.3 (cont.)

can not be modelled as a layer of liquid with interior and exterior interfaces comprised of the same material, then the basic law may not apply. In the figure, note that all variables are normalized. The asymptote comes at the radius where the second bubble has completely collapsed.



<u>Prob. 7.8.1</u> Mass conservation for the lower fluid is represented by



$$[A_\ell(\ell_\ell-\xi_\ell)+A_r(\ell_r+\xi_r)]\rho_b = M_1 \quad (1)$$

and for the upper fluid by

$$[A_\ell(\ell+\xi_\ell)+A_r(\ell_r-\xi_r)]\rho_a = M_2 \quad (2)$$

With the assumption that the velocity has a uniform profile over a given cross-section, it follows that

$$v_\ell = \frac{A_r}{A_\ell} v_r \quad (3)$$

while evaluation of Eqs. 1 and 2 gives

$$\xi_\ell = \frac{A_r}{A_\ell}\xi_r - \frac{M_1}{\rho_b A_\ell} + \frac{A_r}{A_\ell}\ell_r + \ell_\ell \quad (4)$$

$$\xi_\ell = \frac{A_r}{A_\ell}\xi_r + \frac{M_2}{\rho_a A_\ell} - \frac{A_r}{A_\ell}\ell_r - \ell_\ell \quad (5)$$

Bernoulli's equation joining points (2) and (4) through the homogeneous fluid below gives

$$p_2 - \rho_b g\xi_\ell + \frac{1}{2}\rho_b\left(\frac{d\xi_\ell}{dt}\right)^2 - \rho_b(\ell_\ell-\xi_\ell)\frac{d^2\xi_\ell}{dt^2} = p_4 + \rho_b g\xi_r + \frac{1}{2}\rho_b\left(\frac{d\xi_r}{dt}\right)^2 + \rho_b(\ell+\xi_r)\frac{d^2\xi_r}{dt^2} \quad (6)$$

where the approximation made in integrating the inertial term through the transition region should be recognized. Similarly, in the upper fluid,

$$p_3 + \rho_a g\xi_r + \frac{1}{2}\rho_a\left(\frac{d\xi_r}{dt}\right)^2 - \rho_a(\ell_r-\xi_r)\frac{d^2\xi_r}{dt^2} = p_1 - \rho_a g\xi_\ell + \frac{1}{2}\rho_a\left(\frac{d\xi_\ell}{dt}\right)^2 + \rho_a(\ell_\ell+\xi_\ell)\frac{d^2\xi_\ell}{dt^2} \quad (7)$$

These expressions are linked together at the interfaces by the stress-balance and continuity boundary conditions.

$$p_1 = p_2 \;,\; p_3 = p_4 \;,\; v_3 = v_4 \;,\; v_1 = v_2 \quad (8)$$

Thus, subtraction of Eqs. 6 and 7 gives

$$[\rho_b(\xi_r+\ell_r) - \rho_a(\ell_r-\xi_r)]\frac{d^2\xi_r}{dt^2} + \frac{1}{2}(\rho_b-\rho_a)\left(\frac{d\xi_r}{dt}\right)^2 + (\rho_b-\rho_a)g\xi_r$$
$$= [-\rho_b(\ell_\ell-\xi_\ell) - \rho_a(\ell_\ell+\xi_\ell)]\frac{d^2\xi_\ell}{dt^2} + \frac{1}{2}(\rho_b-\rho_a)\left(\frac{d\xi_\ell}{dt}\right)^2 - (\rho_b-\rho_a)g\xi_\ell \quad (9)$$

Prob. 7.8.1(cont.)

Provided that the lengths $\ell_r \gg \xi_r$ and $\ell_\ell \gg \xi_\ell$, the equation of motion therefore takes the form

$$m \frac{d^2\xi_r}{dt^2} + \frac{1}{2}(\rho_b - \rho_a)\left[\left(\frac{d\xi_r}{dt}\right)^2 - \left(\frac{d\xi_\ell}{dt}\right)^2\right] + K\xi_r = 0 \tag{10}$$

where

$$m \equiv \frac{A_r}{A_\ell}\left[\rho_b(\ell_\ell - \xi_\ell) + \rho_a(\ell_\ell + \xi_\ell)\right] + \rho_b(\ell_r + \xi_r) + \rho_a(\ell_r - \xi_r);\ K \equiv g\left(1 + \frac{A_r}{A_\ell}\right)(\rho_b - \rho_a)$$

For still smaller amplitude motions, this expression becomes

$$\left(\frac{A_r}{A_\ell}\ell_\ell + \ell_r\right)(\rho_b + \rho_a)\frac{d^2\xi_r}{dt^2} + g\left(1 + \frac{A_r}{A_\ell}\right)(\rho_b - \rho_a)\xi_r = 0 \tag{11}$$

Thus, the system is stable if $\rho_b > \rho_a$ and given this condition, the natural frequencies are

$$\omega = \left[\frac{g(\rho_b - \rho_a)\left(1 + \frac{A_r}{A_\ell}\right)}{(\rho_b + \rho_a)\left(\frac{A_r}{A_\ell}\ell_\ell + \ell_r\right)}\right]^{1/2} \tag{12}$$

To account for the geometry, this expression obscures the simplicity of what it represents. For example, if the tube is of uniform cross-section, the lower fluid is water and the upper one air, $\rho_b \gg \rho_a$ and the natural frequency is independent of mass density (for the same reason that that of a rigid body pendulum is independent of mass, both the kinetic and potential energies are proportional to the density.) Thus, if $\ell = 1\text{m}$, the frequency is

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{2g}{\ell_\ell + \ell_r}} = \frac{1}{2\pi}\sqrt{\frac{2g}{\ell}} = 0.7\ \text{Hz}$$

Prob. 7.8.2 The problem is similar to the electrical conduction problem of current flow about an insulating cavity obstructing a uniform flow. Guess that the solution is the superposition of one consistent with the uniform flow at infinity and a dipole field to account for the boundary at r=R.

$$\Theta = -Ur\cos\theta + \frac{B}{r^2}\cos\theta \tag{1}$$

Because $v_r=0$ at r=R, $B=-R^3U/2$ and it follows that

$$\Theta = -Ur\cos\theta - \frac{R^3U}{2r}\cos\theta \tag{2}$$

$$v_\theta = -\frac{3}{2}U\sin\theta \tag{3}$$

Because the air is stagnant inside the shell, the pressure there is $P_{in} = P_2 - \rho g h$. At the stagnation point where the air encounters the shell and the hole communicates the interior pressure to the outside, the application of Bernoulli's equation gives

$$\frac{1}{2}\rho v_\theta^2 + \rho g h + p = P_2 \tag{4}$$

where h measures the height from the "ground" plane. In view of Eq. 3 and evaluated in spherical coordinates, this expression becomes

$$p - P_{in} = -\frac{1}{2}\rho v_\theta^2 = -\frac{9}{8}\rho U^2\sin^2\theta \qquad (5)\ (5)$$

To find the force tending to lift the shell off the "ground", compute

$$f_x = -\int_S p n_x da = -\int_{-\pi/2}^{\pi/2}\int_0^{\pi}(p - P_{in})n_x R^2\sin\theta\, d\theta\, d\phi \tag{6}$$

Because $n_x = \sin\theta$, this expression gives

$$f_x = -R^2 2\int_0^{\pi} -\frac{9}{8}\rho U^2\sin^4\theta\, d\theta \tag{7}$$

so that the force is

$$f_x = \rho\pi R^2\left(\frac{27}{64}\right)U^2 \tag{8}$$

Prob. 7.8.3 First, use Eq. 7.8.5 to relate the pressure in the essentially static interior region to the velocity in the cross-section A.

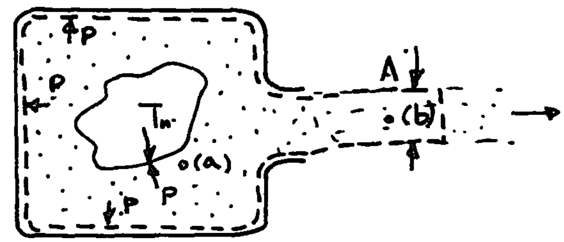
$$p_a + \frac{1}{2}\rho v_a^2 = p_b + \frac{1}{2}\rho v_b^2 \Rightarrow T_n + 0 = 0 + \frac{1}{2}\rho U^2 \tag{1}$$

Second, use the pressure from Sec. 7.4 to write the integral momentum conservation statement of Eq. 7.3.2 as

$$f_x = \oint_S p\, n_x\, da \quad = -\oint_S \rho v_x \bar{v}\cdot\bar{n}\, da = -\rho A U^2 \tag{2}$$

Applied to the surface shown in the figure, this equation becomes

$$f_x = -AU^2\rho \tag{3}$$



The combination of Eqs. 1 and 3 eliminates U as an unknown and gives the required result.

Prob. 7.9.1 See 8.17 for treatment of more general situation which becomes this one in the limit of no volume charge density.

Prob. 7.9.2 (a) By definition, given that the equilibrium velocity is $\bar{v} = \Omega r \bar{i}_\theta$, the vorticity follows as

$$\bar{\omega} = \nabla \times \bar{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\bar{i}_z = 2\Omega\,\bar{i}_z \tag{1}$$

(b) The equilibrium pressure follows from the radial component of the force equation

$$\rho\,(\bar{v}\cdot\nabla\bar{v})_r + \nabla p = 0 \Rightarrow -\rho\Omega^2 r + \frac{\partial p}{\partial r} = 0 \tag{2}$$

Integration gives

$$p = p_o + \frac{1}{2}\rho\Omega^2 r^2 \tag{3}$$

(c) With the laboratory frame of reference given the primed variables, the appropriate equations are

$$\nabla'\cdot\bar{v}' = 0 \tag{4}$$

$$\rho\left(\frac{\partial \bar{v}'}{\partial t'} + \bar{v}'\cdot\nabla\bar{v}'\right) + \nabla' p' = 0 \tag{5}$$

With the recognition that p' and v_θ' have equilibrium parts, these are first linearized to obtain

$$\frac{1}{r'}\frac{\partial}{\partial r'}(r' v_r') + \frac{1}{r'}\frac{\partial v_\theta'}{\partial \theta'} + \frac{\partial v_z'}{\partial z'} = 0 \tag{6}$$

$$\rho\left(\frac{\partial v_r'}{\partial t'} + \Omega\frac{\partial v_r'}{\partial \theta'} - 2\Omega v_\theta'\right) + \frac{\partial p'}{\partial r'} = 0 \tag{7}$$

$$\rho\left(\frac{\partial v_\theta'}{\partial t'} + \Omega\frac{\partial v_\theta'}{\partial \theta'} + 2\Omega v_r'\right) + \frac{1}{r}\frac{\partial p'}{\partial \theta'} = 0 \tag{8}$$

$$\rho\frac{\partial v_z'}{\partial t'} + \frac{\partial p'}{\partial z'} = 0 \tag{9}$$

The transformation of the derivatives is facilitated by the diagram of the dependences of the independent variables given to the right. Thus

$$\begin{tikzcd} r \arrow[r] & r' \\ \theta \arrow[r] \arrow[rdd] & \theta' \\ z \arrow[r] & z' \\ t \arrow[r] & t' \end{tikzcd}$$

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t}\frac{\partial t}{\partial t'} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial t'} = \frac{\partial}{\partial t} - \Omega\frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial r'} = \frac{\partial}{\partial r}, \text{ etc.} \tag{10}$$

Because the variables in Eqs. 6-9 are already linearized, the perturbation

Prob. 7.9.2 (cont.)

part of the azimuthal velocity in the laboratory frame is the same as that in the rotating frame. Thus

$$v_r' = v_r\,,\ v_\theta' \equiv \Omega r + v_\theta'\Big|_{part} = \Omega r + v_\theta\,,\ v_z' = v_z\,,\ p' = p \tag{10}$$

Expressed in the rotating frame of reference, Eqs. 6-9 become

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \tag{11}$$

$$\rho\left(\frac{\partial v_r}{\partial t} - 2\Omega v_\theta\right) + \frac{\partial p}{\partial r} = 0 \tag{12}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + 2\Omega v_r\right) + \frac{1}{r}\frac{\partial p}{\partial \theta} = 0 \tag{13}$$

$$\rho\frac{\partial v_z}{\partial t} + \frac{\partial p}{\partial z} = 0 \tag{14}$$

(d) In the *rotating frame* of reference, it is now assumed that variables take the complex amplitude form

$$\begin{bmatrix} \bar{v} \\ p \end{bmatrix} = \mathcal{R}_e \begin{bmatrix} \hat{\bar{v}} \\ \hat{p} \end{bmatrix} e^{j(\omega t - m\theta - kz)} \tag{15}$$

Then, it follows from Eqs. 22-24 that

$$\hat{v}_r = \frac{1}{\rho}\,\frac{\frac{2jm}{r}\Omega\hat{p} - j\omega\frac{d\hat{p}}{dr}}{(2\Omega)^2 - \omega^2} \tag{16}$$

$$\hat{v}_\theta = -\frac{1}{\rho}\,\frac{\frac{m\omega}{r}\hat{p} - 2\Omega\frac{d\hat{p}}{dr}}{(2\Omega)^2 - \omega^2} \tag{17}$$

$$\hat{v}_z = \frac{k}{\omega\rho}\hat{p} \tag{18}$$

Substitution of these expressions into the continuity equation, Eq. 11, then gives the desired expression for the complex pressure.

$$r^2\frac{d^2\hat{p}}{dr^2} + r\frac{d\hat{p}}{dr} - \hat{p}\left(m^2 + \gamma^2 r^2\right) = 0 \tag{19}$$

where

Prob. 7.9.2 (cont.)

$$\gamma^2 \equiv k^2\left[1 - \frac{(2\Omega)^2}{\omega^2}\right]$$

(e) With the replacement $k^2 \to \gamma^2$, Eq. 19 is the same expression for $\hat{p}$ in cylindrical coordinates as in Sec. 2.16. Either by inspection or by using Eq. 2.16.25, it follows that

$$\hat{p} = \hat{p}^{\alpha}\frac{H_m(j\gamma\beta)J_m(j\gamma r) - J_m(j\gamma\beta)H_m(j\gamma r)}{H_m(j\gamma\beta)J_m(j\gamma\alpha) - J_m(j\gamma\beta)H_m(j\gamma\alpha)}$$
$$+\hat{p}^{\beta}\frac{J_m(j\gamma\alpha)H_m(j\gamma r) - H_m(j\gamma\alpha)J_m(j\gamma r)}{J_m(j\gamma\alpha)H_m(j\gamma\beta) - H_m(j\gamma\alpha)J_m(j\gamma\beta)} \tag{20}$$

From Eq. 16, first evaluated using this expression and then evaluated at $r = \alpha$ and $r = \beta$ respectively, it follows that

$$\begin{bmatrix} \hat{v}_r^{\alpha} \\ \\ \hat{v}_r^{\beta} \end{bmatrix} = \frac{j\omega}{\rho(4\Omega^2 - \omega^2)}\begin{bmatrix} f_m(\beta,\alpha,\gamma) + \dfrac{2\Omega m}{\omega\alpha} & g_m(\alpha,\beta,\gamma) \\ \\ g_m(\beta,\alpha,\gamma) & f_m(\alpha,\beta,\gamma) + \dfrac{2m\Omega}{\beta\omega} \end{bmatrix}\begin{bmatrix} \hat{p}^{\alpha} \\ \\ \hat{p}^{\beta} \end{bmatrix} \tag{21}$$

The inverse of this is the desired transfer relation.

$$\begin{bmatrix} \hat{p}^{\alpha} \\ \\ \hat{p}^{\beta} \end{bmatrix} = \frac{\rho(4\Omega^2 - \omega^2)}{j\omega D}\begin{bmatrix} f_m(\alpha,\beta,\gamma) + \dfrac{2m\Omega}{\beta\omega} & -g_m(\alpha,\beta,\gamma) \\ \\ -g_m(\beta,\alpha,\gamma) & f_m(\beta,\alpha,\gamma) + \dfrac{2\Omega m}{\omega\alpha} \end{bmatrix}\begin{bmatrix} \hat{v}^{\alpha} \\ \\ \hat{v}^{\beta} \end{bmatrix} \tag{22}$$

where

$$D \equiv \left[f_m(\beta,\alpha,\gamma) + \frac{2\Omega m}{\omega\alpha}\right]\left[f_m(\alpha,\beta,\gamma) + \frac{2m\Omega}{\beta\omega}\right] - g_m(\beta,\alpha,\gamma)g_m(\alpha,\beta,\gamma)$$

<u>Prob. 7.11.1</u> For a weakly compressible gas without external force densities, the equations of motion are Eqs. 7.1.3, 7.4.4 (with $\vec{F}_{ex}=0$) and Eq. 7.10.3.

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\bar{v} = 0 \tag{1}$$

$$\rho\left[\frac{\partial\bar{v}}{\partial t} + \bar{v}\cdot\nabla\bar{v}\right] + \nabla p = 0 \tag{2}$$

$$\rho = \rho_o + (p - p_r)/a^2 \tag{3}$$

where ρ_o, a^2 and p_r are constants determined by the static equilibrium. With primes used to indicate perturbations from this equilibrium, the linearized forms of these expressions are

$$\frac{1}{a^2}\frac{\partial p'}{\partial t} + \rho_o\nabla\cdot\bar{v}' = 0 \tag{4}$$

$$\rho_o\frac{\partial\bar{v}'}{\partial t} + \nabla p' = 0 \tag{5}$$

where Eqs. 1 and 3 have been combined.

The divergence of Eq. 5 combines with the time derivative of Eq. 4 to eliminate $\nabla\cdot\bar{v}'$ and give an expression for p' alone.

$$\frac{1}{a^2}\frac{\partial^2 p'}{\partial t^2} = \nabla^2 p' \tag{6}$$

For solutions of the form $p = \mathcal{R}e\,\hat{p}(r)P_n^m(\cos\theta)e^{j(\omega t - m\phi)}$, Eq. 6 reduces to (See Eqs. 2.16.30-2.16.34)

$$P_n^m\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\hat{p}}{dr}\right) + \frac{\hat{p}}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP_n^m}{d\theta}\right) - \frac{m^2P_n^m\hat{p}}{r^2\sin^2\theta} + \frac{\omega^2}{a^2}P_n^m\hat{p} = 0 \tag{7}$$

In view of Eq. 2.16.31, the second and third terms are $-n(n+1)P_n^m\hat{p}$ so that this expression reduces to

$$r^2\frac{d^2\hat{p}}{dr^2} + 2r\frac{d\hat{p}}{dr} + \left[\frac{\omega^2r^2}{a^2} - n(n+1)\right]\hat{p} = 0 \tag{8}$$

Given the solutions to this expression, it follows from Eq. 5 that

$$\hat{v}_r = \frac{j}{\omega\rho_o}\frac{d\hat{p}}{dr} \tag{9}$$

provides the velocity components.

Substitution into Eq. 8 shows that with $u \equiv \frac{\omega r}{a}$, solutions to Eq. 8 are

$$j_n(u) \equiv \sqrt{\frac{\pi}{2u}}\,J_{n+\frac{1}{2}}(u)\quad ; \quad h_n(u) \equiv \sqrt{\frac{\pi}{2u}}\,H_{n+\frac{1}{2}}(u)$$

(j_n and h_n are spherical Bessel functions of first and third kind. See Abramowitz, M. and Stegun, I.A., Handbook of Mathematical Functions, National

Prob. 7.11.1 (cont.)

Bureau of Standards, 1964, p437.) As is clear from its definition, $h_n(u)$ is singular as $u \to 0$.

The appropriate linear combination of these solutions can be written by inspection as

$$\hat{p} = \hat{p}^{\alpha}\frac{\left[\dfrac{j_n\left(\frac{\omega r}{a}\right)}{j_n\left(\frac{\omega\beta}{a}\right)} - \dfrac{h_n\left(\frac{\omega r}{a}\right)}{h_n\left(\frac{\omega\beta}{a}\right)}\right]}{\left[\dfrac{j_n\left(\frac{\omega\alpha}{a}\right)}{j_n\left(\frac{\omega\beta}{a}\right)} - \dfrac{h_n\left(\frac{\omega\alpha}{a}\right)}{h_n\left(\frac{\omega\beta}{a}\right)}\right]} + \hat{p}^{\beta}\frac{\left[\dfrac{j_n\left(\frac{\omega r}{a}\right)}{j_n\left(\frac{\omega\alpha}{a}\right)} - \dfrac{h_n\left(\frac{\omega r}{a}\right)}{h_n\left(\frac{\omega\alpha}{a}\right)}\right]}{\left[\dfrac{j_n\left(\frac{\omega\beta}{a}\right)}{j_n\left(\frac{\omega\alpha}{a}\right)} - \dfrac{h_n\left(\frac{\omega\beta}{a}\right)}{h_n\left(\frac{\omega\alpha}{a}\right)}\right]} \tag{10}$$

Thus, from Eq. 9 it follows that

$$\hat{v}_r = \frac{j}{\omega\rho_0}\left\{\frac{\omega}{a}\,\frac{j_n'\left(\frac{\omega r}{a}\right)h_n\left(\frac{\omega\beta}{a}\right) - h_n'\left(\frac{\omega r}{a}\right)j_n\left(\frac{\omega\beta}{a}\right)}{j_n\left(\frac{\omega\alpha}{a}\right)h_n\left(\frac{\omega\beta}{a}\right) - h_n\left(\frac{\omega\alpha}{a}\right)j_n\left(\frac{\omega\beta}{a}\right)}\,\hat{p}^{\alpha} - \frac{\omega}{a}\,\frac{j_n'\left(\frac{\omega r}{a}\right)h_n\left(\frac{\omega\alpha}{a}\right) - h_n'\left(\frac{\omega r}{a}\right)j_n\left(\frac{\omega\alpha}{a}\right)}{h_n\left(\frac{\omega\beta}{a}\right)j_n\left(\frac{\omega\alpha}{a}\right) - j_n\left(\frac{\omega\beta}{a}\right)h_n\left(\frac{\omega\alpha}{a}\right)}\,\hat{p}^{\beta}\right\} \tag{11}$$

where j_n' and h_n' signify derivatives with respect to the arguments. Evaluation of Eq. 11 at the respective boundaries gives transfer relations

$$\begin{bmatrix}\hat{v}_r^{\alpha}\\ \hat{v}_r^{\beta}\end{bmatrix} = \frac{j}{\omega\rho_0}\begin{bmatrix} f_n(\beta,\alpha) & g_n(\alpha,\beta)\\ g_n(\beta,\alpha) & f_n(\alpha,\beta)\end{bmatrix}\begin{bmatrix}\hat{p}^{\alpha}\\ \hat{p}^{\beta}\end{bmatrix} \tag{12}$$

where

$$f_n(x,y) \equiv -\frac{\omega}{a}\,\frac{h_n\left(\frac{\omega}{a}x\right)j_n'\left(\frac{\omega}{a}y\right) - j_n\left(\frac{\omega x}{a}\right)h_n'\left(\frac{\omega y}{a}\right)}{j_n\left(\frac{\omega}{a}x\right)h_n\left(\frac{\omega}{a}y\right) - j_n\left(\frac{\omega y}{a}\right)h_n\left(\frac{\omega}{a}x\right)}$$

$$g_n(x,y) \equiv -\frac{\omega}{a}\,\frac{h_n\left(\frac{\omega}{a}x\right)j_n'\left(\frac{\omega}{a}x\right) - j_n\left(\frac{\omega}{a}x\right)h_n'\left(\frac{\omega}{a}x\right)}{j_n\left(\frac{\omega}{a}x\right)h_n\left(\frac{\omega}{a}y\right) - j_n\left(\frac{\omega}{a}y\right)h_n\left(\frac{\omega}{a}x\right)}$$

Prob. 7.11.1 (cont.)

Inversion of these relations gives

$$\begin{bmatrix} \hat{p}^{\alpha} \\ \hat{p}^{\beta} \end{bmatrix} = -j\omega\rho_0 \frac{\begin{bmatrix} f_n(\alpha,\beta) & -g_n(\alpha,\beta) \\ -g_n(\beta,\alpha) & f_n(\beta,\alpha) \end{bmatrix}}{f_n(\beta,\alpha) f_n(\alpha,\beta) - g_n(\beta,\alpha) g_n(\alpha,\beta)} \begin{bmatrix} \hat{v}_r^{\alpha} \\ \hat{v}_r^{\beta} \end{bmatrix} \tag{13}$$

and this expression becomes

$$\begin{bmatrix} \hat{p}^{\alpha} \\ \hat{p}^{\beta} \end{bmatrix} = -j\omega\rho_0 \begin{bmatrix} F_n(\beta,\alpha) & G_n(\alpha,\beta) \\ G_n(\beta,\alpha) & F_n(\alpha,\beta) \end{bmatrix} \begin{bmatrix} \hat{v}_r^{\alpha} \\ \hat{v}_r^{\beta} \end{bmatrix} \tag{14}$$

where

$$F_n(x,y) \equiv \frac{a}{\omega} \frac{h_n\left(\frac{\omega y}{a}\right) j_n'\left(\frac{\omega x}{a}\right) - j_n\left(\frac{\omega y}{a}\right) h_n'\left(\frac{\omega x}{a}\right)}{j_n'\left(\frac{\omega x}{a}\right) h_n'\left(\frac{\omega y}{a}\right) - j_n'\left(\frac{\omega y}{a}\right) h_n'\left(\frac{\omega x}{a}\right)}$$

$$G_n(x,y) \equiv -\frac{a}{\omega} \frac{h_n\left(\frac{\omega x}{a}\right) j_n'\left(\frac{\omega x}{a}\right) - j_n\left(\frac{\omega x}{a}\right) h_n'\left(\frac{\omega x}{a}\right)}{h_n'\left(\frac{\omega x}{a}\right) j_n'\left(\frac{\omega y}{a}\right) - j_n'\left(\frac{\omega x}{a}\right) h_n'\left(\frac{\omega y}{a}\right)}$$

With a rigid wall at r=R it follows from Eq. 14 that there can then only be a response if

$$F_n(0,R) = \frac{a}{\omega} \frac{j_n\left(\frac{\omega R}{a}\right)}{j_n'\left(\frac{\omega R}{a}\right)} \longrightarrow \infty \tag{15}$$

so that the desired eigenvalue equation is

$$j_n'\left(\frac{\omega R}{a}\right) = 0 \tag{16}$$

This is easy to see without the transfer relations because in this case Eq. 10 is replaced by simply

$$\hat{p} = \hat{p}^{\alpha} \frac{j_n\left(\frac{\omega r}{a}\right)}{j_n\left(\frac{\omega R}{a}\right)} \tag{17}$$

so that it follows from Eq. 9 that

$$\hat{v}_r = \frac{j}{\omega\rho_0} \hat{p}^{\alpha} \left(\frac{\omega}{a}\right) \frac{j_n'\left(\frac{\omega r}{a}\right)}{j_n\left(\frac{\omega R}{a}\right)}$$

For $\hat{p}^{\alpha}$ to be finite at r=R but $\hat{v}_r^{\alpha} = 0$ there, Eq. 16 must hold. Roots to this expression are tabulated (Abromowitz and Stegun, p468).

Prob. 7.12.1 It follows from Eq. (f) of Table 7.9.1 in the limit $\beta \to 0$ that

$$\hat{p}^{\alpha} = j(\omega - kU)\rho F_m(0,R)\hat{v}_r^{\alpha} \tag{1}$$

where

$$F_m(0,R) \to \frac{J_m(j\gamma R)}{j\gamma R\, J_m'(j\gamma R)} \tag{2}$$

It follows that there can be a finite pressure response at the wall even if there is no velocity there if

$$\begin{aligned} \gamma R = 0 \quad &\Rightarrow \quad \omega - kU = \pm ak \quad (n=0) \\ J_m'(j\gamma R) = 0 \quad &\Rightarrow \quad j\gamma R = \alpha_n \ , \ n \neq 0, \pm 1, \pm 2 \cdots \end{aligned} \tag{3}$$

The zero mode is the principal mode (propagation down to zero frequency)

$$k = \frac{\omega}{U \pm a} \tag{4}$$

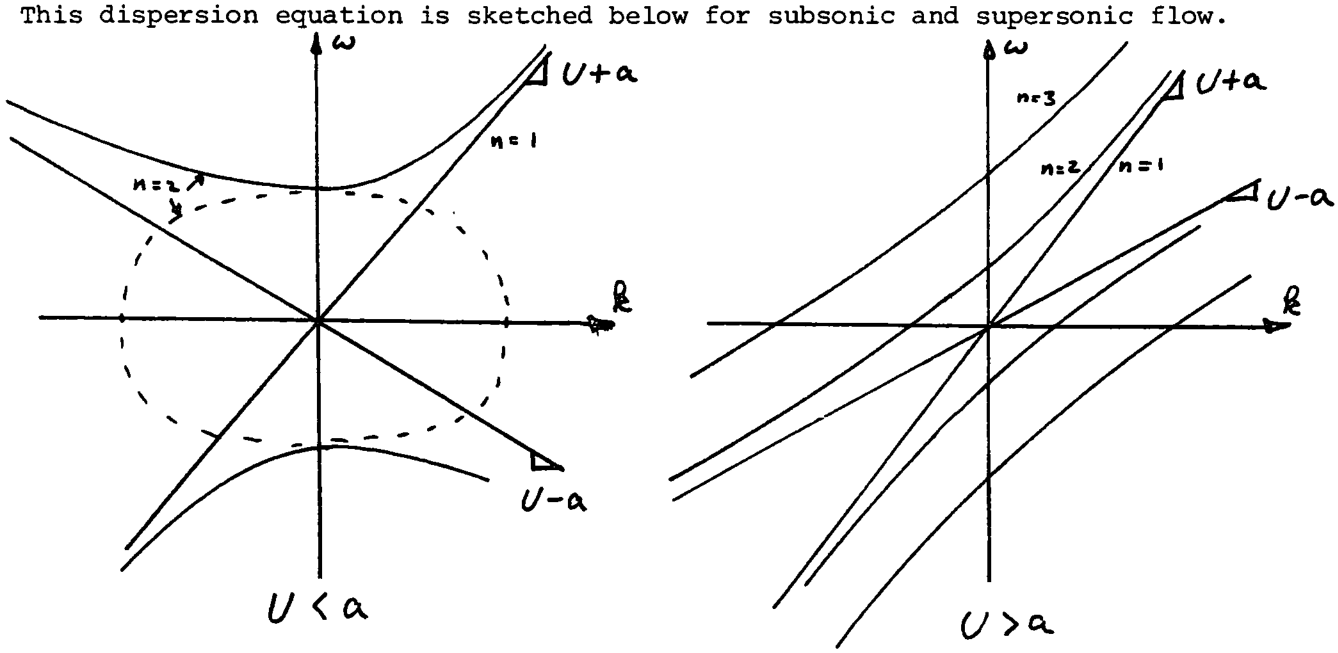
while the higher order modes have a dispersion equation that follows from the roots of Eq. 3b and the definition of γ .

$$-\frac{a^2\alpha_n^2}{R^2} = a^2k^2 - (\omega - kU)^2 \tag{5}$$

Solution of k gives the wavenumbers of the spatial modes

$$k = \frac{-\omega U \pm \sqrt{a^2\omega^2 - (a^2 - U^2)a^2\frac{\alpha_n^2}{R^2}}}{(a^2 - U^2)} \tag{6}$$

This dispersion equation is sketched below for subsonic and supersonic flow.



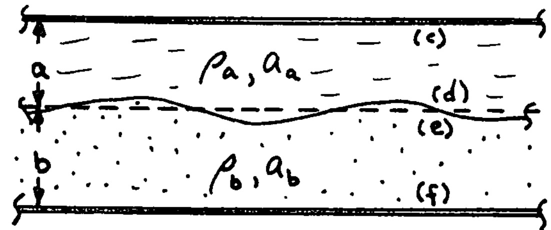
<u>Prob. 7.12.2</u> Boundary conditions are

$$\hat{v}_x^c = 0 \tag{1}$$

$$\hat{v}_x^d = \hat{v}_x^e \tag{2}$$

$$\hat{p}^d = \hat{p}^e \tag{3}$$

$$\hat{v}_x^f = 0 \tag{4}$$



With these conditions incorporated from the outset, the transfer relations (Eqs. (c) of Table 7.9.1) for the respective regions are

$$\begin{bmatrix} \hat{p}^c \\ \hat{p}^d \end{bmatrix} = \frac{j\omega\rho_a}{\gamma_a}\begin{bmatrix} -\coth\gamma_a a & \dfrac{1}{\sinh\gamma_a a} \\ \dfrac{-1}{\sinh\gamma_a a} & \coth\gamma_a a \end{bmatrix}\begin{bmatrix} 0 \\ \hat{v}_x^d \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} \hat{p}^d \\ \hat{p}^f \end{bmatrix} = \frac{j\omega\rho_b}{\gamma_b}\begin{bmatrix} -\coth\gamma_b b & \dfrac{1}{\sinh\gamma_b b} \\ \dfrac{-1}{\sinh\gamma_b b} & \coth\gamma_b b \end{bmatrix}\begin{bmatrix} \hat{v}_x^d \\ 0 \end{bmatrix} \tag{6}$$

where $\gamma_a^2 = k^2 - \omega^2/a_a^2$ and $\gamma_b^2 = k^2 - \omega^2/a_b^2$. By equating Eqs. 5b and 6a it follows that

$$j\frac{\omega\rho_a}{\gamma_a}\coth\gamma_a a = -j\frac{\omega\rho_b}{\gamma_b}\coth\gamma_b b \tag{7}$$

With the definitions of γ_a and γ_b, this expression is the desired dispersion equation relating ω and k. Given a real ω , the wavenumbers of the spatial modes are in general complex numbers satisfying the complex equation, Eq. 7. For long waves, a principal mode propagates through the system with a phase velocity that combines those of the two regions. That is, for $|\gamma_a a| \ll 1$ and $|\gamma_b b| \ll 1$ Eq. 7 becomes

$$\frac{\rho_a}{\gamma_a^2 a} = -\frac{\rho_b}{\gamma_b^2 b} \Rightarrow \frac{a}{\rho_a}\left[k^2 - \left(\frac{\omega}{a_a}\right)^2\right] = -\frac{b}{\rho_b}\left[k^2 - \left(\frac{\omega}{a_b}\right)^2\right] \tag{8}$$

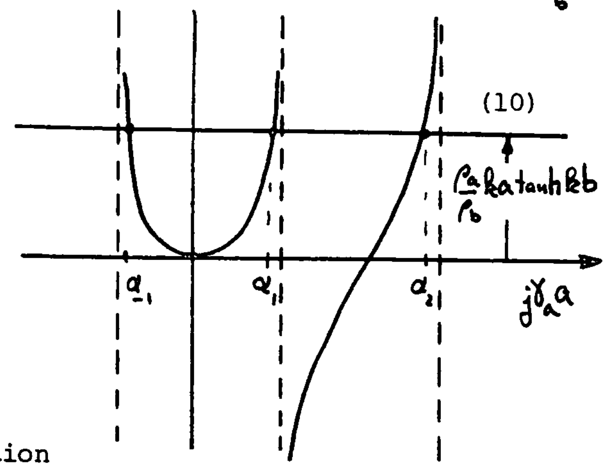
and it follows that

$$k = \pm\frac{\omega}{a_c} \quad ; \quad a_c \equiv \sqrt{\frac{\left[\dfrac{a}{\rho_a} + \dfrac{b}{\rho_b}\right]}{\left[\dfrac{a}{\rho_a a_a^2} + \dfrac{b}{\rho_b a_b^2}\right]}} \tag{9}$$

Prob. 7.12.2(cont.)

A second limit is of interest for propagation of acoustic waves in a gas over a liquid. The liquid behaves in a quasi-static fashion for the lowest order modes because on time scales of interest waves propagate through the liquid essentially instantaneously. Thus, the liquid acts as a massive load comprising one wall of a guide for the waves in the air. In this limit, $a_a \ll a_b$ and $k^2 \gg \frac{\omega^2}{a_b^2}$ $\Rightarrow \gamma_b \approx k$ and Eq. 7 becomes

$$\frac{\rho_b}{\rho_a(ka)}\coth kb = -\frac{\coth \gamma_a a}{\gamma_a a} \qquad (10)$$



This expression can be solved graphically, as illustrated in the figure, because Eq. 10 can be written so as to make evident real roots.

$$\frac{\rho_a}{\rho_b} ka \tanh kb = (j\gamma_a a)\tan(j\gamma_a a) \qquad (11)$$

Given these roots, it follows from the definition of γ_a that the wavenumbers of the associated spatial modes are

$$k = \pm\left(\frac{\omega^2}{a_a^2} - \frac{\alpha_n^2}{a^2}\right)^{1/2} \qquad (11)$$

Prob. 7.13.1 The objective here is to establish some rapport for the elastic solid. Whether subjected to shear or normal stresses, it can deform in such a way as to balance these stresses with no further displacements. Thus, it is natural to expect stresses to be related to displacements rather than velocities. (Actually strains rather than strain-rates.) That a linearized description does not differentiate between $\xi(\bar{r}_0,t)$ interpreted as the displacement of the particle that is at $\bar{r}_0$ or was at $\bar{r}_0$ (and is now at $\bar{r}_0+\bar{\xi}(\bar{r}_0,t)$) can be seen by simply making a Taylor's expansion.

$$\xi_i(\bar{r}_0+\bar{\xi},t) = \xi_i(\bar{r}_0) + \left.\frac{\partial \xi_i}{\partial x_j}\right|_{\bar{r}_0} \xi_j + \cdots \tag{1}$$

Terms that are quadradic or more in the components of $\bar{\xi}$ are negligible

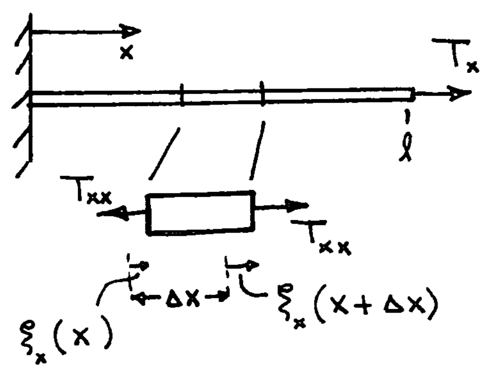
Because the measured result is observed for various spacings, d, the suggestion is that an incremental slice of the material, shown analogously in Fig. 7.13.1, can be described by

$$T_{zx} = G_s\left[\frac{\xi_z(x+\Delta x) - \xi_z(x)}{\Delta x}\right] \tag{2}$$

In the limit $\Delta x \to 0$, the one-dimensional shear-stress displacement relation follows

$$T_{zx} = G_s \frac{\partial \xi_z}{\partial x} \tag{3}$$

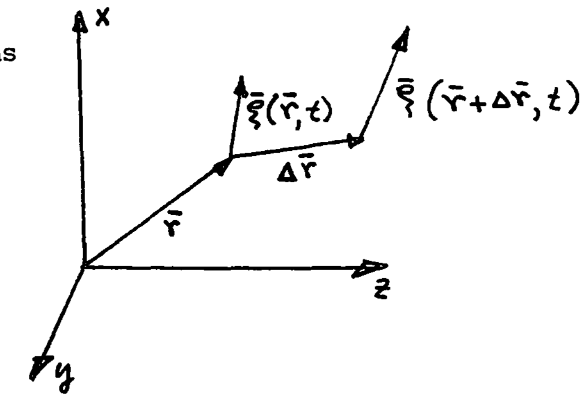
For dilatational motions, it is helpful to discern what can be expected by considering the one-dimensional extension of the thin rod shown in the sketch. That the measured result holds independent of the initial length, ℓ, suggests that the relation should hold for a section of length Δx as well. Thus,



$$T_{xx} = E_s\left[\frac{\xi_x(x+\Delta x) - \xi_x(x)}{\Delta x}\right] \tag{4}$$

In the limit $\Delta x \to 0$, the stress- displacement relation for a <u>thin</u> rod follows.

$$T_{xx} = E_s \frac{\partial \xi_x}{\partial x} \tag{5}$$



Prob. 7.14.1 Consider the relative deformations of material having the initial relative displacement $\Delta\bar{r}$, as shown in the sketch. Taylor's expansion gives

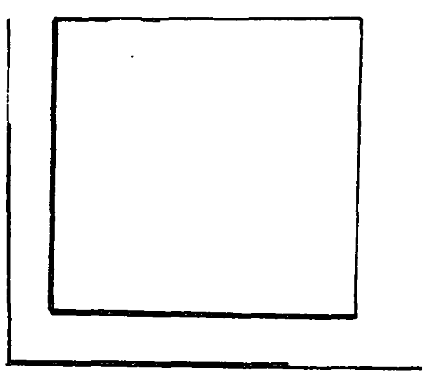
$$\xi_i(\bar{r}+\Delta\bar{r})-\xi_i(\bar{r})=\xi_i(\bar{r})+\frac{\partial\xi_i}{\partial x_j}\Delta x_j-\xi_i(\bar{r}) \quad (1)$$

Terms are grouped so as to identify the rotational part of the deformation and exclude it from the definition of the strain.

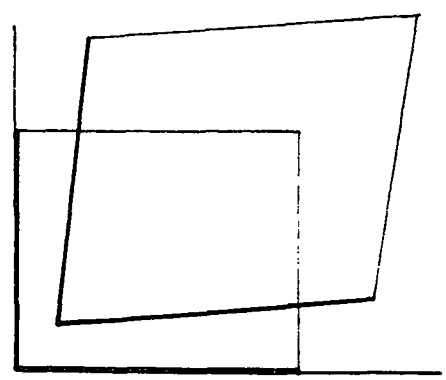
$$\xi_i(\bar{r}+\Delta\bar{r})-\xi_i(\bar{r})=\frac{1}{2}\left[\frac{\partial\xi_i}{\partial x_j}-\frac{\partial\xi_j}{\partial x_i}\right]\Delta x_j + e_{ij}\Delta x_j;\ e_{ij}\equiv\frac{1}{2}\left[\frac{\partial\xi_i}{\partial x_j}+\frac{\partial\xi_j}{\partial x_i}\right] \quad (2)$$

Thus, the strain is defined as describing that part of the deformation that can be expected to be directly related to the local stress.

• The sketches below respectively show the change in shape of a rectangle attached to the material as it suffers pure dilatational and shear deformations.



normal strain



shear strain

Prob. 7.15.1 Arguments follow those given, with $\overset{o}{e}_{ij} \rightarrow e_{ij}$. To make Eq. 6.5.17 become Eq. (b) of the table, it is clear that

$$k_1 - k_2 = 2G_r \;;\; k_2 = \lambda_s \tag{1}$$

The new coefficient is related to G and E by considering the thin rod experiment. Because the transverse stress components were zero, the normal component of stress and strain are related by

$$\begin{bmatrix} T_{xx} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} k_1 & k_2 & k_2 \\ k_2 & k_1 & k_2 \\ k_2 & k_2 & k_1 \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \end{bmatrix} \tag{2}$$

Given T_{xx}, the longitudinal and transverse strain components are determined from these three equations. Solution for e_{xx} gives

$$e_{xx} = \frac{(k_1 + k_2)}{k_1(k_1 + k_2) - 2k_2^2} T_{xx} \tag{3}$$

and comparison of this expression to that for the thin rod shows that

$$E_r = \frac{(2G_r + \lambda_s)(G_r + \lambda_s) - \lambda_s^2}{G_r + \lambda_s} \tag{4}$$

Solution of this expression for λ_s gives Eq. (f) of the table.

It also follows from Eqs. 2 that

$$-\frac{e_{yy}}{e_{xx}} = -\frac{(k_2^2 - k_1 k_2)}{(k_1 + k_2)(k_1 - k_2)} = \frac{k_2}{k_1 + k_2} \tag{5}$$

With k_1 and k_2 expressed using Eqs. 1 and then the expression for λ_s in terms of G and E, Eq. g of the table follows.

Prob. 7.15.2 In general

$$e_{ij} = a_{ik} a_{jl} e_{kl} \tag{1}$$

In particular, the sum of the diagonal elements in the primed frame is

$$e'_{nn} = a_{nk} a_{nl} e_{kl} \tag{2}$$

It follows from Eq. 3.9.14 and the definition of a_{ij} that $a_{ki}a_{kj} = \delta_{ij}$. Thus, Eq. 2 becomes the statement to be proven

$$e'_{nn} = \delta_{kl} e_{kl} = e_{nn} \tag{3}$$

Prob. 7.15.3 From Eq. 7.15.20 it follows that

$$S_{ij} = \begin{bmatrix} p & 0 & \frac{\eta U}{d} \\ 0 & p & 0 \\ \frac{\eta U}{d} & 0 & p \end{bmatrix} \tag{1}$$

Thus, Eq. 7.15.5 becomes

$$\begin{bmatrix} p-T & 0 & \frac{\eta U}{d} \\ 0 & p-T & 0 \\ \frac{\eta U}{d} & 0 & p-T \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \tag{2}$$

which reduces to

$$-(p-T)^3 + \left(\frac{\eta U}{d}\right)^2 (p-T) = 0 \tag{3}$$

Thus, the principal stresses are

$$T = p \quad , \quad T = p \pm \sqrt{\eta U/d} \tag{4}$$

From Eq. 7.15.5c it follows that

$$n_1 = \pm n_3 \tag{5}$$

so that the normal vectors to the two nontrivial principal planes are

$$\bar{n} = \frac{1}{\sqrt{2}}\left(\bar{i}_x \pm \bar{i}_z\right) \tag{6}$$

Prob. 7.16.1 Equation d of the table states Newton's law for incremental motions. Substitution of Eq. b for T_{ij} and of Eq. a for e_{ij} gives

$$\frac{\partial T_{ij}}{\partial x_j} = G_s \frac{\partial}{\partial x_j}\left(\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i}\right) + \lambda_s \frac{\partial}{\partial x_i}\left(\frac{\partial \xi_k}{\partial x_k}\right) \quad (1)$$

Manipulations are now made with the vector identity

$$\nabla \times \nabla \times \bar{\xi} = \nabla(\nabla \cdot \bar{\xi}) - \nabla^2 \bar{\xi} \quad (2)$$

in mind. In view of the desired form of the equation of motion, Eq. 1 is written as

$$\frac{\partial T_{ij}}{\partial x_j} = (2G_s + \lambda)\frac{\partial}{\partial x_i}\left(\frac{\partial \xi_k}{\partial x_k}\right) - 2G_s \frac{\partial}{\partial x_i}\left(\frac{\partial \xi_k}{\partial x_k}\right) + G_s \frac{\partial}{\partial x_j}\left(\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i}\right) \quad (3)$$

Half of the second term cancels with the last, so that the expression becomes

$$\frac{\partial T_{ij}}{\partial x_j} = (2G_s + \lambda)\frac{\partial}{\partial x_i}\left(\frac{\partial \xi_k}{\partial x_k}\right) - G_s\left[\frac{\partial}{\partial x_i}\left(\frac{\partial \xi_k}{\partial x_k}\right) - \frac{\partial^2 \xi_i}{\partial x_j \partial x_j}\right] \quad (4)$$

In vector form, this is equivalent to

$$\nabla \cdot \bar{\bar{T}} = (2G_s + \lambda_s)\nabla(\nabla \cdot \bar{\xi}) - G_s[\nabla(\nabla \cdot \bar{\xi}) - \nabla^2 \bar{\xi}] \quad (5)$$

Finally, the identity of Eq. 2 is used to obtain

$$\nabla \cdot \bar{\bar{T}} = (2G_s + \lambda_s)\nabla(\nabla \cdot \bar{\xi}) - G_s \nabla \times \nabla \times \bar{\xi} \quad (6)$$

and the desired equation of incremental motion is obtained.

Prob. 7.18.1 Because $\bar{A}_s$ is solenoidal, $\nabla \times \nabla \times \bar{A}_s = -\nabla^2 \bar{A}_s$ and so substitution of $\bar{\xi}$ into the equation of motion gives

$$\nabla \times \left[\rho \frac{\partial^2 \bar{A}_s}{\partial t^2} - G_s \nabla^2 \bar{A}_s - \bar{G}\right] - \nabla\left[\rho \frac{\partial^2 \psi_s}{\partial t^2} - (2G + \lambda)\nabla^2 \psi_s + \mathcal{E}\right] = 0 \quad (1)$$

The equation is therefore satisfied if

$$\frac{\partial^2 \bar{A}_s}{\partial t^2} = v_s^2 \nabla^2 \bar{A}_s + \frac{\bar{G}}{\rho} \quad ; \quad v_s \equiv \sqrt{G_s/\rho} \quad (2)$$

$$\frac{\partial^2 \psi_s}{\partial t^2} = v_c^2 \nabla^2 \psi_s - \frac{\mathcal{E}}{\rho} \quad ; \quad v_c \equiv \sqrt{(2G_s + \lambda_s)/\rho} \quad (3)$$

That $\bar{A}_s$ represent rotational (shearing) motions is evident from taking the curl of the deformation

$$\nabla \times \bar{\xi} = \nabla \times [\nabla \times \bar{A}_s] - \nabla \times \nabla \psi_s = -\nabla^2 \bar{A}_s \quad (4)$$

Similarly, the divergence is represented by ψ_s alone. These classes of deformation propagate with distinct velocities and are uncoupled in the material volume. However, at a boundary there is in general coupling between the two modes.

Prob. 7.18.2 Subject to no external forces, the equation of motion for the particle is simply

$$\frac{4}{3}\rho_p \pi R^3 \frac{dU}{dt} + 6\pi\eta R U = 0 \tag{1}$$

Thus, with U_o the initial velocity,

$$U = U_o \exp(-t/\tau) \tag{2}$$

where $\tau \equiv (2/9)(\rho_p R^2/\eta)$

Prob.7.19.1 There are two ways to obtain the stress tensor. First, observe that the divergence of the given S_{ij} is the mechanical force density on the right in the incompressible force equation.

$$\frac{\partial S_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}(-p\,\delta_{ij}) + G_s \frac{\partial}{\partial x_j}\left(\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i}\right) \tag{1}$$

Because $\partial\xi_j/\partial x_j \equiv \nabla\cdot\vec{\xi}=0$, this expression becomes

$$\frac{\partial S_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + G_s \frac{\partial^2 \xi_i}{\partial x_j \partial x_j} \tag{2}$$

which is recognized as the right hand side of the force equation.

As a second approach, simply observe from Eq. (b) of Table P7.16.1 that the required S_{ij} is obtained if $\lambda_s \nabla\cdot\vec{\xi} \rightarrow -p$ and e_{ij} is as given by Eq. (a) of that Table.

One way to make the analogy is to write out the equations of motion in terms of complex amplitudes.

$j\omega\rho\hat{v}_x = -\dfrac{d\hat{p}}{dx} + \eta\left(\dfrac{d^2\hat{v}_x}{dx^2} - k^2\hat{v}_x\right)$	$(j\omega)^2\rho\hat{\xi}_x = -\dfrac{d\hat{p}}{dx} + G_s\left(\dfrac{d^2\hat{\xi}_x}{dx^2} - k^2\hat{\xi}_x\right)$	(3)
$j\omega\rho\hat{v}_y = jk\hat{p} + \eta\left(\dfrac{d^2\hat{v}_y}{dx^2} - k^2\hat{v}_y\right)$	$(j\omega)^2\rho\hat{\xi}_y = jk\hat{p} + G_s\left(\dfrac{d^2\hat{\xi}_y}{dx^2} - k^2\hat{\xi}_y\right)$	(4)
$\dfrac{d\hat{v}_x}{dx} - jk\hat{v}_y = 0$	$\dfrac{d\hat{\xi}_x}{dx} - jk\hat{\xi}_y = 0$	(5)
$\hat{S}_{xx} = -\hat{p} + \eta\dfrac{d\hat{v}_x}{dx}$	$\hat{S}_{xx} = -\hat{p} + G_s\dfrac{d\hat{\xi}_x}{dx}$	(6)
$\hat{S}_{yx} = \eta\left(\dfrac{d\hat{v}_y}{dx} - jk\hat{v}_x\right)$	$\hat{S}_{yx} = G_s\left(\dfrac{d\hat{\xi}_y}{dx} - jk\hat{\xi}_x\right)$	(7)

The given substitution then turns the left side equations (for the incompressible fluid mechanics) into those on the right (for the incompressible solid mechanics).

Prob. 7.19.2 The laws required to represent the elastic displacements and stresses are given in Table P7.16.1. In terms of $\bar{A}_s$ and Ψ_s as defined in Prob. 7.18.1, Eq. (e) becomes

$$\nabla\times\left[\rho\frac{\partial^2\bar{A}_s}{\partial t^2}+G_s\nabla\times\nabla\times\bar{A}_s\right]-\nabla\left[\rho\frac{\partial^2\Psi_s}{\partial t^2}-(2G_s+\lambda_s)\nabla^2\Psi_s\right]=0 \tag{1}$$

Given that because $\nabla\cdot\bar{A}_s=0, \nabla\times\nabla\times\bar{A}_s=-\nabla^2\bar{A}_s$, this expression is satisfied if

$$\frac{\partial^2\bar{A}_s}{\partial t^2}=\frac{G_s}{\rho}\nabla^2\bar{A}_s \Rightarrow \frac{\partial^2\bar{A}}{\partial t^2}=\frac{G_s}{\rho}\left(\frac{\partial^2 A}{\partial x^2}+\frac{\partial^2 A}{\partial y^2}\right) \tag{2}$$

$$\frac{\partial^2\Psi_s}{\partial t^2}=\frac{2G_s+\lambda_s}{\rho}\nabla^2\Psi_s \Rightarrow \frac{\partial^2\Psi}{\partial t^2}=\frac{2G_s+\lambda_s}{\rho}\left(\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}\right) \tag{3}$$

In the second equations, $\bar{A}_s=A(x,y,t)\bar{i}_z$, $\Psi_s=\Psi(x,y,t)$, to represent the two-dimensional motions of interest.

Given solutions to Eqs. 2 and 3, $\bar{\xi}$ is evaluated.

$$\xi_x=\frac{\partial A}{\partial y}-\frac{\partial\Psi}{\partial x} \tag{4}$$

$$\xi_y=-\frac{\partial A}{\partial x}-\frac{\partial\Psi}{\partial y} \tag{5}$$

The desired stress components then follow from Eqs. (a) and (b) from Table P7.16.1.

$$S_{xx}=(2G_s+\lambda_s)\frac{\partial\xi_x}{\partial x}+\lambda_s\frac{\partial\xi_y}{\partial y} \tag{6}$$

$$S_{yx}=G_s\left(\frac{\partial\xi_y}{\partial x}+\frac{\partial\xi_x}{\partial y}\right) \tag{7}$$

In particular, solutions of the form $A=\mathrm{Re}\,\hat{A}(x)\,e^{j(\omega t-k_y)}$ and

Prob. 7.19.2 (cont.)

$\psi = R_e \hat{\psi}(x) e^{j(\omega t - ky)}$ are substituted into Eqs. 2 and 3 to obtain

$$\frac{d^2\hat{A}}{dx^2} - \gamma_s^2 \hat{A} = 0 \tag{8}$$

$$\frac{d^2\hat{\psi}}{dx^2} - \gamma_c^2 \hat{\psi} = 0 \tag{9}$$

where $\gamma_s^2 = k^2 - \omega^2\rho/G_s$ and $\gamma_c^2 = k^2 - \omega^2\rho/(2G_s + \lambda_s)$.

With the proviso that γ_s and γ_c have positive real parts,

$$\hat{A} = \hat{A}_1 e^{\pm\gamma_s x} \; ; \; \hat{\psi} = \hat{\psi}_1 e^{\pm\gamma_c x} \tag{10}$$

are solutions appropriate to infinite half spaces. The upper signs refer to a lower half space while the lower ones refer to an upper half space.

It follows from Eq. 10 that the displacements of Eqs. 4 and 5 are

$$\hat{\xi}_x = -jk\hat{A}_1 e^{\pm\gamma_s x} \mp \gamma_c \hat{\psi}_1 e^{\pm\gamma_c x} \tag{11}$$

$$\hat{\xi}_y = \mp\gamma_s \hat{A}_1 e^{\pm\gamma_s x} + jk\hat{\psi}_1 e^{\pm\gamma_c x} \tag{12}$$

These expressions are now used to trade-in the $(\hat{A}_1, \hat{\psi}_1)$ on the displacements evaluated at the interface.

$$\begin{bmatrix} \hat{\xi}_x^{\alpha\beta} \\ \hat{\xi}_y^{\alpha\beta} \end{bmatrix} \quad \begin{bmatrix} -jk & \mp\gamma_c \\ \mp\gamma_s & jk \end{bmatrix} \begin{bmatrix} \hat{A}_1 \\ \hat{\psi}_1 \end{bmatrix} \tag{13}$$

Inversion of this expression gives

Prob. 7.19.2 (cont.)

$$\begin{bmatrix} \hat{A}_1 \\ \hat{\Psi}_1 \end{bmatrix} = \frac{1}{k^2 - \gamma_c \gamma_s} \begin{bmatrix} jk & \pm\gamma_c \\ \pm\gamma_s & -jk \end{bmatrix} \begin{bmatrix} \hat{\xi}_x^{\alpha,\beta} \\ \hat{\xi}_y^{\alpha,\beta} \end{bmatrix} \tag{14}$$

In terms of complex amplitudes, Eqs. 6 and 7 are

$$\hat{S}_{xx} = (2G_s + \lambda_s)\frac{d\hat{\xi}_x}{dx} - jk\lambda_s \hat{\xi}_y \tag{15}$$

$$\hat{S}_{yx} = G_s\left[\frac{d\hat{\xi}_y}{dx} - jk\hat{\xi}_x\right] \tag{16}$$

and these in turn are evaluated using Eqs. 11 and 12. The resulting expressions are evaluated at $x = 0$ to give

$$\begin{bmatrix} \hat{S}_{xx}^{\alpha,\beta} \\ \hat{S}_{yx}^{\alpha,\beta} \end{bmatrix} = \begin{bmatrix} \mp 2jG_s k\gamma_s & [-(2G_s+\lambda_s)\gamma_c^2 + k^2\lambda_s] \\ -G_s(\gamma_s^2 + k^2) & \pm jG_s k 2\gamma_c \end{bmatrix} \begin{bmatrix} \hat{A}_1 \\ \hat{\Psi}_1 \end{bmatrix} \tag{17}$$

Finally, the transfer relations follow by replacing the column matrix on the right by the right-hand side of Eq. 14, and multiplying out the two 2x2 matrices. Note that the definitions, $v_c^2 \equiv (2G_s + \lambda_s)/\rho$, $v_s^2 \equiv G_s/\rho$ (and hence $v_c^2 - 2v_s^2 = \lambda_s/\rho$) from Prob. 7.18.1 have been used.

Prob. 7.19.3 (a) The boundary conditions are on the stress. Because only perturbations are involved, $\hat{S}^{\alpha}_{xx}$ and $\hat{S}^{\beta}_{yx}$ are therefore zero. It follows that the determinant of the coefficients of $(\hat{\xi}^{\alpha}_{x}, \hat{\xi}^{\beta}_{y})$ is therefore zero. Thus, the desired dispersion equation is

$$\gamma_s v_c^2(\gamma_c^2 - k^2)v_s^2\gamma_c(\gamma_s^2 - k^2) + k^2 v_s^2(\gamma_s^2 + k^2 - 2\gamma_c\gamma_s)[k^2(v_c^2 - 2v_s^2) - v_c^2\gamma_c^2 + 2\gamma_c\gamma_s v_s^2] = 0 \tag{1}$$

This simplifies to the given expression provided that the definitions of γ_s^2 and γ_c^2 are used to eliminate v_c^2 through the condition $v_s^2(\gamma_s^2 - k^2) = v_c^2(\gamma_c^2 - k^2)$.

(b) Substitution of $\gamma_s^2 = k^2 - \omega^2/v_s^2$, $\gamma_c^2 = k^2 - \omega^2/v_c^2$ into the square of the dispersion equation gives

$$\left(2k^2 - \frac{\omega^2}{v_s^2}\right)^4 - 16\left(k^2 - \frac{\omega^2}{v_s^2}\right)\left(k^2 - \frac{\omega^2}{v_c^2}\right)k^4 = 0 \tag{2}$$

Division by k^8 gives

$$(2 - \underline{\omega}^2)^4 - 16(1 - \underline{\omega}^2)\left(1 - \frac{v_s^2}{v_c^2}\underline{\omega}^2\right) = 0 \tag{3}$$

where $\underline{\omega} \equiv \omega/k v_s$ and it is clear that the only parameter is v_s/v_c. Multiplied out, this expression becomes the given polynomial.

(c) Given a valid root to Eq. 3 (one that makes $\text{Re}\,\gamma_s > 0$ and $\text{Re}\,\gamma_c > 0$, $\underline{\omega} = \alpha$, it follows that

$$\omega = \alpha v_s k \tag{4}$$

Thus, the phase velocity, αv_s , is independent of k.

(d) From Prob. 7.18.1

$$\frac{v_s^2}{v_c^2} = G_s/(2G_s + \lambda_s) \tag{5}$$

Prob. 7.19.3 (cont.)

while from Eq. g of Table P7.16.1

$$E_s = (\nu_s + 1) 2 G_s \tag{6}$$

Thus, E_s is eliminated from Eq. f of that table to give

$$\lambda_s = 2\nu_s G_s/(1 - 2\nu_s) \tag{7}$$

The desired expression follows from substitution of this expression for λ_s in Eq. 5.

Prob. 7.19.4 (a) With the force density included, Eq. 1 becomes

$$\nabla^2 \left(\rho \frac{\partial A_v}{\partial t} - \eta \nabla^2 A_v - G \right) = 0 \tag{1}$$

In terms of complex amplitudes, this expression in turn is

$$\left(\frac{d^2}{dx^2} - k^2 \right) \left[\frac{d^2 \hat{A}}{dx^2} - \gamma^2 \hat{A} + \frac{\hat{G}(x)}{\eta} \right] = 0 \tag{2}$$

The solution that makes the quantity in brackets [ ] vanish is now called $\hat{A}_p(x)$ and the total solution is $\hat{A} = \hat{A}_H + \hat{A}_p$ with associated velocity and stress functions of the form $\hat{v}_x = (\hat{v}_x)_H + (\hat{v}_x)_p$ and $\hat{S}_{xx} = (\hat{S}_{xx})_H + (\hat{S}_{xx})_p$. The transfer relations, Eq. 7.19.13, still relate the homogeneous solutions, so

$$\begin{bmatrix} \hat{S}_{xx}^{\alpha} - (\hat{S}_{xx}^{\alpha})_p \\ \hat{S}_{xx}^{\beta} - (\hat{S}_{xx}^{\beta})_p \\ \hat{S}_{yx}^{\alpha} - (\hat{S}_{yx}^{\alpha})_p \\ \hat{S}_{yx}^{\beta} - (\hat{S}_{yx}^{\beta})_p \end{bmatrix} = \eta [P_{ij}] \begin{bmatrix} \hat{v}_x^{\alpha} - (\hat{v}_x^{\alpha})_p \\ \hat{v}_x^{\beta} - (\hat{v}_x^{\beta})_p \\ \hat{v}_y^{\alpha} - (\hat{v}_y^{\alpha})_p \\ \hat{v}_y^{\beta} - (\hat{v}_y^{\beta})_p \end{bmatrix} \tag{3}$$

With the particular stress solutions shifted to the right and the velocity components separated, this expression is equivalent to that given.

(b) For the example where $\hat{G} = F_o x$,

Prob. 7.19.4 (cont.)

$$(\hat{v}_x)_p = \frac{-jkF_o}{\eta\gamma^2}x\ ;\ (v_y)_p = \frac{-F_o}{\eta\gamma^2}\ ;\ (\hat{p})_p = 0;\ (\hat{S}_{xx})_p = -\frac{2jkF_o}{\gamma^2};\ (\hat{S}_{yx})_p = -\frac{k^2F_o}{\gamma^2}x \tag{4}$$

Thus, evaluation of Eq. 3 gives

$$\begin{bmatrix} \hat{S}_{xx}^{\alpha} \\ \hat{S}_{xx}^{\beta} \\ \hat{S}_{yx}^{\alpha} \\ \hat{S}_{yx}^{\beta} \end{bmatrix} = \eta\left[P_{ij}\right]\begin{bmatrix} \hat{v}_x^{\alpha} \\ \hat{v}_x^{\beta} \\ \hat{v}_y^{\alpha} \\ \hat{v}_y^{\beta} \end{bmatrix} - \frac{kF_o}{\gamma^2}\begin{bmatrix} 2j \\ 2j \\ k\Delta \\ 0 \end{bmatrix} - \frac{F_o}{\gamma^2}\left[P_{ij}\right]\begin{bmatrix} -jk\Delta \\ 0 \\ -1 \\ -1 \end{bmatrix} \tag{5}$$

Prob. 7.19.5 The temporal modes follow directly from Eq. 13, because the velocities are zero at the respective boundaries. Thus, unless the root happens to be trivial, for the response to be finite, F=0. Thus, with Δ = d, the required eigenfrequency equation is

$$\frac{2\gamma}{k}(1-\cosh\gamma d\cosh kd)+\sinh\gamma d\sinh kd\left[\left(\frac{\gamma}{k}\right)^2+1\right]=0 \tag{1}$$

where once γ is found from this expression, the frequency follows from the definition

$$\gamma \equiv \sqrt{k^2 + \frac{j\omega\rho}{\eta}} \tag{2}$$

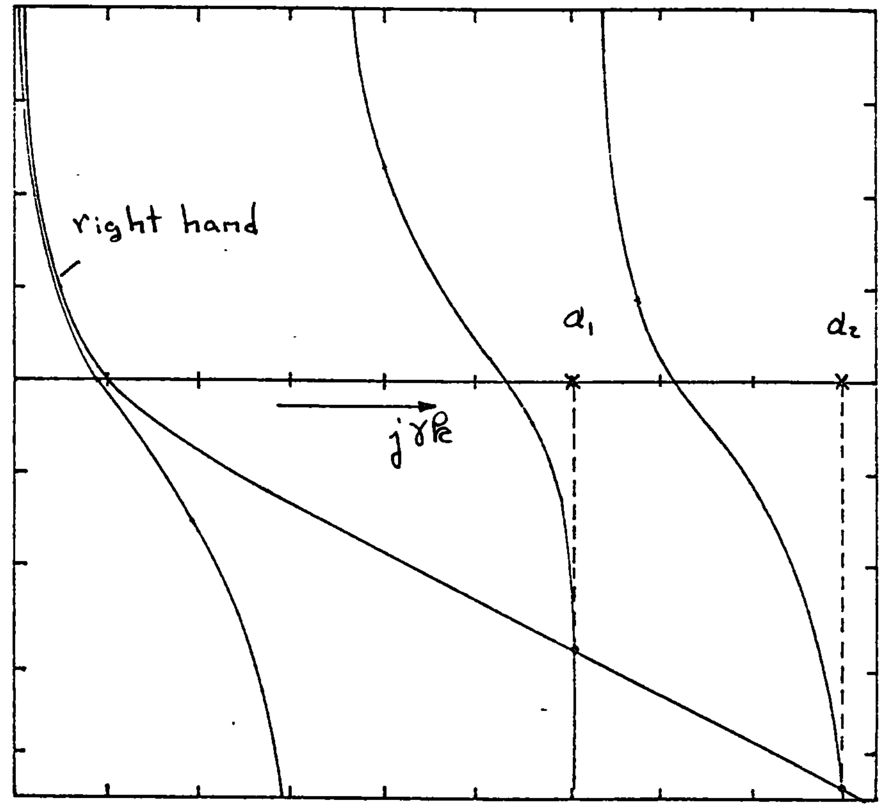
Note that Eq. 1 can be written as

$$\frac{\cos\left(j\frac{\gamma}{k}kd\right)\cosh kd - 1}{\sin\left(\frac{j\gamma}{k}kd\right)\sinh kd} = \frac{1-\left(\frac{j\gamma}{k}\right)^2}{2\left(\frac{j\gamma}{k}\right)} \tag{3}$$

The right-hand side of this expression can be plotted once and for all, as shown in the figure. To plot the left-hand side as a function of $j\gamma/k$, it is necessary to specify kd. For the case where kd=1, the plot is as shown in the figure. From the graphical solution, roots $j\gamma/k = \alpha_n$ follow. The corresponding eigenfrequency follows from Eq. 2 as

Prob. 7.19.5 (cont.)

$$j\omega_n = \frac{\eta k^2}{\rho}(d_n^2 - 1)$$



Prob. 7.20.1 The analogy is clear if the force and stress equations are compared. The appropriate fluid equation in the creep flow limit is Eq. 7.18.12.

$$\nabla p = \eta \nabla^2 \bar{v} \qquad \Big| \qquad \nabla p = G_s \nabla^2 \bar{\xi} \tag{1}$$

$$S_{ij} = -p + \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad \Big| \qquad S_{ij} = -p + G_s\left(\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i}\right) \tag{2}$$

To see that this limit is one in which times of interest are long compared to the time for propagation of either a compressional or a shear wave through a length of interest, write Eq. (e) of Table P7.16.1 in normalized form

$$\frac{\partial^2 \bar{\xi}}{\partial t^2} = \frac{2G_s + \lambda_s}{\rho l^2}\tau^2 \nabla(\nabla\cdot\bar{\xi}) - \frac{G_s \tau^2}{l^2}\nabla\times\nabla\times\bar{\xi} \tag{3}$$

where (see Prob. 7.18.1 exploration of wave dynamics)

$$t = \underline{t}\,\tau \quad ; \qquad v_c \equiv \sqrt{(2G_s+\lambda_s)/\rho}$$

$$(x,y,z) = (\underline{x},\underline{y},\underline{z})\,l \quad ; \qquad v_s \equiv \sqrt{G_s/\rho}$$

and observe that the inertial term is ignorable if

$$\frac{(l/\tau)^2}{v_c^2} << 1 \quad ; \quad \frac{(l/\tau)^2}{v_s^2} << 1 \tag{4}$$

Prob. 7.20.1 (cont.)

With the identification $p \equiv (2G_s+\lambda_s)\nabla\cdot\bar{\xi}$, the fully quasistatic elastic equations result. Note that in this limit, it is understood that $\nabla\cdot\bar{v}=0$ and $\nabla\cdot\bar{\xi}=0$

<u>Prob. 7.21.1</u> In Eq. 7.20.17, $\tilde{v}_r^{\alpha}=0$, $\tilde{v}_\theta^{\beta}=0$ and $n=1$. Thus,

$$\tilde{\Lambda}_1=\frac{UR^2}{2}, \tilde{\Lambda}_2=\frac{R^2}{4}U, \tilde{\Lambda}_3=-\frac{3R^2U}{4}, \tilde{\Lambda}_4=0 \tag{1}$$

and so Eq. 7.20.13 becomes

$$\tilde{\Lambda}=\frac{R^2U}{2}\left[\left(\frac{r}{R}\right)^2+\frac{1}{2}\left(\frac{R}{r}\right)-\frac{3}{2}\left(\frac{r}{R}\right)\right] \tag{2}$$

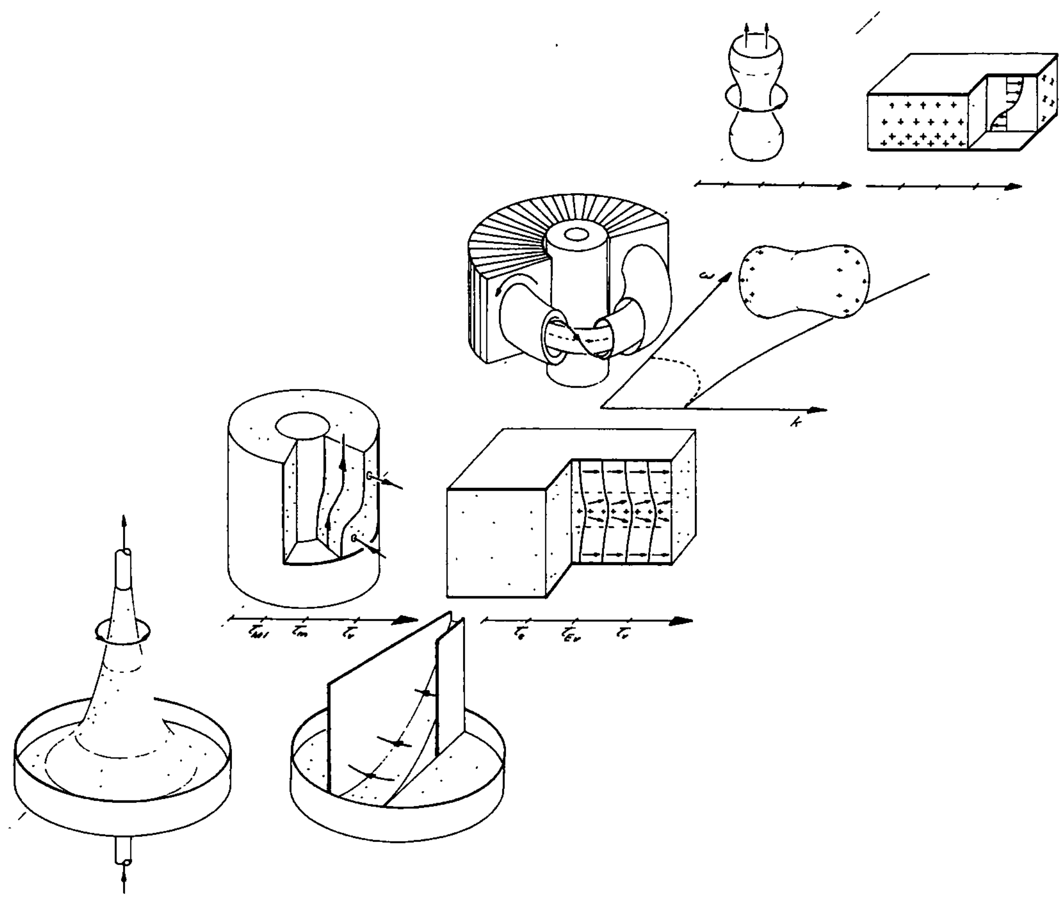
The θ dependence is given by Eq. 10 as $\sin\theta P_1'(\cos\theta)$ so finally the desired stream function is Eq. 5.5.5.

<u>Prob. 7.21.2</u> The analogy discussed in Prob. 7.20.1 applies so that the transfer relations are directly applicable (with the appropriate substitutions) to the evaluation of Eq. 7.21.1. Thus, $U\rightarrow\Xi$ and $\eta\rightarrow G_s$. Just as the rate of fall of a sphere in a highly viscous fluid can be used to deduce the viscosity through the use of Eq. 4, the shear modulus can be deduced by observing the displacement of a sphere subject to the force f_z.

$$f_z = 6\pi G_s R\,\Xi \tag{1}$$

8

Statics and Dynamics of Systems Having a Static Equilibrium



<u>Prob. 8.3.1</u> In the fringing region near the edges of the electrodes (at a distance large compared to the electrode spacing) the electric field is

$$\bar{E} = -\frac{V_o}{2\pi r}\bar{i}_\theta \tag{1}$$

This field is unaltered if the dielectric assumes a configuration that is essentially independent of θ. In that case, the electric field is everywhere tangential to the interface, continuity of tangential $\bar{E}$ is satisfied and there is no normal $\bar{E}$ (and hence $\bar{D}$) to be concerned with. In the force density and stress-tensor representation of Eq. 3.7.19 (Table 3.10.1) there is no electric force density in the homogeneous bulk of the liquid. Thus, Bernoulli's equation applies without a coupling term. With the height measured from the fluid level outside the field region, points (a) and (b) just above and below the interface at an arbitrary point are rel ted to the pressure at infinity by

$$p_a + \rho_a g \xi = p_\infty \tag{2}$$

$$p_b + \rho_b g \xi = p_\infty \tag{3}$$

The pressure at infinity has been taken as the same in each fluid because there is no surface force density acting in that field-free region. At the interfacial position denoted by (a) and (b), stress equilibrium in the normal direction requires that

$$[\![p]\!]\delta_{nj}n_j = [\![T_{nj}]\!]n_j \tag{4}$$

Thus, if $\rho_a \ll \rho_b$, it follows from Eqs. 2-4 that

$$\rho_b g \xi = [\![T_{nj}]\!]n_j \quad ; \quad T_{nj} = E_n D_j - \delta_{nj}W' = -\delta_{nj}W' \tag{5}$$

To evaluate the coenergy density, W', use is made of the constitutive law.

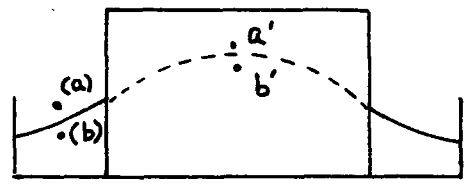
$$W' = \int_0^E D\,\delta E = \int_0^E \left(\epsilon_o E_\theta + \frac{E_\theta}{\alpha_1\sqrt{\alpha_2^2 + E_\theta^2}}\right) dE_\theta = \frac{1}{2}\epsilon_o E_\theta^2 + \frac{2\sqrt{\alpha_2^2 + E_\theta^2}}{2\alpha_1} - \frac{\alpha_2}{\alpha_1} \tag{6}$$

Thus, Eq. 5 can be solved for the interfacial position.

$$\xi = \frac{1}{\rho_b g}[\![-W']\!] = \frac{1}{\rho_b g \alpha_1}\left\{\sqrt{\alpha_2^2 + \left(\frac{V_o}{2\pi r}\right)^2} - \alpha_2\right\} \tag{7}$$

Prob. 8.3.2 Because the liquid is homogeneous, the electromechanical coupling is, according to Eq. 3.8.14 of Table 3.10.1, confined to the interface. To evaluate the stress, note that

$$W' = \int_0^E D\,\delta E = \frac{1}{2}\epsilon_o E^2 + \frac{\alpha_1}{\alpha_2}\ln\cosh \alpha_2 E$$



Hence, with points positioned as shown in the figure, Bernoulli's equation requires that

$$-P_a = -P_{a'} \tag{2}$$

$$P_b + \rho g \xi_o = P_{b'} + \rho g \xi \tag{3}$$

and stress balance at the two interfacial positions requires that

$$P_a = P_b \tag{4}$$

$$-P_{a'} + P_{b'} = -\frac{\alpha_1}{\alpha_2}\ln\cosh(\alpha_2 E) \tag{5}$$

Addition of these last four expressions eliminates the pressure. Substitution for E with $V_o/s(z)$ then gives the required result

$$\xi - \xi_o = \frac{\alpha_1}{\rho g \alpha_2}\ln\cosh\left(\frac{\alpha_2 V_o}{s(z)}\right) \tag{6}$$

Note that the simplicity of this result depends on the fact that regardless of the interfacial position, the electric field at any given z is simply the voltage divided by the spacing.

<u>Prob. 8.4.1</u> (a) From Table 2.18.1, the normal flux density at the surface of the magnets is related to A by $B_x = B_o \cos ky = \partial A/\partial y$. There are no magnetic materials below the magnets, so their fields extend to $x \to -\infty$. It follows that the imposed magnetic field has the vector potential (z directed)

$$A = \frac{B_o}{k} \sin ky \, e^{k(x-d)} \tag{1}$$

Given that $\xi = \xi_o$ at y=0 where A=0, Eq. 8.4.18 is adapted to the case at hand. hand.

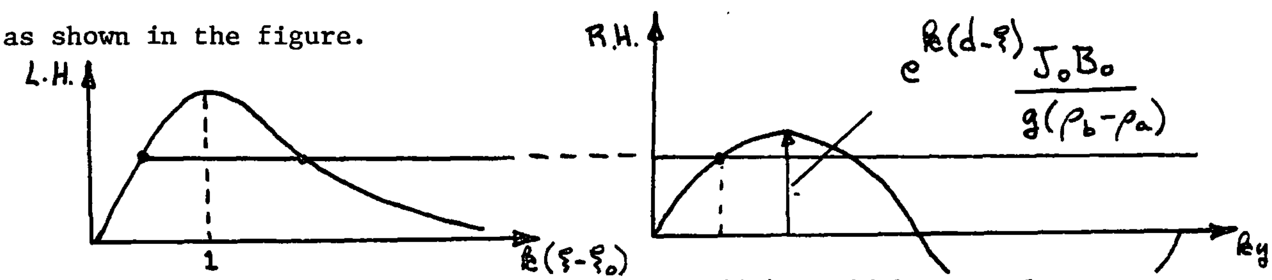
$$\lambda = -\frac{B_o}{k} \sin ky \, e^{k(\xi-d)} \tag{2}$$

and it follows from Eq. 8.4.19 with $\hat{\xi}_\infty \to \hat{\xi}_c$ that

$$\xi = \xi_o + \frac{J_o B_o \sin ky}{kg(\rho_b - \rho_a)} e^{k(\xi-d)} \tag{3}$$

Variables can be regrouped in this expression to obtain the given $\xi(y)$.

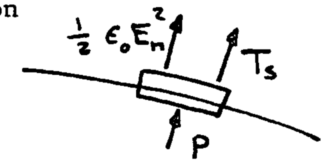
(b) Sketches of the respective sides of the implicit expression are as shown in the figure.



The procedure (either graphically or numerically) would be to select a y, evaluate the expression on the right, and then read off the deformation relative to ξ_o from the expression as represented on the left. The peak in the latter curve comes at $k(\xi-\xi_o)=1$ where its value is 1/e. If the two solutions are interpreted as being stable and unstable to left and right respectively, it follows that if the peak in the curve on the right is just high enough to make these solutions join, there should be an instability. This critical condition follows as

$$J_o B_o / g(\rho_b - \rho_a) = \exp[-k(d-\xi_o)-1]$$

Prob. 8.4.2 (a) Stress equilibrium in the normal direction at the interface requires that



$$p + \frac{1}{2}\epsilon_o E_n^2 - \gamma\,\nabla\cdot\bar{n} = 0 \qquad (1)$$

The normal vector is related to the interfacial deflection by

$$\bar{n} = \left(\bar{i}_x - \frac{\partial\xi}{\partial y}\bar{i}_y\right)\left[1 + \left(\frac{\partial\xi}{\partial y}\right)^2\right]^{-\frac{1}{2}} \qquad (2)$$

In the long-wave limit, the electric field at the interface is essentially

$$E_n \simeq -\frac{V}{d-\xi} \qquad (3)$$

Finally, Bernoulli's equation evaluated at the interface where the height, is ξ becomes

$$p + \rho g\xi = 0 + \rho g b \Rightarrow p = \rho g(b-\xi) \qquad (4)$$

These last three expressions are substituted into Eq. 1 to give the required relation

$$\gamma\frac{d}{dy}\left\{\left[1+\left(\frac{d\xi}{dy}\right)^2\right]^{-\frac{1}{2}}\frac{d\xi}{dy}\right\} + \frac{1}{2}\epsilon_o\frac{V^2}{(d-\xi)^2} - \rho g(\xi-b) = 0 \qquad (5)$$

(b) For small perturbations of ξ from b, let $\xi = b+\xi'$ where ξ' is "small". Then, the linearized form of Eq. 5 is

$$\gamma\frac{d^2\xi'}{dy^2} + \frac{1}{2}\epsilon_o V^2\left[\frac{1}{(d-b)^2} + \frac{2\xi'}{(d-b)^3}\right] - \rho g\xi' = 0 \qquad (6)$$

With the "drive" put on the right, this expression is

$$\frac{d^2\xi'}{dy^2} - \frac{\xi'}{\ell_y^2} = \frac{-\epsilon_o V^2}{2(d-b)^2} \qquad (7)$$

where

$$\ell_y \equiv \left[\frac{\rho g}{\gamma} - \frac{\epsilon_o V^2}{\gamma(d-b)^3}\right]^{-\frac{1}{2}} \qquad (8)$$

is real to insure stability of the interface. To satisfy the asymptotic condition as $y\to\infty$, the increasing exponential must be zero. Thus, the

Prob. 8.4.2 (cont.)

combination of particular and homogeneous solutions that satisfies the boundary condition at y=0 is

$$\xi' = \frac{\epsilon V^2 \ell_y}{2(d-b)^2}\left(1 - e^{-y/\ell_y}\right) \tag{8}$$

(c) The multiplication of Eq. 5 by $u \equiv d\xi/dy$ gives

$$u\frac{d}{dy}\left\{(1+u^2)^{-1/2}u\right\} + \frac{dP}{dy} = 0 \tag{9}$$

where

$$P \equiv \frac{1}{2}\left[\epsilon_0\frac{V^2}{d-\xi} - \rho g(\xi-b)^2\right]$$

To integrate, define

$$v = (1+u^2)^{-1/2}u \tag{10}$$

so that

$$u = (1-v^2)^{-1/2}v \tag{11}$$

Then, Eq. 9 can be written as

$$\frac{v}{\sqrt{1-v^2}}dv + dP = 0 \tag{12}$$

and integration gives

$$-\sqrt{1-v^2} + P = \text{const.} \tag{13}$$

This expression can be written in terms of $d\xi/dy \equiv u$ by using Eq. 10.

$$\frac{-1}{\sqrt{1+\left(\frac{d\xi}{dy}\right)^2}} + P = \text{const.} \tag{14}$$

Because $d\xi/dy \to 0$ as $\xi \to \xi_o$, the constant is $P(\xi_o) - 1$ and Eq. 14 becomes

$$\frac{1}{\sqrt{1+\left(\frac{d\xi}{dy}\right)^2}} = -P(\xi_o) + P(\xi) + 1 \tag{15}$$

Solution for $d\xi/dy$ leads to the integral expression

$$\int_b^{\xi}\frac{d\xi}{\sqrt{[1+P(\xi_o)-P(\xi)]^{-2}-1}} = \int_0^y dy \tag{16}$$

Note that the lower limit is set by the boundary condition at y=0.

Prob. 8.6.1 In view of Eq. 31 from problem solution 7.9.2, the requirement that $\hat{v}_r^d = 0$ be zero with $d=R$ but $\beta \to 0$ shows that if $\hat{p}^d$ is to be finite then

$$f_o(0, R, \gamma) = 0 \tag{1}$$

provided that $\omega \neq \pm 2\Omega$. By the definition of this function, given in Table 2.16.2, this is the statement that

$$-j\gamma \frac{J_o'(j\gamma R)}{J_o(j\gamma R)} = 0 = \frac{-j\gamma J_1(j\gamma R)}{J_o(j\gamma R)} \tag{2}$$

So the eigenvalue problem is reduced to finding the roots, X_{ok}, of

$$J_1(j\gamma R) = 0 \tag{3}$$

In view of the definition of γ , the eigenfrequencies are then written in terms of these roots by solving

$$\gamma^2 = -\frac{X_{ok}^2}{R^2} \equiv k^2 \left[1 - \frac{(2\Omega)^2}{\omega^2}\right] \tag{4}$$

for ω .

$$\omega_o = \frac{\pm 2\Omega}{\sqrt{1 + \frac{X_{ok}^2}{(kR)^2}}} \tag{5}$$

(b) According to this dispersion equation, waves having the same frequency have wavenumbers that are negatives. Thus, waves traveling in the $\pm z$ directions can be superimposed to obtain standing pressure waves that vary as $\cos kz$. According to Eq. 14, if p is proportional to $\cos kz$ then $v_z \propto \sin kz$ and the conditions that $v_z(0)=0$, $v_z(\ell)=0$ are satisfied if $k = n\pi/\ell, n=0,1,2,\cdots$. For these modes, which satisfy both longitudinal and transverse boundary conditions, the resonance frequencies are therefore

$$\omega_{ok} = \frac{\pm 2\Omega}{\sqrt{1 + \frac{X_{ok}^2 \ell^2}{(n\pi R)^2}}} \tag{6}$$

Problem 8.7.1 The total potential, distinguished from the perturbation potential by a prime, is $\Phi' = -E_o y + \Phi$. Thus,

$$\frac{D\Phi'}{Dt} = \frac{\partial\Phi'}{\partial t} + \bar{v}\cdot\nabla\Phi' = \frac{\partial\Phi}{\partial t} + v_x\frac{\partial\Phi}{\partial x} + v_y\left(-E_o + \frac{\partial\Phi}{\partial y}\right) = 0 \qquad (1)$$

to linear terms, this becomes

$$\frac{\partial\Phi}{\partial t} - E_o v_y = 0 \qquad (2)$$

which will be recognized as the limit $\sigma \to \infty$ of Eq. 8.7.6 integrated twice on x.

Problem 8.7.2 What is new about these laws is the requirement that the current linked by a surface of fixed identity be conserved. In view of the generalized Leibnitz rule, Eq. 2.6.4 and Stoke's Theorem, Eq. 2.6.3, integral condition (a) requires that

$$\frac{d}{dt}\int_S \bar{J}_f\cdot\bar{n}\,da = \oint_S\left[\frac{\partial\bar{J}_f}{\partial t} + (\nabla\cdot\bar{J}_f)\bar{v}\right]\cdot\bar{n}\,da + \int_S \nabla\times(\bar{J}_f\times\bar{v})\cdot\bar{n}\,da \qquad (3)$$

The laws are MQS, so $\bar{J}_f$ is solenoidal and it follows from Eq. 3 that

$$\frac{\partial\bar{J}_f}{\partial t} - \nabla\times(\bar{v}\times\bar{J}_f) = 0 \qquad (4)$$

With the understanding that ρ is a constant, and that $\bar{B} = \mu_o\bar{H}$, the remaining laws are standard.

Problem 8.7.3 Note that $\bar{v}$ and $\bar{J}_f$ are automatically solenoidal if they take the given form. The x component of Eq. (c) from Prob. 8.7.2 is also an identity while the y and z components are

$$\frac{\partial J_y}{\partial t} - J_o\frac{\partial v_y}{\partial x} = 0 \qquad (1)$$

$$\frac{\partial J_z}{\partial t} - J_o\frac{\partial v_z}{\partial x} = 0 \qquad (2)$$

Similarly, the x component of Eq. (d) from Prob. 8.7.2 is an identity while the y and z components are

$$\rho\frac{\partial v_y}{\partial t} = B_o J_z + \eta\frac{\partial^2 v_y}{\partial x^2} \qquad (3)$$

$$\rho\frac{\partial v_z}{\partial t} = -B_o J_y + \eta\frac{\partial^2 v_z}{\partial x^2} \qquad (4)$$

Because $\bar{B}$ is imposed, Ampere's Law is not required unless perturbations in the magnetic field are of interest.

Prob. 8.7.3(cont.)

In terms of complex amplitudes $v_y = \mathrm{Re}\,\hat{v}_y \exp j\omega t$, Eqs. 1 and 2 show that

$$\hat{J}_y = -j\frac{J_o\gamma}{\omega}\hat{v}_y;\quad \hat{J}_z = -j\frac{J_o\gamma}{\omega}\hat{v}_z;\quad \hat{v}_y = A e^{\gamma x} \tag{5}$$

Substituted into Eqs. 3 and 4, these relations give

$$\begin{bmatrix} (\eta\gamma^2 - j\omega\rho) & -j\frac{J_o\gamma B_o}{\omega} \\ j\frac{J_o\gamma B_o}{\omega} & (\eta\gamma^2 - j\omega\rho) \end{bmatrix}\begin{bmatrix} \hat{v}_y \\ \hat{v}_z \end{bmatrix} = 0 \tag{6}$$

The dispersion equation follows from setting the determinant of the coefficients equal to zero.

$$(\eta\gamma^2 - j\omega\rho)\frac{\omega}{\gamma} = \pm J_o B_o \tag{7}$$

with the normalization $\tau_v \equiv \Delta^2\rho/\eta,\ \tau_{MV} \equiv \eta/J_o B_o\Delta,\ \underline{\gamma} = \gamma\Delta$

it follows that

$$\gamma = \pm\gamma_{\substack{1\\2}}\ ;\ \gamma_{\substack{1\\2}} \equiv \left[\frac{1}{2}\frac{1}{\omega\tau_{MV}} \pm \sqrt{\left(\frac{1}{2\omega\tau_{MV}}\right)^2 + j\omega\tau_v}\,\right] \tag{8}$$

Thus, solutions take the form

$$\hat{v}_z = \hat{A}_1 e^{\gamma_1 x} + \hat{A}_2 e^{-\gamma_1 x} + \hat{A}_3 e^{\gamma_2 x} + \hat{A}_4 e^{-\gamma_2 x} \tag{9}$$

From Eq. 6(a) and the dispersion equation, Eq. 8, it follows from Eq. 9 that

$$\hat{v}_y = j\hat{A}_1 e^{\gamma_1 x} - j\hat{A}_2 e^{-\gamma_1 x} + j\hat{A}_3 e^{\gamma_2 x} - j\hat{A}_4 e^{-\gamma_2 x} \tag{10}$$

The shear stress can be written in terms of these same coefficients using Eq. 9.

$$\hat{S}_{zx} = \eta\left(\gamma_1\hat{A}_1 e^{\gamma_1 x} - \gamma_1\hat{A}_2 e^{-\gamma_1 x} + \gamma_2\hat{A}_3 e^{\gamma_2 x} - \gamma_2\hat{A}_4 e^{-\gamma_2 x}\right) \tag{11}$$

Similarly, from Eq. 10,

$$\hat{S}_{yx} = \eta\left(j\gamma_1\hat{A}_1 e^{\gamma_1 x} + j\gamma_1\hat{A}_2 e^{-\gamma_1 x} + j\gamma_2\hat{A}_3 e^{\gamma_2 x} + j\gamma_2\hat{A}_4 e^{-\gamma_2 x}\right) \tag{12}$$

Evaluated at the respective α and β surfaces, where x = Δ and x=0, Eqs. 9 and 10 show that

Prob. 8.7.3(cont.)

$$\begin{bmatrix}\hat{v}_y^\alpha\\ \hat{v}_y^\beta\\ \hat{v}_z^\alpha\\ \hat{v}_z^\beta\end{bmatrix}=[Q_{ij}]\begin{bmatrix}\hat{A}_1\\ \hat{A}_2\\ \hat{A}_3\\ \hat{A}_4\end{bmatrix};\ Q_{ij}\equiv\begin{bmatrix} j e^{\underline{\gamma}_1} & -j e^{-\underline{\gamma}_1} & j e^{\underline{\gamma}_2} & -j e^{-\underline{\gamma}_2}\\ j & -j & j & -j\\ e^{\underline{\gamma}_1} & e^{-\underline{\gamma}_1} & e^{\underline{\gamma}_2} & e^{-\underline{\gamma}_2}\\ 1 & 1 & 1 & 1\end{bmatrix} \tag{13}$$

Similarly, from Eqs. 11 and 12, evaluation at the surfaces gives

$$\begin{bmatrix}\hat{S}_{yx}^\alpha\\ \hat{S}_{yx}^\beta\\ \hat{S}_{zx}^\alpha\\ \hat{S}_{zx}^\beta\end{bmatrix}=[U_{ij}]\begin{bmatrix}\hat{A}_1\\ \hat{A}_2\\ \hat{A}_3\\ \hat{A}_4\end{bmatrix};\ U_{ij}\equiv\eta\begin{bmatrix} j\gamma_1 e^{\underline{\gamma}_1} & j\gamma_1 e^{-\underline{\gamma}_1} & j\gamma_2 e^{\underline{\gamma}_2} & j\gamma_2 e^{-\underline{\gamma}_2}\\ j\gamma_1 & j\gamma_1 & j\gamma_2 & j\gamma_2\\ \gamma_1 e^{\underline{\gamma}_1} & -\gamma_1 e^{-\underline{\gamma}_1} & \gamma_2 e^{\underline{\gamma}_2} & -\gamma_2 e^{-\underline{\gamma}_2}\\ \gamma_1 & -\gamma_1 & \gamma_2 & -\gamma_2\end{bmatrix} \tag{14}$$

The transfer relations follow from inversion of 13 and multiplication with 14

$$\begin{vmatrix}\hat{S}_{yx}^\alpha\\ \hat{S}_{yx}^\beta\\ \hat{S}_{zx}^\alpha\\ \hat{S}_{zx}^\beta\end{vmatrix}=[W_{ij}]\begin{vmatrix}\hat{v}_y^\alpha\\ \hat{v}_y^\beta\\ \hat{v}_z^\alpha\\ \hat{v}_z^\beta\end{vmatrix};\ [W_{ij}]=[Q_{ij}]^{-1}[U_{ij}] \tag{15}$$

All required here are the temporal eigen-frequencies with the velocities constrained to zero at the boundaries. To this end, Eq. 13 is manipulated to take the form (note that $A_1e^{x}+A_2e^{-x}\equiv(A_1+A_2)\cosh x+(A_1-A_2)\sinh x$)*

$$\begin{bmatrix}\hat{v}_y^\alpha\\ \hat{v}_y^\beta\\ \hat{v}_z^\alpha\\ \hat{v}_z^\beta\end{bmatrix}=\begin{bmatrix} j\cosh\underline{\gamma}_1 & j\sinh\underline{\gamma}_1 & j\cosh\underline{\gamma}_2 & j\sinh\underline{\gamma}_2\\ j & 0 & j & 0\\ \sinh\underline{\gamma}_1 & \cosh\underline{\gamma}_1 & \sinh\underline{\gamma}_2 & \cosh\underline{\gamma}_2\\ 0 & 1 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{A}_1-\hat{A}_2\\ \hat{A}_1+\hat{A}_2\\ \hat{A}_3-\hat{A}_4\\ \hat{A}_3+\hat{A}_4\end{bmatrix} \tag{16}$$

The condition that the determinant of the coefficients vanish is then

$$\cosh\underline{\gamma}_1\cosh\underline{\gamma}_2-\sinh\underline{\gamma}_1\sinh\underline{\gamma}_2\equiv\cosh(\underline{\gamma}_1-\underline{\gamma}_2)=1 \tag{17}$$

*Transformation suggested by Mr. Rick Ehrlich.

Prob. 8.7.3(cont.)

This expression is identical to $\cos j(\underline{\gamma}_1 - \underline{\gamma}_2) = 1$ and therefore has solutions

$$j(\underline{\gamma}_1 - \underline{\gamma}_2) = 2n\pi, \quad n = 0, 1, 2 \cdots \tag{18}$$

With the use of Eq. 8, an expression for the eigenfrequencies follows

$$2j\left[\left(\frac{1}{2\omega\tau_{MV}}\right)^2 + j\omega\tau_v\right]^{1/2} = 2n\pi \tag{19}$$

Manipulation and substitution $s \equiv j\omega$ shows that this is a cubic in s.

$$s^3\tau_v + (n\pi)^2 s^2 - \frac{1}{4\tau_{MV}^2} = 0 \tag{20}$$

If the viscosity is high enough that inertial effects can be ignored, the ordering of characteristic times is as shown in Fig. 1

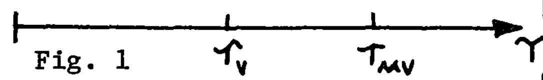


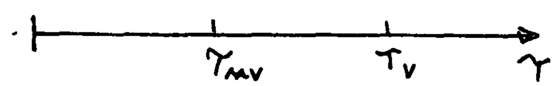
Fig. 1

Then, there are two roots to Eq. 20 given by setting τ_v =0 and solving for s.

$$s = \pm 1/2\tau_{MV} n\pi \tag{21}$$

Thus, there is an instability having a growth rate typified by the magneto-viscous time $2n\pi\tau_{MV}$.

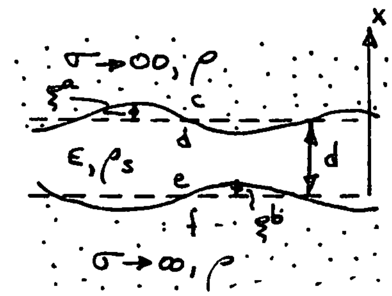
In the opposite extreme, where inertial effects are dominant, the ordering of times is as shown in Fig. 2 and the middle term in Eq. 20 is negligible compared to the other two. In this case,



$$s = 1/(4\tau_{MV}^2\tau_v)^{1/3} = \left(\frac{J_o^2 B_o^2}{4\eta\rho}\right)^{1/3} \tag{22}$$

Note that substitution back into Eq. 20 shows that the approximation is in fact self-consistent. The system is again unstable, this time with a growth rate determined by a time that is between τ_v and τ_{MV}.

Prob. 8.7.4 The particle velocity is simply U=bE = $2a\epsilon E^2/\eta$. Thus, the time required to traverse the distance 2a is $2aU = \eta/\epsilon E^2$.



Prob. 8.10.1 With the designations indicated in the figure, first consider the bulk relations. The perturbation electric field is confined to the insulating layer, where

$$\begin{bmatrix} \hat{e}_x^d \\ \hat{e}_x^e \end{bmatrix} = k \begin{bmatrix} -\coth kd & \dfrac{1}{\sinh kd} \\ \dfrac{-1}{\sinh kd} & \coth kd \end{bmatrix} \begin{bmatrix} \hat{\Phi}^d \\ \hat{\Phi}^e \end{bmatrix} \tag{1}$$

The transfer relations for the mechanics are applied three times. Perhaps it is best to first write the second of the following relations, because the transfer relations for the infinite half spaces (with it understood that $k > 0$) follow as limiting cases of the general relations.

$$\hat{p}^c = \frac{j\omega\rho}{k}\hat{v}_x^c = -\frac{\omega^2\rho}{k}\hat{\xi}^a \tag{2}$$

$$\begin{bmatrix} \hat{p}^d \\ \hat{p}^e \end{bmatrix} = \frac{j\omega\rho_s}{k} \begin{bmatrix} -\coth kd & \dfrac{1}{\sinh kd} \\ \dfrac{-1}{\sinh kd} & \coth kd \end{bmatrix} \begin{bmatrix} \hat{v}_x^d \\ \hat{v}_x^e \end{bmatrix} = -\frac{\omega^2\rho_s}{k} \begin{bmatrix} -\coth kd & \dfrac{1}{\sinh kd} \\ \dfrac{-1}{\sinh kd} & \coth kd \end{bmatrix} \begin{bmatrix} \hat{\xi}^a \\ \hat{\xi}^b \end{bmatrix} \tag{3}$$

$$\hat{p}^f = -\frac{j\omega\rho}{k}\hat{v}^f = \frac{\omega^2\rho}{k}\hat{\xi}^b \tag{4}$$

Now, consider the boundary conditions. The interfaces are perfectly conducting, so

$$\bar{n}\times\bar{E} = 0 \Rightarrow -E_o\frac{\partial\xi}{\partial z} = e_z \tag{5}$$

In terms of the potential, this becomes

$$\hat{\Phi}^a = E_o\hat{\xi}^a \tag{6}$$

Similarly,

$$\hat{\Phi}^b = E_o\hat{\xi}^b \tag{7}$$

Stress equilibrium for the x direction is

$$[\![p]\!]n_x = [\![T_{xj}]\!]n_j - \gamma\nabla\cdot\bar{n}\,n_x \tag{8}$$

In particular,

$$(\Pi_c + p'^c) - (\Pi_d + p'^d) = -\frac{\epsilon}{2}(E_o + e_x)^2 + \gamma\left(\frac{\partial^2\xi}{\partial y^2} + \frac{\partial^2\xi}{\partial z^2}\right) \tag{9}$$

Hence, in terms of complex amplitudes, stress equilibrium for the upper interface is

Prob. 8.10.1(cont.)

$$-\hat{p}^c + \hat{p}^d - \epsilon E_o \hat{e}_x^d - k^2\gamma\hat{\xi}^a = 0 \tag{10}$$

Similarly, for the lower interface,

$$-\hat{p}^e + \hat{p}^f + \epsilon E_o \hat{e}_x^e - k^2\gamma\hat{\xi}^b = 0 \tag{11}$$

Now, to put these relations together and obtain a dispersion equation, insert Eqs. 5 and 6 into Eq. 1. Then, Eqs. 1-4 can be substituted into Eqs. 9 and 10, which become

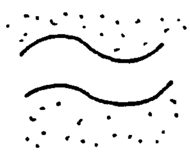
$$\begin{bmatrix} \left[\frac{\omega^2\rho}{k} + \frac{\omega^2\rho_s}{k}\coth kd + \epsilon E_o^2 k \coth kd - k^2\gamma\right] & \left[\frac{-\omega^2\rho_s}{k\sinh kd} - \frac{\epsilon E_o^2 k}{\sinh kd}\right] \\ \left[\frac{-\omega^2\rho_s}{k\sinh kd} - \frac{\epsilon_o E_o^2 k}{\sinh kd}\right] & \left[\frac{\omega^2\rho}{k} + \frac{\omega^2\rho_s}{k}\coth kd + \epsilon_o E_o^2 k\coth kd - k^2\gamma\right] \end{bmatrix} \begin{bmatrix} \hat{\xi}^a \\ \hat{\xi}^b \end{bmatrix} = 0 \tag{12}$$

For the kink mode ($\xi^a = \xi^b$), both of these expressions are satisfied if

$$\frac{\omega^2}{k}\left(\rho + \rho_s\coth kd - \frac{\rho_s}{\sinh kd}\right) + \epsilon E_o^2 k\left[\coth kd - \frac{1}{\sinh kd}\right] - k^2\gamma = 0 \tag{13}$$

With the use of the identity $(\cosh u - 1)/\sinh u = \tanh u/2$, this expression reduces to

$$\frac{\omega^2}{k}\left(\rho + \rho_s\tanh\frac{kd}{2}\right) = \gamma k^2 - \epsilon E_o^2 k\tanh\frac{kd}{2} \tag{14}$$

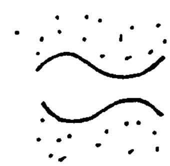


For the sausage mode ($\xi^a = -\xi^b$), both are satisfied if

$$\frac{\omega^2}{k}\left(\rho + \rho_s\coth kd + \frac{\rho_s}{\sinh kd}\right) + \epsilon E_o^2 k\left[\coth kd + \frac{1}{\sinh kd}\right] - k^2\gamma = 0 \tag{15}$$

and because $(\cosh u + 1)/\sinh u = \coth u/2$

$$\frac{\omega^2}{k}\left(\rho + \rho_s\coth\frac{kd}{2}\right) = \gamma k^2 - \epsilon E_o^2 k\coth\frac{kd}{2} \tag{16}$$



In the limit $kd \ll 1$, Eqs. 14 and 16 become

$$\frac{\omega^2}{k}\left(\rho + \rho_s\frac{kd}{2}\right) = \left(\gamma - \frac{\epsilon E_o^2 d}{2}\right)k^2 \tag{17}$$

$$\frac{\omega^2}{k}\left(\rho + \frac{2\rho_s}{kd}\right) = \gamma k^2 - \frac{2\epsilon E_o^2}{d} \tag{18}$$

Prob. 8.10.1(cont.)

Thus, the effect of the electric field on the kink mode is equivalent to having a field dependent surface tension with $\gamma \rightarrow \gamma - \epsilon E_o^2 d/2$

The sausage mode is unstable at $k \rightarrow 0$ (infinite wavelength) with $E_o=0$ while the kink mode is unstable at $E_o = \sqrt{2\gamma/\epsilon d}$. If the insulating liquid filled in a hole between regions filled by high conductivity liquid, the hole boundaries would limit the values of possible k's. Then there would be a threshold value of E_o.

Prob. 8.11.1 (a) In static equilibrium, $\bar{H}$ is tangential to the interface and hence not affected by the liquid. Thus, $\bar{H} = \bar{i}_\theta H_o(R/r)$ where $H_o=I/2\pi R$. The surface force densities due to magnetization and surface tension are held in equilibrium by the pressure jump $(\mu_a \equiv \mu_o,\ \mu_b \equiv \mu)$

$$\Pi_a - \Pi_b = -\frac{1}{2}(\mu_a - \mu_b)H_o^2 - \frac{\gamma}{R} \tag{1}$$

(b) Perturbation boundary conditions at the interface are, at $r = R+\xi$

$$\bar{n}\cdot[\![\mu\bar{H}]\!] = \left(\bar{i}_r - \frac{1}{R}\frac{\partial\xi}{\partial\theta}\bar{i}_\theta - \frac{\partial\xi}{\partial z}\bar{i}_z\right)\cdot\left([\![\mu h_r]\!]\bar{i}_r + [\![\mu(H_o\frac{R}{r} + h_\theta)]\!]\bar{i}_\theta + [\![\mu h_z]\!]\bar{i}_z\right) \tag{2}$$

which to linear terms requires

$$[\![\mu\hat{h}_r]\!] = -j[\![\mu]\!]\frac{H_o m}{R}\hat{\xi} \tag{3}$$

and $\bar{n}\times[\![\bar{H}]\!]=0$ which to linear terms requires that $[\![h_\theta]\!]=0$ and $[\![h_z]\!]=0$

These are represented by

$$[\![\hat{\psi}]\!] = 0 \tag{4}$$

With $n_j[\![p]\!] = [\![T_{rj}]\!]n_j + T_s n_r$, stress equilibrium for the interface requires that

$$[\![p]\!] = -\frac{1}{2}[\![\mu(H_o\frac{R}{R+\xi} + h_\theta)^2]\!]_{R+\xi} - \gamma\nabla\cdot\bar{n} \tag{5}$$

To linear terms, this expression becomes Eq. (1) and

$$[\![\hat{p}]\!] = [\![\mu]\!]\frac{H_o^2}{R}\hat{\xi} - [\![\mu]\!]H_o j\frac{m\hat{\psi}^a}{R} + \frac{\gamma}{R^2}[(1-m^2) - (kR)^2]\hat{\xi} \tag{6}$$

where use has been made of $\hat{h}_\theta = jm\hat{\psi}/R$

Perturbation fields are assumed to decay to zero as $r\rightarrow\infty$ and to be finite at $r = 0$. Thus, bulk relations for the magnetic field are (Table 2.16.2)

Prob. 8.11.1 (cont.)

$$\hat{\psi}^a = \hat{h}_r^a / f_m(\infty, R) \tag{7}$$

$$\hat{\psi}^b = \hat{h}_r^b / f_m(0, R) \tag{8}$$

From Eqs. (3) and (4) together with these last two expressions, it follows that

$$\hat{\psi}^a = -jm[\![\mu]\!] H_o \hat{\xi} / R[\mu_a f_m(\infty, R) - \mu_b f_m(0, R)] \tag{9}$$

This expression is substituted into Eq. (6), along with the bulk relation for the perturbation pressure, Eq. (f) of Table 7.9.1, to obtain the desired dispersion equation.

$$-\omega^2 \rho F_m(0,R) = (\mu_b - \mu_a)\frac{H_o^2}{R} + m^2 [\![\mu]\!]^2 H_o^2 \frac{1}{R^2[\mu_a f_m(\infty,R) - \mu_b f_m(0,R)]} - \frac{\gamma}{R^2}\left[(1-m^2) - (kR)^2\right] \tag{10}$$

(c) Remember (from Sec. 2.17) that $F_m(0,R)$ and $f_m(0,R)$ are negative while $f_m(\infty,R)$ is positive. For $\mu_b > \mu_a$, the first "imposed field" term on the right stabilizes. The second "self-field" term stabilizes regardless of the permeabilities, but only influences modes with finite m. Thus, sausage modes can "exchange" with no change in the self-fields. Clearly, all modes $m \neq 0$ are stable. To stabilize the m=0 mode,

$$(\mu_b - \mu_a)\frac{H_o^2}{R} > \frac{\gamma}{R^2} \tag{11}$$

(d) In the m=0 mode the mechanical deformations are purely radial. Thus, the rigid boundary introduced by the magnet does not interfere with the motion. Also, the perturbation magnetic field is zero, so there is no difficulty satisfying the field boundary conditions on the magnet surface. (Note that the other modes are altered by the magnet). In the long wave limit, Eq. 2.16.28 gives $F_o(0,R) = f_o^{-1}(0,R) \rightarrow (-k^2R/2)^{-1}$ and hence, Eq. (10) becomes simply

$$\omega^2 = (\mu - \mu_o)\frac{H_o^2}{\rho} k^2 \tag{12}$$

Thus, waves propagate in the z direction with phase velocity $\sqrt{(\mu - \mu_o)H_o^2/\rho}$

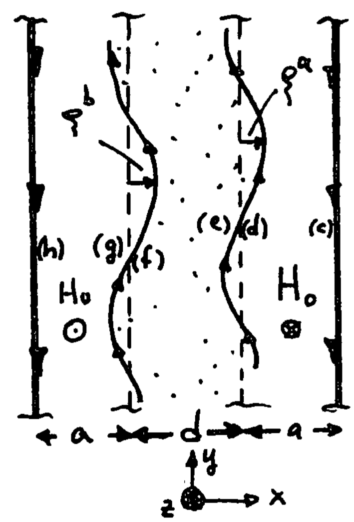
Prob. 8.11.1 (cont.)

Resonances occur when the longitudinal wavenumbers are multiples of n

Thus, the resonance frequencies are

$$f_n = \frac{n H_o}{2\ell}\sqrt{\frac{\mu-\mu_o}{\rho}} \tag{13}$$

Prob. 8.12.1 In the vacuum regions to either side of the fluid sheet the magnetic fields take the form



$$\bar H = -H_o\bar i_z + \bar h \tag{1}$$

$$\bar H = H_o \bar i_z + \bar h \tag{2}$$

where $\bar h = -\nabla\psi$.

In the regions to either side, the mass density is negligible, and so the pressure there can be taken as zero. In the fluid, the pressure is therefore

$$p = \frac{1}{2}\mu_o H_o^2 + \mathrm{Re}\,\hat p\, e^{j(\omega t - k_y y - k_z z)} \tag{3}$$

where p is the perturbation associated with departures of the fluid from static equilibrium. Boundary conditions reflect the electromechanical coupling and are consistent with fields governed by Laplace's equation in the vacuum regions and fluid motions governed by Laplace's equation in the layer. That is one boundary condition on the magnetic field at the surfaces bounding the vacuum, and one boundary condition on the fluid mechanics at each of the deformable interfaces. First, because $\bar n\cdot\bar B = 0$ on the perfectly conducting interfaces,

$$\hat h_x^c = 0 \tag{4}$$

$$\left[\bar i_x - \frac{\partial\xi}{\partial y}\bar i_y - \frac{\partial\xi}{\partial z}\bar i_z\right]\cdot\left[-H_o\bar i_z + \bar h\right] = 0 \Rightarrow \hat h_x^d = jk_z\hat\xi^a H_o \tag{5}$$

$$\hat h_x^g = -jk_z\hat\xi^b H_o \tag{6}$$

$$\hat h_x^h = 0 \tag{7}$$

In the absence of surface tension, stress balance requires that

$$[\![p]\!]n_x = [\![T_{xj}]\!]n_j \Rightarrow \tag{8}$$

In particular, to linear terms at the right interface

$$\hat p^e = -\mu_o H_o \hat h_z^d = -jk_z\mu_o H_o\hat\psi^d \tag{9}$$

Prob. 8.12.1(cont.)

Similarly, at the left interface

$$\hat{p}^f = \mu_o H_o \hat{h}_z^g = jk_z \mu_o H_o \hat{\psi}^g \tag{10}$$

In evaluating these boundary conditions, the amplitudes are evaluated at the unperturbed position of the interface. Hence, the coupling between interfaces through the bulk regions can be represented by the transfer relations. For the fields, Eqs. (a) of Table 2.16.1 (in the magnetic analogue) give

$$\begin{bmatrix} \hat{\psi}^c \\ \hat{\psi}^d \end{bmatrix} = \frac{1}{k}\begin{bmatrix} -\coth ka & \frac{1}{\sinh ka} \\ \frac{-1}{\sinh ka} & \coth ka \end{bmatrix}\begin{bmatrix} \hat{h}_x^c \\ \hat{h}_x^d \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} \hat{\psi}^g \\ \hat{\psi}^h \end{bmatrix} = \frac{1}{k}\begin{bmatrix} -\coth ka & \frac{1}{\sinh ka} \\ \frac{-1}{\sinh ka} & \coth ka \end{bmatrix}\begin{bmatrix} \hat{h}_x^g \\ \hat{h}_x^h \end{bmatrix} \tag{12}$$

For the fluid layer, Eqs. (c) of Table 7.9.1 become

$$\begin{bmatrix} \hat{p}^e \\ \hat{p}^f \end{bmatrix} = \frac{j\omega\rho}{k}\begin{bmatrix} -\coth kd & \frac{1}{\sinh kd} \\ \frac{-1}{\sinh kd} & \coth kd \end{bmatrix}\begin{bmatrix} \hat{v}_x^e \\ \hat{v}_x^f \end{bmatrix} \tag{13}$$

Because the fluid has a static equilibrium, at the interfaces, $\hat{v}_x^e = j\omega\hat{\xi}^a$, $\hat{v}_x^f = j\omega\hat{\xi}^b$. It sounds more complicated then it really is to make the following substitutions. First, Eqs. 4-7 are substituted into Eqs. 11 and 12. In turn, Eqs. 11b and 12a are used in Eqs. 9 and 10. Finally these relations are entered into Eqs. 13 which are arranged to give

$$\begin{bmatrix} -\frac{\omega^2\rho}{k}\coth kd + \mu_o\frac{H_o^2 k_z^2}{k}\coth ka & \frac{\omega^2\rho}{k}\frac{1}{\sinh kd} \\ -\frac{\omega^2\rho}{k}\frac{1}{\sinh kd} & \frac{\omega^2\rho}{k}\coth kd - \mu_o\frac{H_o^2 k_z^2}{k}\coth ka \end{bmatrix}\begin{bmatrix} \hat{\xi}^a \\ \hat{\xi}^b \end{bmatrix} = 0 \tag{14}$$

For the kink mode, note that setting $\hat{\xi}^a = \hat{\xi}^b$ insures that both of Eqs. 14 are satisfied if*

$$*\tanh\tfrac{1}{2}u = \frac{\cosh u - 1}{\sinh u} = \frac{\sinh u}{\cosh u + 1}$$

Prob. 8.12.1(cont.)

$$\frac{\omega^2\rho}{k}\tanh\frac{kd}{2} = \frac{\mu_o H_o^2 k_z^2}{k}\coth ka \tag{15}$$

Similarly, if $\hat{\xi}^a = -\hat{\xi}^b$, so that a sausage mode is considered, both equations are satisfied if

$$\frac{\omega^2\rho}{k}\coth\frac{kd}{2} = \frac{\mu_o H_o^2 k_z^2}{k}\coth ka \tag{16}$$

These last two expression comprise the dispersion equations for the respective modes. It is clear that both of the modes are stable. Note however that perturbations propagating in the y direction (k_z=0) are only neutrally stable. This is the "interchange" direction discussed with Fig. 8.12.3. Such perturbations result in no change in the magnetic field between the fluid and the walls and in no change in the surface current. As a result, there is no perturbation magnetic surface force density tending to restore the interface.

Problem 8.12.2

Stress equilibrium at the interface requires that

$$-\Pi - P_d' + P_e' - T_{rr}\Big|_{R+\xi} = 0 \Rightarrow \hat{p}^d = -\mu_o H_o^2 \frac{\hat{\xi}}{R} + \mu_o H_o \hat{h}_\theta^e \; ; \; \Pi = \frac{1}{2}\mu_o H_o^2 \tag{1}$$

Also, at the interface flux is conserved, so

$$\bar{n}\cdot\bar{H}\Big|_{R+\xi} = 0 \Rightarrow \hat{h}_r^e = -j\frac{H_o m}{R}\hat{\xi} \tag{2}$$

While at the inner rod surface

$$\hat{h}_r^f = 0 \tag{3}$$

At the outer wall,

$$\hat{\xi}^c = 0 \tag{4}$$

Bulk transfer relations are

$$\begin{bmatrix} \hat{p}^c \\ \hat{p}^d \end{bmatrix} = -\rho\omega^2 \begin{bmatrix} F_m(R,a) & G_m(a,R) \\ G_m(R,a) & F_m(a,R) \end{bmatrix} \begin{bmatrix} 0 \\ \hat{\xi} \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} \hat{h}_\theta^e \\ \hat{h}_\theta^f \end{bmatrix} = \frac{jm}{R} \begin{bmatrix} F_m(b,R) & G_m(R,b) \\ G_m(b,R) & F_m(R,b) \end{bmatrix} \begin{bmatrix} \hat{h}_r^e \\ 0 \end{bmatrix} \tag{6}$$

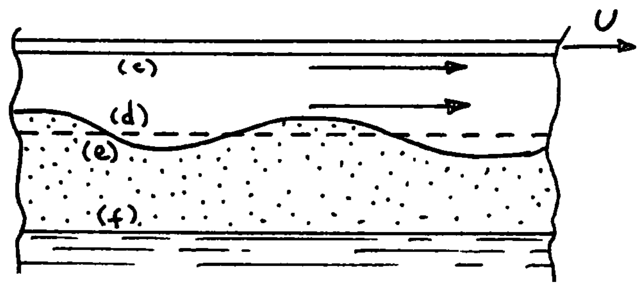
The dispersion equation follows by substituting Eq. (1) for $\hat{p}^d$ in Eq. (5b) with $\hat{h}_\theta^e$ substituted from Eq. (6a). On the right in Eq. (5b), Eq. (2) is substituted. Hence,

$$\frac{-\mu_o H_o^2}{R}\hat{\xi} + \mu_o H_o jm F_m(b,R)\left(\frac{-jH_o m}{R}\right)\hat{\xi} = -\rho\omega^2 F_m(a,R)\hat{\xi} \tag{7}$$

Thus, the dispersion equation is

$$\omega^2 = \frac{\mu_o H_o^2}{\rho R F_m(a,R)}\left[1 - \frac{m^2}{R}F_m(b,R)\right] \tag{8}$$

From the reciprocity energy conditions discussed in Sec. 2.17, $F_m(a,R) > 0$ and $F_m(b,R) < 0$, so Eq. 8 gives real values of ω regardless of k. The system is stable.



Problem 8.12.3 In static equilibrium $\bar{v}=0$,

$$\Pi_a - \Pi_b = -\tfrac{1}{2}\mu_o H_o^2 \quad \text{and}$$

$$p = \Pi_b - \rho g x \tag{1}$$

With positions next to boundaries denoted as shown in the figure, boundary conditions from top to bottom are as follows. For the conducting sheet backed by an infinitely permeable material, Eq. (a) of Table 6.3.1 reduces to

$$k^2 \hat{h}_y^c = -\mu_o \sigma_s k_y (\omega - k_y U)\hat{h}_x^c \tag{2}$$

The condition that the normal magnetic flux vanish at the deformed interface is to linear terms

$$\hat{h}_x^d + j k_y H_o \hat{\xi} = 0 \tag{3}$$

The perturbation part of the stress balance equation for the interface is

$$-\hat{p}^e = -\mu_o H_o \hat{h}_y^d - k^2 \gamma \hat{\xi} - \rho g \hat{\xi} \tag{4}$$

In addition, continuity and the definition of the interface require that $\hat{v}_x = j\omega\hat{\xi}$

Finally, the bottom is rigid, so

$$\hat{v}_x^f = 0 \tag{5}$$

Bulk relations for the perturbations in magnetic field follow from Eqs. (a) of Table 2.16.1

$$\begin{bmatrix} \hat{h}_x^c \\ \hat{h}_x^d \end{bmatrix} = \frac{k}{j k_y} \begin{bmatrix} -\coth ka & \dfrac{1}{\sinh ka} \\ \dfrac{-1}{\sinh ka} & \coth ka \end{bmatrix} \begin{bmatrix} \hat{h}_y^c \\ \hat{h}_y^d \end{bmatrix} \tag{6}$$

where $\hat{h}_y = j k_y \hat{\psi}$ has been used.

Problem 8.12.3 (cont.)

The mechanical perturbation bulk relations follow from Eqs. (c) of Table 7.9.1

$$\begin{bmatrix} \hat{p}^e \\ \\ \hat{p}^f \end{bmatrix} = \frac{j\omega\rho}{k} \begin{bmatrix} -\coth kb & \dfrac{1}{\sinh kb} \\ \\ \dfrac{-1}{\sinh kb} & \coth kb \end{bmatrix} \begin{bmatrix} \hat{v}_x^e \\ \\ \hat{v}_x^f \end{bmatrix} \tag{7}$$

where

$$\hat{v}_x^e = j\omega\hat{\xi} \tag{8}$$

Equations 2 and 6a give

$$\hat{H}_y^c = \frac{j\mu_o\sigma_s k(\omega - k_y U)\hat{H}_y^d}{\sinh ka\left[k^2 + j\sigma_s\mu_o k(\omega - k_y U)\coth ka\right]} \tag{9}$$

This expression combines with Eqs. 3 and 6b to show that

$$kH_o\hat{\xi} = \frac{k}{k_y}\left\{\frac{-j\mu_o\sigma_s k(\omega - kU)}{\sinh^2 ka\left[k^2 + j\sigma_s\mu_o k(\omega - k_y U)\coth ka\right]} + \coth ka\right\}\hat{H}_y^d \tag{10}$$

Thus, the stress balance equation, Eq. 4, can be evaluated using $\hat{H}_y^d$ from Eq. 10 along with $\hat{p}^e$ from Eq. 7a, Eq. 5 and Eq. 8. The coefficient of $\hat{\xi}$ is the desired dispersion equation.

$$\frac{\omega^2\rho}{k}\coth kb = \rho g + k^2\gamma + \mu_o H_o^2 \frac{k_y^2}{k}\tanh ka\left\{\frac{1 + j\dfrac{\mu_o\sigma_s}{k}(\omega - k_y U)\coth ka}{1 + j\dfrac{\mu_o\sigma_s}{k}(\omega - k_y U)\tanh ka}\right\} \tag{11}$$

Prob. 8.12.4 The development of this section leaves open the configuration beyond the radius r=a. Thus, it can be readily adapted to include the effect of the lossy wall. The thin conducting shell is represented by the boundary condition of Eq. (b) from Table 6.3.1.

$$j\left(\frac{m^2}{a^2}+k^2\right)(\hat{\psi}^e-\hat{\psi}^b)=\sigma_s\mu_o\omega\,\hat{h}_r^b \tag{1}$$

where (e) denotes the position just outside the shell. The region outside the shell is free space and described by the magnetic analogue of Eq. (b) from Table 2.16.2.

$$\hat{\psi}^e=F_m(\infty,a)\hat{h}_r^e=F_m(\infty,a)\hat{h}_r^b \tag{2}$$

Equations 8.12.4a and 8.12.7 combine to represent what is "seen" looking inward from the wall.

$$\hat{\psi}^b=F_m(R,a)\hat{h}_r^b-jG_m(a,R)\left(\frac{mH_t}{R}+kH_a\right)\hat{\xi} \tag{3}$$

Thus, substitution of Eqs. 2 and 3 into Eq. 1 gives

$$\hat{h}_r^b=\frac{G_m(a,R)\left(\frac{mH_t}{R}+kH_a\right)\hat{\xi}}{j[F_m(\infty,a)-F_m(R,a)]-\mu_o\sigma_s\omega)/(\frac{m^2}{a^2}+k^2)} \tag{4}$$

Finally, this expression is inserted into Eq. 8.12.11 to obtain the desired dispersion equation.

$$\omega^2\rho F_m(0,R)=\frac{\mu_o H_t^2}{R}-\mu_o\left(\frac{m}{R}H_t+kH_a\right)^2F_m(a,R)$$
$$-j\frac{\mu_o\left(\frac{m}{R}H_t+kH_a\right)^2G_m(R,a)G_m(a,R)}{j[F_m(\infty,a)-F_m(R,a)]-\mu_o\sigma_s\omega)/(\frac{m^2}{a^2}+k^2)} \tag{5}$$

The wall can be regarded as perfectly conducting provided that the last term is negligible compared to the one before it. First, the conduction term in the denominator must dominate the energy storage term.

$$\frac{\mu_o\sigma_s|\omega|}{\left(\frac{m^2}{a^2}+k^2\right)}>F_m(\infty,a)-F_m(R,a)>0 \tag{6}$$

Prob. 8.12.4(cont.)

Second, the last term is then negligible if

$$\frac{\mu_o \sigma_s |\omega|}{\left(\frac{m^2}{a^2} + k^2\right)} > -G_m(R,a)G_m(a,R)/F_m(a,R) > 0 \tag{7}$$

In general, the dispersion equation is a cubic in ω and describes the coupling of the magnetic diffusion mode on the wall with the surface Alfvén waves propagating on the perfectly conducting column. However, in the limit where the wall is highly resistive, a simple quadratic expression is obtained for the damping effect of the wall on the surface waves. With the second term in the denominator small compared to the first, $(a+b)^{-1} \simeq a^{-1} - b/a^2$ and

$$-\rho F_m(0,R)(j\omega)^2 + B(j\omega) + K = 0 \tag{8}$$

where an effective spring constant is

$$K = \frac{-\mu_o H_t^2}{R} + \mu_o\left(\frac{m}{R} H_t + kH_a\right)^2 F_m(a,R) + \frac{\mu_o\left(\frac{m}{R} H_t + kH_a\right)^2 G_m(R,a)G_m(a,R)}{F_m(\infty,a) - F_m(R,a)} \tag{9}$$

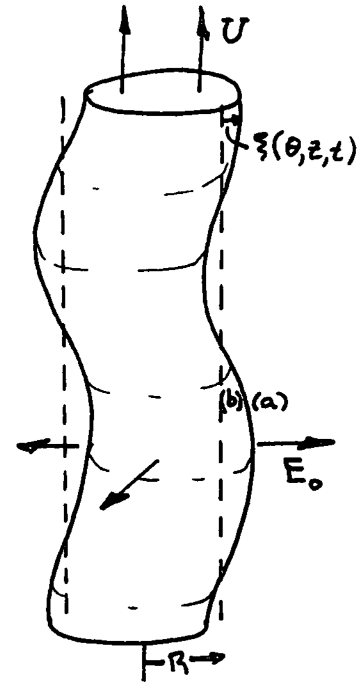
and an effective damping coefficient is

$$B = \frac{-\mu_o\left(\frac{m}{R} H_t + kH_a\right)^2 G_m(R,a)G_m(a,R)}{[F_m(\infty,a) - F_m(R,a)]^2} \frac{\mu_o \sigma_s}{\left(\frac{m^2}{a^2} + k^2\right)} \tag{10}$$

Thus, the frequencies (given by Eq. 8) are

$$j\omega = \frac{-B \pm \sqrt{B^2 - (-\rho F_m(0,R))4K}}{2[-\rho F_m(0,R)]} \tag{11}$$

Note that $F_m(0,R) < 0$, $F_m(a,R) > 0$, $F_m(\infty,a) - F_m(R,a) > 0$ and $G_m(R,a)G_m(a,R) < 0$. Thus, the wall produces damping.



Prob. 8.13.1 In static equilibrium, the radial stress balance becomes

$$[\![p]\!] = [\![T_{rr}]\!] - \gamma\left(\frac{1}{R_1}+\frac{1}{R_2}\right) \quad (1)$$

so that the pressure jump under this condition is

$$[\![\Pi]\!] = \frac{1}{2}\epsilon_o E_o^2 - \frac{\gamma}{R} \quad (2)$$

In the region surrounding the column, the electric field intensity takes the form

$$\bar{E} = E_o \frac{R}{r}\bar{i}_r + \bar{e}\ ;\ \bar{e} = -\nabla\Phi \quad (3)$$

while inside the column the electric field is zero and the pressure is given by

$$p = \Pi_b + p'(r,\theta,z,t) = \Pi_b + Re\,\hat{p}(r)\,e^{j(\omega t - m\theta - kz)} \quad (4)$$

Electrical boundary conditions require that the perturbation potential vanish as r becomes large and that the tangential electric field vanish on the deformable surface of the column.

$$\bar{n}\times\bar{E}\Big|_{r=R+\xi} = 0 \cong \begin{bmatrix} \bar{i}_r & \bar{i}_\theta & \bar{i}_z \\ 1 & -\frac{1}{R}\frac{\partial\xi}{\partial\theta} & -\frac{\partial\xi}{\partial z} \\ E_o\frac{R}{r}+e_r & e_\theta & e_z \end{bmatrix} \Rightarrow e_z = -E_o\frac{\partial\xi}{\partial z} \quad (5)$$

In terms of complex amplitudes, with $\hat{e}_z = jk\hat{\Phi}$,

$$\hat{\Phi}^a = E_o\hat{\xi} \quad (6)$$

Stress balance in the radial direction at the interface requires that (with some linearization) $(p_a' \approx 0)$

$$\Pi_a - \Pi_b - p_b' = \frac{1}{2}\epsilon_o\left[E_o\frac{R}{(R+\xi)} + e_r\right]^2 + (T_s)_r \quad (7)$$

To linear terms, this becomes (Eqs. (f) and (h), Table 7.6.2 for $\bar{T}_s$)

$$\hat{p}_b = \frac{\epsilon_o E_o^2}{R}\hat{\xi} - \epsilon_o E_o\hat{e}_r^a - \frac{\gamma}{R^2}\left(1 - m^2 - (kR)^2\right)\hat{\xi} \quad (8)$$

Bulk relations representing the fields surrounding the column and the fluid within are Eq. (a) of Table 2.16.2 and (f) of Table 7.9.1

Prob. 8.13.1(cont.)

$$\hat{e}_r^a = f_m(\infty,R)\hat{\Phi}^a \tag{10}$$

$$\hat{p}^b = j(\omega - kU)\rho F_m(0,R)\hat{v}_r \tag{11}$$

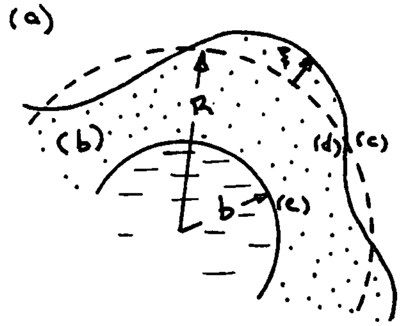
Recall that $\hat{v}_r = j(\omega - kU)\hat{\xi}$, and it follows that Eqs. 9,10 and 6 can be substituted into the stress balance equation to obtain

$$-(\omega - kU)^2\rho F_m(0,R)\hat{\xi} = \frac{\epsilon_o E_o^2}{R}\hat{\xi} - \epsilon_o E_o^2 f_m(\infty,R)\hat{\xi} - \frac{\gamma}{R^2}(1 - m^2 - k^2R^2)\hat{\xi} \tag{12}$$

If the amplitude is to be finite, the coefficients must equilibrate. The result is the dispersion equation given with the problem.

Problem 8.13.2

The equilibrium is static with the distribution of electric field intensity



$$E_r = \frac{q}{4\pi r^2}\begin{Bmatrix} \frac{1}{\epsilon_0} \\ \frac{1}{\epsilon} \end{Bmatrix}\begin{matrix} ;R<r \\ ;b<r<R \end{matrix} \quad (1)$$

and difference between equilibrium pressures required to balance the electric surface force density and surface tension

$$\Pi_b - \Pi_a = \frac{2\gamma}{R} - \frac{1}{2}\frac{q^2}{16\pi^2 R^2}\left[\frac{\epsilon-\epsilon_0}{\epsilon\epsilon_0}\right] \quad (2)$$

With the normal given by Eq. 8.17.18, the perturbation boundary conditions require $\bar{n}\times[\![\bar{E}]\!]=0$ at the interface.

$$\hat{\Phi}^c - \hat{\Phi}^d - \hat{\xi}\frac{q}{4\pi R^2}\left[\frac{\epsilon-\epsilon_0}{\epsilon\epsilon_0}\right] = 0 \quad (3)$$

that the jump in normal $\bar{D}$ be zero,

$$\epsilon_0 \hat{e}_r^c - \epsilon \hat{e}_r^d = 0 \quad (4)$$

and that the radial component of the stress equilibrium be satisfied

$$-(\hat{p}^c - \hat{p}^d) - \frac{2q^2}{(4\pi)^2 R^5}\frac{(\epsilon-\epsilon_0)}{\epsilon\epsilon_0}\hat{\xi} + \frac{q}{4\pi R^2}(\hat{e}_r^c - \hat{e}_r^d) - \frac{\gamma}{R^2}(n-1)(n+2)\hat{\xi} = 0 \quad (5)$$

In this last expression, it is assumed that Eq. (2) holds for the equilibrium stress. On the surface of the solid perfectly conducting core,

$$\hat{\xi}^e = 0 \quad ; \quad \hat{\Phi}^e = 0 \quad (6)$$

Mechanical bulk conditions require (from Eq. 8.12.25) (F(b,R) <0 for $R>b$

$$\hat{p}^c = 0 \quad ; \quad \hat{p}^d = -\omega^2\rho F(b,R)\hat{\xi} \quad (7)$$

while electrical conditions in the respective regions require (Eq. 4.8.16)*

$$\varepsilon_0 \hat{e}_r^c = \frac{\varepsilon_0(n+1)}{R}\hat{\Phi}^c \quad ; \quad \varepsilon \hat{e}_r^d = \epsilon f(b,R)\hat{\Phi}^d \quad (1*0$$

Now, Eqs. (7) and (8) are respectively used to substitute for $\hat{p}^c, \hat{p}^d, \hat{e}_r^c$ & $\hat{e}_r^d$ in Eqs. (5) and (4) to make Eqs. (3)-(5) become the three expressions

* $\lim_{b\to 0} f(b,R) = -n/R$; $f(b,R) < 0$ for $R>b$

Problem 8.13.2 (cont.)

$$\begin{bmatrix} 1 & -1 & \dfrac{-q(\epsilon-\epsilon_o)}{4\pi R^2\epsilon\epsilon_o} \\ \epsilon_o\dfrac{(n+1)}{R} & -\epsilon f(b,R) & 0 \\ \dfrac{q(n+1)}{4\pi R^3} & -\dfrac{qf(b,R)}{4\pi R^2} & -\omega^2\rho F(b,R)-\dfrac{2q^2(\epsilon-\epsilon_o)}{(4\pi)^2R^5\epsilon\epsilon_o}-\dfrac{\gamma(n-1)(n+2)}{R^2} \end{bmatrix}\begin{bmatrix}\hat{\Phi}^c \\ \hat{\Phi}^d \\ \hat{\xi}\end{bmatrix}=0 \qquad (9)$$

The determinant of the coefficients gives the required dispersion equation which can be solved for the inertial term to obtain

$$-\omega^2\rho F_n(b,R)=\frac{2q^2(\epsilon-\epsilon_o)}{(4\pi)^2R^5\epsilon\epsilon_o}+\frac{\gamma}{R^2}(n-1)(n+2) + \frac{q^2(\epsilon-\epsilon_o)^2(n+1)f(b,R)}{(4\pi)^2R^4\epsilon\epsilon_o[-\epsilon f(b,R)R+\epsilon_o(n+1)]} \qquad (10)$$

The system will be stable if the quantity on the right is positive. In the limit $b \ll R$, this comes down to the requirement that for instability

$$\Gamma\frac{\epsilon_o}{\epsilon}\left\{2\frac{(\epsilon-\epsilon_o)}{\epsilon_o}-\left(\frac{\epsilon}{\epsilon_o}-1\right)^2\frac{(n+1)n}{\frac{\epsilon}{\epsilon_o}n+(n+1)}\right\}+(n-1)(n+2)<0 \qquad (11)$$

or

$$\Gamma > \frac{(n-1)(n+2)}{\left[\left(\frac{\epsilon}{\epsilon_o}-1\right)^2\frac{(n+1)n}{\frac{\epsilon}{\epsilon_o}n+n+1}-2\left(\frac{\epsilon}{\epsilon_o}-1\right)\right]\frac{\epsilon_o}{\epsilon}} \qquad (12)$$

where

$$\Gamma \equiv \frac{q^2}{\gamma(4\pi)^2R^3\epsilon_o}$$

and it is clear from Eq. (11) that for cases of interest, the denominator of Eq. (12) is positive.

Problem 8.13.2 (cont.)

The figure shows how the conditions for incipient instability can be calculated given ϵ/ϵ_o. What is plotted is the right hand side of Eq. (2). In the range where this function is positive, it has an asymptote which can be found by setting the denominator of Eq. (12) to zero

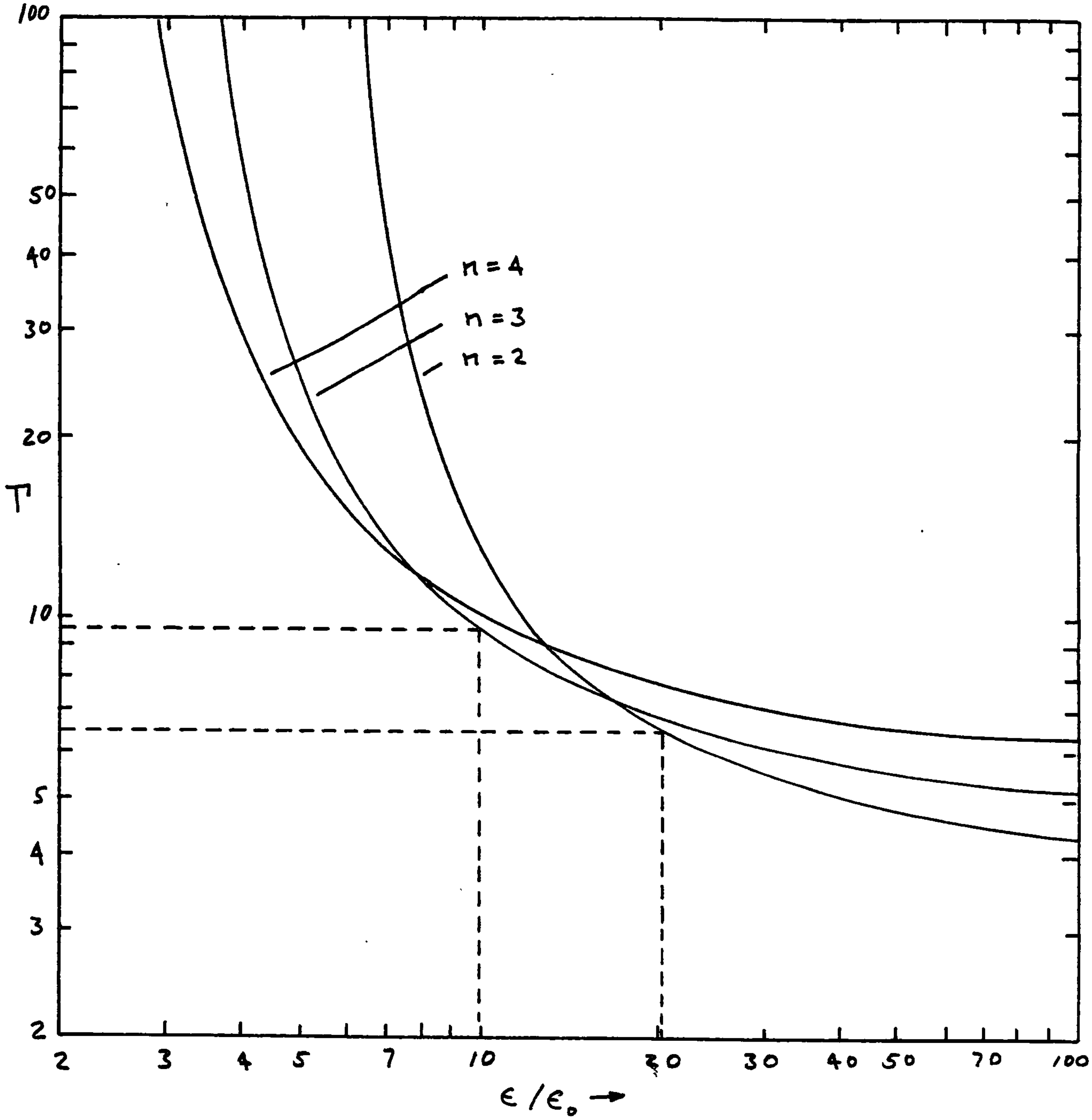
$$\left(\frac{\epsilon}{\epsilon_o}\right)_n = \frac{n^2+3n+2}{n(n-1)} \tag{13}$$

The asymptote in the horizontal direction is the limit of Eq. (12) as $\epsilon/\epsilon_o \to \infty$

$$\Gamma_n = n+2 \tag{14}$$

The curves are for the lowest mode numbers n = 2,3,4 and give an idea of how higher modes would come into play. To use the curves, take ϵ/ϵ_o =20 as an example. Then, it is clear that the first mode to become unstable is n=2 and that instability will occur as the charge is made to exceed about a value such that Γ = 6.5. Similarly, for ϵ/ϵ_o=10, the first mode to become unstable is n=3, and to make this happen, the value of Γ must be Γ=9.6. The higher order modes should be drawn in to make the story complete, but it appears that as ϵ/ϵ_o is reduced, the most critical mode number is increased, as is also the value of Γ required to obtain the instability.

Problem 8.13.2 (cont.)



Prob. 8.14.1 As in Sec. 8.14, the bulk coupling can be absorbed in the pressure. This is because in the bulk the only external force is

$$\bar{F} = -\nabla \mathcal{E} \quad ; \quad \mathcal{E} \equiv q \Phi \tag{1}$$

where $q = Q/\frac{4}{3}\pi R^3$ is uniform throughout the bulk of the drop. Thus, the bulk force equation is the same as for no bulk coupling if $p \to \Pi \equiv p + q\Phi$. In terms of equilibrium and perturbation quantities,

$$\pi = p_o(r) + q\Phi_o(r) + \text{Re}\,\hat{\pi}(r) P_n^m(\cos\theta) e^{j(\omega t - m\phi)} \tag{2}$$

where $\Pi = p_o(r) + q\Phi_o(r)$ and $\hat{\pi}$ plays the role $\hat{p}$ in the mechanical transfer relations. Note that from Gauss' Law, $\Phi_o = qr^2/6\epsilon_o$, and that because the drop is in static equilibrium, $d\Pi/dr = 0$ and Π is independent of r. Thus, for a solid sphere of liquid, Eq. (i) of Table 7.9.1 becomes

$$\hat{\pi}^b = j\omega\rho_b F_n(0, R)\hat{v}_r^b \tag{3}$$

In the outside fluid, there is no charge density and this same transfer relation becomes

$$\hat{p}^a = j\omega\rho_a F_n(\infty, R)\hat{v}_r^a \tag{4}$$

At each point in the bulk, where deformations leave the charge distribution uniform, the perturbation electric field is governed by Laplace's equation. Thus, Eq. (a) of Table 2.16.3 becomes

$$\hat{e}_r^a = f_n(\infty, R)\hat{\Phi}^a \tag{4}$$

$$\hat{e}_r^b = f_n(0, R)\hat{\Phi}^b \tag{5}$$

Boundary conditions are written in terms of the surface displacement

$$\hat{v}_x^a = \hat{v}_x^b = j\omega\hat{\xi} \tag{6}$$

Prob. 8.14.1 (cont.)

Because there is no surface force density (The permittivity is ϵ_o in each region and there is no free surface charge density.)

$$\llbracket p \rrbracket \Big|_{r=R+\xi} = T_s \tag{7}$$

This requires that

$$\Pi_a + R_e \hat{p}^a P_n^m e^{j(\omega t - m\phi)} - \left\{ \Pi_b - g\Phi_o \Big|_{R+\xi} + R_e(\hat{\pi}^b - g\hat{\Phi}^b) P_n^m e^{j(\omega t - m\phi)} \right\} = T_s \tag{8}$$

Continuation of the linearization gives

$$\Pi_a - \Pi_b + g\Phi_o(R) = -\frac{2\gamma}{R} \tag{9}$$

for the static equilibrium and

$$\hat{p}^a - \hat{\pi}^b - \frac{g^2 R}{3\epsilon_o}\hat{\xi} + g\hat{\Phi}^b = -\frac{\gamma}{R^2}(n-1)(n+2)\hat{\xi} \tag{10}$$

for the perturbation. In this last expression. Eq. (1) of Table 7.6.2 has been used to express the surface tension force density on the right.

That the potential is continuous at r=R is equivalent to the condition that $\bar{n} \times \llbracket \bar{E} \rrbracket = 0$ there. This requires that

$$\begin{bmatrix} \bar{i}_r & \bar{i}_\theta & \bar{i}_\phi \\ 1 & -\frac{1}{r}\frac{\partial\xi}{\partial\theta} & \frac{-1}{r\sin\theta}\frac{\partial\xi}{\partial\phi} \\ \llbracket E_o(r) + e_r \rrbracket & \llbracket e_\theta \rrbracket & \llbracket e_\phi \rrbracket \end{bmatrix} = 0 \Rightarrow \llbracket e_\theta \rrbracket + \frac{1}{R}\frac{\partial\xi}{\partial\theta}\llbracket E_o \rrbracket = 0 \tag{11}$$

where the second expression is the ϕ component of the first. It follows from Eq. (11) that

$$-\llbracket \hat{\Phi} \rrbracket + \hat{\xi}\llbracket E_o \rrbracket = 0 \tag{12}$$

and finally, because $\llbracket E_o \rrbracket = 0$

$$\hat{\Phi}^a - \hat{\Phi}^b = 0 \tag{13}$$

The second electrical condition requires that $\bar{n} \cdot \llbracket \bar{D} \rrbracket = 0$, which becomes

$$\llbracket \epsilon_o E_o \rrbracket \Big|_{R+\xi} + \llbracket \epsilon_o e_r \rrbracket = 0 \tag{14}$$

Prob. 8.14.1 (cont.)

Linearization of the equilibrium term gives

$$[\![\frac{dE_o}{dr}]\!]\xi + [\![e_r]\!] = 0 \tag{15}$$

Note that outside, $E_o = R^3 q/3\epsilon_o r^2$ while inside, $E_o = q r/3\epsilon_o$. Thus, Eq. 15 becomes

$$-\frac{q}{\epsilon_o}\hat{\xi} + \hat{e}_r^a - \hat{e}_r^b = 0 \tag{16}$$

Equations 4 and 5, with Eq. 13, enter into Eq. 16 to give

$$-\frac{q}{\epsilon_o}\hat{\xi} + [f_n(\infty,R) - f_n(0,R)]\hat{\Phi}^b = 0 \tag{17}$$

which is solved for $\hat{\Phi}^b$. This can then be inserted into Eq. 10, along with $\hat{p}^a$ and $\hat{\pi}^b$ given by Eqs. 2 and 3 and Eq. 6 to obtain the desired dispersion equation

$$\omega^2[\rho_a F_n(\infty,R) - \rho_b F_n(0,R)] = \frac{\gamma}{R^2}(n-1)(n+2) - \frac{q^2 R}{3\epsilon_o} + \frac{q^2}{\epsilon_o[f_n(\infty,R) - f_n(0,R)]} \tag{18}$$

The functions $F_n(\infty,R) > 0$, $F_n(0,R) < 0$ and $f_n(\infty,R) - f_n(0,R) = (2n+1)/R$ so it follows that the imposed field (second term on the right) is destabilizing, and that the self-field (the third term on the right) is stabilizing. In spherical geometry, the surface tension term is stabilizing for all modes of interest, $n > 1$.

All modes first become unstable (as Q is raised) as the term on the right in Eq. 18 passes through zero. With $q \equiv Q/\frac{4}{3}\pi R^3$, this condition is therefore $(n \neq 1)$

$$Q^2 = \frac{8}{3}\pi^2 \epsilon_o \gamma R^3 (n+2)(2n+1) \tag{19}$$

The n=0 mode is not allowed because of mass conservation. The n=1 mode, which represents lateral translation, is marginally stable, in that it gives

Prob. 8.14.1 (cont.)

$\omega=0$ in Eq. 18. The n=1 mode has been excluded from Eq. 19. For $n>0$, Q^2 is a monotonically increasing function of n in Eq. 19, so the first unstable mode is n=2. Thus, the most critical displacement of the interfaces have the three relative surface displacements shown in Table 2.16.3 for P_2^m. The critical charge is

$$Q_c = \sqrt{\frac{160}{3}\pi^2 R^3 \gamma \epsilon_o} = 7.3\pi\sqrt{\epsilon_o \gamma R^3}$$

Note that this charge is slightly lower than the critical charge on a perfectly conducting sphere drop (Rayleigh's limit, Eq. 8.13.11).

Prob. 8.14.2 The configuration is as shown in Fig. 8.14.2 of the text, except that each region has its own uniform permittivity. This complication evidences itself in the linearization of the boundary conditions, which is somewhat more complicated because of the existence of a surface force density due to the polarization.

The x-component of the condition of stress equilibrium for the interface is in general

$$-[\![p]\!]n_x + [\![T_{xj}]\!]n_j + T_s = 0 \tag{1}$$

This expression becomes

$$-[\![-q\Phi_o + \Pi - \rho g x]\!]_{x=\xi} - [\![p']\!]_{x=0} + [\![\frac{\epsilon}{2}(E_o+e_x)^2]\!]_{x=\xi} + \gamma\left(\frac{\partial^2\xi}{\partial y^2}+\frac{\partial^2\xi}{\partial z^2}\right)=0 \tag{2}$$

Note that $E_o=E_o(x)$, so that there is a perturbation part of E_o^2 evaluated at the interface, namely $\xi E_o\, dE_o/dx$. Thus, with the equilibrium part of Eq. 2 cancelled out, the remaining part is

$$[\![q\frac{d\Phi_o}{dx}]\!]\hat{\xi} + g\hat{\xi}(\rho_a-\rho_b) - (\hat{p}^d-\hat{p}^e) + [\![\epsilon E_o \hat{e}_x]\!] + [\![\epsilon E_o \frac{dE_o}{dx}]\!]\hat{\xi} - \gamma k^2\hat{\xi} = 0 \tag{3}$$

It is the bulk relations written in terms of $\hat{\pi}$ that are available, so this expression is now written using the definition $\hat{p} = \hat{\pi} - q\hat{\Phi}$. Also, $d\Phi_o/dx = -E_o$ and $\epsilon\, dE_o/dx = q$, so Eq. 3 becomes

$$g(\rho_a-\rho_b)\hat{\xi} - [\![\hat{\pi}]\!] + [\![q\hat{\Phi}]\!] + [\![\epsilon E_o \hat{e}_x]\!] - \gamma k^2\hat{\xi} = 0 \tag{4}$$

Prob. 8.14.2 (cont.)

The first of the two electrical boundary conditions is

$$\bar{n}\times[\![\bar{E}]\!]\Big|_{x=\xi}=0 \Rightarrow [\![e_y]\!]+[\![E_o]\!]\frac{\partial\xi}{\partial y}=0 \tag{5}$$

and to linear terms this is

$$-[\![\hat{\Phi}]\!]+[\![E_o]\!]\hat{\xi}=0 \tag{6}$$

The second condition is

$$\bar{n}\cdot[\![\epsilon\bar{E}]\!]\Big|_{x=\xi}=0 \Rightarrow [\![\epsilon\hat{e}_x]\!]+[\![\epsilon\frac{dE_o}{dx}]\!]\hat{\xi}=0 \tag{7}$$

By Gauss Law, $\epsilon dE_o/dx=q$ and so this expression becomes

$$[\![\epsilon\hat{e}_x]\!]+[\![q]\!]\hat{\xi}=0 \tag{8}$$

These three boundary conditions, Eqs. 4, 6 and 8, are three equations in the unknowns $\hat{\xi},\hat{\pi}^d,\hat{\pi}^e,\hat{\Phi}^d,\hat{\Phi}^e,\hat{e}_x^d,\hat{e}_x^e$. Four more relations are provided by the electrical and mechanical bulk relations, Eqs. 12b, 13a, 14b and 15a, which are substituted into these boundary conditions to give

$$\begin{bmatrix} g(\rho_a-\rho_b)-\gamma k^2+\frac{\omega^2}{k}(\rho_a\coth ka+\rho_b\coth kb) & q_a+\epsilon_a E_o k\coth ka & -q_b+\epsilon_b E_o k\coth kb \\ [\![E_o]\!] & -1 & 1 \\ [\![q]\!] & \epsilon_a k\coth ka & \epsilon_b k\coth kb \end{bmatrix}\begin{bmatrix}\hat{\xi}\\ \hat{\Phi}^d \\ \hat{\Phi}^e\end{bmatrix}=0 \tag{9}$$

This determinant reduces to the desired dispersion equation.

Prob. 8.14.2 (cont.)

$$\frac{\omega^2}{k}(\rho_a \coth ka + \rho_b \coth kb) = g(\rho_b - \rho_a) + \gamma k^2 + E_a q_a - E_b q_b$$
$$+ \frac{(q_a - q_b)^2}{\epsilon_a \coth ka + \epsilon_b \coth kb} - \frac{2(\epsilon_b - \epsilon_a)(q_a E_a \coth kb + q_b E_b \coth ka)}{\epsilon_a \coth ka + \epsilon_b \coth kb} \quad (10)$$
$$- \frac{k(\epsilon_a - \epsilon_b)^2 E_a E_b}{\epsilon_a \tanh kb + \epsilon_b \tanh ka}$$

In the absence of convection, the first and second terms on the right represent the respective effects of gravity and capillarity. The third term on the right is an imposed field effect of the space charge, due to the interaction of the space charge with fields that could largely be imposed by the electrodes. By contrast, the fourth term, which is also due to the space-charge interaction, is proportional to the square of the space-charge discontinuity at the interface, and can, therefore, be interpreted as a self-field term, where the interaction is between the space charge and the field produced by the space charge. This term is present, even if the electric field intensity at the interface were to vanish. The fifth and sixth terms are clearly due to polarization, since they would not be present if the permittivities were equal. In the absence of any space-charge densities, only the sixth term would remain, which always tends to destabilize the interface. However, by contrast with the example of Sec. 8.10, the fifth term is one due to both the polarizability and the space charge. That is, E_a and E_b include effects of the space-charge. (See "Space-Charge Dynamics of Liquids", Phys. Fluids, 15 (1972), p. 1197.)

Problem 8.15.1

Because the force density is a pure gradient, Equation 7.8.11 is applicable. With $B_o = \mu_o I/2\pi r = -\frac{\partial A}{\partial r}$, it follows that $A = -(\mu_o I/2\pi r)\ln(r/R)$ so that $\mathcal{E} = -J_o A$ and Equation 7.8.11 becomes

$$p = \Pi - \frac{J_o \mu_o I}{2\pi} \ln\left(\frac{r}{R}\right) + \rho \frac{\partial \Theta}{\partial t} \tag{1}$$

Note that there are no self-fields giving rise to a perturbation field, as in Section 8.14. There are also no surface currents, so the pressure jump at the interface is equilibrated by the surface tension surface force density.

$$\Pi_a - \Pi_b = -\frac{\gamma}{R} \tag{2}$$

while the perturbation requires that

$$\frac{J_o \mu_o I}{2\pi} \ln\left(\frac{R+\xi}{R}\right) - p'^{b} = \gamma\left[\frac{\xi}{R^2} + \frac{1}{R^2}\frac{\partial^2 \xi}{\partial \theta^2}\right] \tag{3}$$

Linearization of the first term on the left $(\ln(1+x) \sim x)$, substitution to obtain complex amplitudes and use of the pressure-velocity relation for a column of fluid from Table 7.9.1 then gives an expression that is homogeneous $\hat{\xi} D(\omega, m) = 0$. Thus the dispersion equation, $D(\omega, m) = 0$, is

$$-\omega^2 \rho F_m(0,R) = -\frac{\gamma}{R^2}(1-m^2) + \frac{\mu_o J_o I}{2\pi R} \tag{4}$$

(c) Recall from Section 2.17 that $F_m(0,R) < 0$ and that the m = 0 mode is excluded because there is no z dependence. Surface tension therefore only tends to stabilize. However, in the m = 1 mode (which is a pure translation of the column) it has no effect and stability is determined by the electromechanical term. It follows that the m = 1 mode is unstable if $J_o I < 0$. Higher order modes become unstable for $-J_o I = (m^2-1)2\pi\gamma/\mu_o R$. Conversely,

Problem 8.15.1 (cont.)

all modes are stable if $J_o I > 0$. With J_o and I of the same sign, the $\bar{J}\times\mu_o\bar{H}$ force density is radially inward. The uniform current density fills regions of fluid extending outward providing an incremental increase in the pressure (say at r = R) of the fluid at any fixed location. The magnetic field is equivalent in its effect to a radially directed gravity that is inward if $J_o I > 0$.

Problem 8.16.1 In static equilibrium

$$S_{xx} = -p = \begin{cases} -\Pi_o & ; x>0 \\ -\Pi_o + \rho g x + \frac{1}{2}\epsilon_o E_o^2 ; & x<0 \end{cases} \tag{1}$$

In the bulk regions, where there is no electromechanical coupling, the stress-velocity relations of Eq. 7.19.19 apply

$$\begin{bmatrix} \hat{S}_{xx}^e \\ \hat{S}_{yx}^e \end{bmatrix} = \eta \begin{bmatrix} \frac{\gamma}{k}(\gamma+k) & -j(\gamma-k) \\ j(\gamma-k) & \gamma+k \end{bmatrix} \begin{bmatrix} \hat{v}_x^e \\ \hat{v}_y^e \end{bmatrix} \tag{2}$$

and the flux-potential relations, Eq. (a) of Table 2.16.1, show that

$$\hat{E}_x^d = k\hat{\Phi}^d \tag{3}$$

The crux of the interaction is represented by the perturbation boundary conditions. Stress equilibrium in the x direction requires that

$$[\![S_{xj}]\!] n_j + [\![T_{xj}]\!] n_j - \gamma_a \nabla\cdot\bar{n}\, n_x = 0 \tag{4}$$

With the use of Eq. (d) of Table 7.6.2 and $\hat{\xi} = \hat{v}_x^b/j\omega$, the linearized version of this condition is

$$\frac{j\rho g}{k}\frac{\hat{v}_x^e}{(\omega/k)} + \epsilon_o E_o \hat{e}_x^d + j\gamma_a k \frac{\hat{v}_x^e}{(\omega/k)} - \hat{S}_{xx}^e = 0 \tag{5}$$

The stress equilibrium in the y direction requires that

$$[\![S_{yj}]\!] n_j + [\![T_{yj}]\!] n_j - \gamma_a (\nabla\cdot\bar{n}) n_y = 0 \tag{6}$$

and the linearized form of this condition is

$$\epsilon_o E_o \hat{e}_y^d - \frac{\epsilon_o E_o^2 k}{\omega}\hat{v}_x^e - \hat{S}_{yx}^e = 0 \tag{7}$$

Prob. 8.16.1 (cont.)

The tangential electric field must vanish on the interface, so

$$\hat{e}_y^d = \frac{E_o k}{\omega}\hat{v}_x^e \tag{8}$$

and from this expression and Eq. 7, it follows that the latter condition can be replaced with

$$\hat{S}_{yx}^e = 0 \tag{9}$$

Equations 2 and 3 combine with Eqs. 5 and 9 to give the homogeneous equations

$$\begin{bmatrix} \frac{j\rho g}{\omega} - j\frac{\epsilon_o E_o^2 k}{\omega} + j\frac{\gamma_a k^2}{\omega} - \eta\frac{\gamma}{k}(\gamma+k) & j\eta(\gamma-k) \\ j(\gamma-k) & (\gamma+k) \end{bmatrix}\begin{bmatrix} \hat{v}_x^e \\ \hat{v}_y^e \end{bmatrix} = 0 \tag{10}$$

Multiplied out, the determinant becomes the desired dispersion equation.

$$j\omega\eta\frac{[k(\gamma-k)^2 - \gamma(\gamma+k)^2]}{k(\gamma+k)} = -(\epsilon_o k E_o^2 - \gamma_a k^2 - \rho g) \tag{11}$$

With the use of the definition $\gamma^2 \equiv k^2 + j\omega\rho/\eta$, this expression becomes

$$-\frac{j\omega\eta}{k}\left(\frac{4k^2\gamma}{\gamma+k} + \frac{j\omega\rho}{\eta}\right) = \rho g + k^2\gamma_a - \epsilon_o k E_o^2 \tag{12}$$

Now, in the limit of low viscosity, $k/\gamma \rightarrow 0$ and Eq. 12 become

$$\omega^2\frac{\rho}{k} - j4k\eta\omega - (\rho g + k^2\gamma_a - \epsilon_o k E_o^2) = 0 \tag{13}$$

which can be solved for ω.

$$\omega = j\frac{2k^2\eta}{\rho} \pm \sqrt{-\left(\frac{2k^2\eta}{\rho}\right)^2 + \frac{k}{\rho}(\rho g + k^2\gamma_a - \epsilon_o k E_o^2)} \tag{14}$$

Note that in this limit, the rate of growth depends on viscosity, but the field for incipience of instability does not.

In the high viscosity limit, $\gamma \approx k + j\omega\rho/2\eta k$ and Eq. 12 become

$$-\frac{j\omega\rho}{k}\left[\frac{4k^2\left(k + \frac{j\omega\rho}{2\eta k}\right)}{2k + \frac{j\omega\rho}{2\eta k}} + \frac{j\omega\rho}{\eta}\right] = \rho g + k^2\gamma_a - \epsilon_o k E_o^2 \tag{15}$$

Prob. 8.16.1 (cont.)

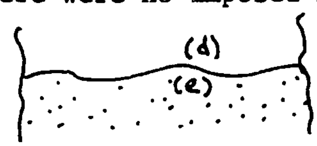
Further expansion of the denominator reduces this expression to

$$\frac{3}{2}\frac{\omega^2 \rho}{k} = 2j\omega\eta k + \rho g + k^2\gamma_R - \epsilon_o k E_o^2 \tag{16}$$

Again, viscosity effects the rate of growth, but not the conditions for incipience of instability.

Problem 8.16.2 In static equilibrium, there is no surface current, and so the distribution of pressure is the same as if there were no imposed $\bar{H}$.

$$S_{xx} = -p = \begin{cases} -\Pi_o & ; x>0 \\ -\Pi_o + \rho g x & ; x<0 \end{cases} \tag{1}$$



The perfectly conducting interface is to be modeled by its boundary conditions. The magnetic flux density normal to the interface is taken as continuous.

$$\bar{n}\cdot[\![\bar{B}]\!] = 0 \tag{2}$$

With this understood, consider the consequences of flux conservation for a surface of fixed identity in the interface (Eqs. 2.6.4 and 6.2.4).

$$\frac{d}{dt}\int_S \bar{B}\cdot\bar{n}\,da = \int_S \left[\frac{\partial\bar{B}}{\partial t} + \nabla\times(\bar{B}\times\bar{v})\right]\cdot\bar{n}\,da = 0 \tag{3}$$

Linearized, and in view of Eq. 2, this condition becomes

$$\frac{\partial H_x}{\partial t} = -H_o\frac{\partial v_y}{\partial y} \Rightarrow \hat{H}_x^d = \hat{H}_x^e = H_o k\,\hat{v}_y^e/\omega \tag{4}$$

Bulk conditions in the regions to either side of the interface represent the fluid and fields without a coupling. The stress-velocity conditions for the lower half-space are Eqs. 2.19.19.

$$\begin{bmatrix}\hat{S}_{xx}^e \\ \hat{S}_{yx}^e\end{bmatrix} = \eta\begin{bmatrix}\frac{\gamma}{k}(\gamma+k) & -j(\gamma-k) \\ j(\gamma-k) & \gamma+k\end{bmatrix}\begin{bmatrix}\hat{v}_x^e \\ \hat{v}_y^e\end{bmatrix}$$

While the flux-potential relations for the magnetic fields, Eqs. (a) of Table 2.16.1, reduce to

$$\hat{B}_x^d = \mu_o k\,\hat{\psi}^d = -j\mu_o\hat{H}_y^d\,;\ \hat{B}_x^e = -\mu_o k\,\hat{\psi}^e = j\mu_o\hat{H}_y^e \tag{6}$$

Prob. 8.16.2 (cont.)

Boundary conditions at the interface for the fields are the linearized versions of Eqs. 2 and 4. For the fluid, stress balance in the x direction requires

$$\frac{j\rho g}{\omega}\hat{v}_x^e + j\frac{\gamma_a k^2}{\omega}\hat{v}_x^e - \hat{S}_{xx}^e = 0 \tag{7}$$

where $\hat{v}_x^e = j\omega\hat{\xi}$. Stress balance in the y direction requires

$$-\hat{S}_{yx}^e + 2\mu_o H_o \hat{H}_y^d = 0 \tag{8}$$

$$\begin{bmatrix} j\frac{\rho g}{\omega} + j\frac{\gamma_a k^2}{\omega} - \eta\left(\frac{\gamma}{k}\right)(\gamma+k) & j\eta(\gamma-k) \\ j\eta(\gamma-k) & \eta(\gamma+k) - j2\mu_o H_o^2\frac{k}{\omega} \end{bmatrix}\begin{bmatrix}\hat{v}_x^e \\ \hat{v}_y^e\end{bmatrix} = 0 \tag{9}$$

It follows that the required dispersion equation is

$$\left[\rho g + \gamma_a k^2 + j\eta\left(\frac{\gamma}{k}\right)(\gamma+k)\omega\right]\left[\eta(\gamma+k) - j2\mu_o H_o^2\frac{k}{\omega}\right] - j\eta^2(\gamma-k)^2\omega = 0 \tag{10}$$

In the low viscosity limit, $\gamma \sim \sqrt{j\omega\rho/\eta} + \frac{1}{2}\sqrt{\eta/j\omega\rho}\,k^2$ and therefore the last term goes to zero as $\eta \to 0$ so that the equation factors into the dispersion equations for two modes. The first, the transverse mode, is represented by the first term in brackets in Eq. 10, which can be solved to give the dispersion equation for a gravity-capillary mode with no coupling to the magnetic field.

$$\omega^2 = gk + \frac{\gamma_a k^2}{\rho} \tag{11}$$

The second term in brackets becomes the dispersion equation for the mode involving dilatations of the interface.

$$\omega = \omega_c\left[\frac{\sqrt{3}}{2} + \frac{j}{2}\right]; \quad \omega_c \equiv \left[\frac{2\mu_o H_o^2 k}{\sqrt{\eta\rho}}\right]^{2/3} \tag{12}$$

If $\omega > \omega_c$, then in the second term in brackets of Eq. 10, $\eta(\gamma+k) > 2\mu_o H_o^2 k/\omega$ and the dispersion equation is as though there were no electromechanical coupling. Thus, for $\omega \gg \omega_c$ the damping effect of viscosity is much as in Problem 8.16.1. In the opposite extreme, if $\omega \ll \omega_c$, then the second term

Prob. 8.16.2 (cont.)

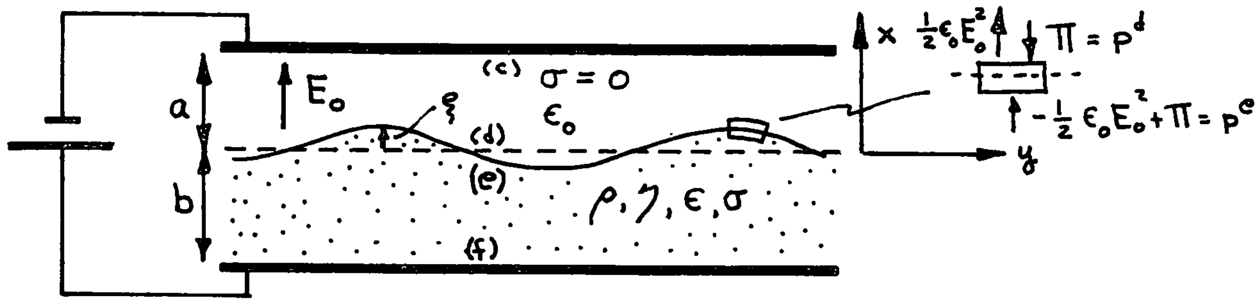
has $\eta(\gamma+k)<<2\mu_o H_o^2 k/\omega$ and is approximated by the magnetic field term. In this case, Eq. 10 is approximated by

$$\rho g + \gamma_a k^2 - \frac{\omega^2\rho}{\eta} + j\eta\omega + j\frac{\omega^3\rho\eta}{2\mu_0 H_o^2 k} = 0 \tag{13}$$

In the limit of very high H_o, the last term is negligible and the remainder of the equation can be used to approximate the damping effect of viscosity.

Certainly the model is not meaningful unless the magnetic diffusion time based on the sheet thickness and the wavelength is small compared to times of interest. Suggested by Eq. 6.10.2 in the limit $d\to\infty$ is a typical magnetic diffusion time $\mu\sigma a/k$, where a is the thickness of the "perfectly" conducting layer.

Prob. 8.16.3 A cross-section of the configuration is shown in the figure.



In static equilibrium, the electric field intensity is

$$\bar{E} = \begin{cases} E_o \bar{i}_x & x > 0 \\ 0 & x < 0 \end{cases} \tag{1}$$

and in accordance with the stress balance shown in the figure, the mechanical stress, S_{xx}, reduces to simply the negative of the hydrodynamic pressure.

$$S_{xx} = -p = \begin{cases} -\Pi \\ \rho g x + \frac{1}{2}\epsilon_o E_o^2 - \Pi \end{cases} \tag{2}$$

Electrical bulk conditions reflecting the fact that $\bar{E} = -\nabla\Phi$ where Φ satisfies Laplace's equation both in the air-gap and in the liquid layer are Eqs. (b) from Table 2.16.1. Incorporated at the outset are the boundary conditions $\hat{\Phi}^c = 0$ and $\hat{\Phi}^f = 0$, reflecting the fact that the upper and lower electrodes are highly conducting.

$$\hat{e}_x^d = k \coth ka \, \hat{\Phi}^d \tag{3}$$

$$\hat{e}_x^e = -k \coth kb \, \hat{\Phi}^e \tag{4}$$

The mechanical bulk conditions, reflecting mass conservation and force equilibrium for the liquid, which has uniform mass density and viscosity, are Eqs. 7.20.6. At the outset, the boundary conditions at the lower electrode requiring that both the tangential and normal liquid velocities be zero are incorporated in writing these expressions $(\hat{v}_x^f = 0, \ \hat{v}_y^f = 0)$.

$$\hat{S}_{xx}^e = \eta P_{11} \hat{v}_x^e + \eta P_{13} \hat{v}_y^e \tag{5}$$

$$\hat{S}_{yx}^e = \eta P_{31} \hat{v}_x^e + \eta P_{33} \hat{v}_y^e \tag{6}$$

Prob. 8.16.3 (cont.)

Boundary conditions at the upper and lower electrodes have already been included in writing the bulk relations. The conditions at the interface remain to be written, and of course represent the electromechanical coupling. Charge conservation for the interface, Eq. 23 of Table 2.10.1 and Gauss'law, require that

$$\frac{\partial \sigma_f}{\partial t} = -\nabla_\Sigma \cdot (\sigma_f \bar{v}) - \bar{n}\cdot [\![\sigma \bar{E}]\!] \tag{7}$$

where by Gauss'law $\sigma_f = \bar{n}\cdot[\![\epsilon \bar{E}]\!]$.

Linearized and written in terms of the complex amplitudes, this requires that

$$j\omega(\epsilon_o \hat{e}_x^d - \epsilon \hat{e}_x^e) = jk\epsilon_o E_o \hat{v}_y^e + \sigma \hat{e}_x^e \tag{8}$$

The tangential electric field at the interface must be continuous. In linearized form this requires that

$$[\![e_y]\!] + \frac{\partial \xi}{\partial y} E_o = 0 \tag{9}$$

Because $\hat{\xi} = \hat{v}_x / j\omega$ and $\hat{e}_y = jk\hat{\Phi}$, this condition becomes

$$\hat{\Phi}^d - \hat{\Phi}^e - \frac{\hat{v}_x^e}{j\omega} E_o = 0 \tag{10}$$

In general, the balance of pressure and viscous stresses (represented by S_{ij}), of the Maxwell stress and of the surface tension surface force density, require that

$$[\![S_{ij}]\!] n_j + [\![T_{ij}]\!] n_j + n_i \gamma \frac{\partial^2 \xi}{\partial y^2} = 0 \tag{11}$$

With i=x (the x component of the stress balance) this expression requires that to linear terms

$$[\![S_{xx}]\!] + [\![S_{xy}]\!]\left(-\frac{\partial \xi}{\partial y}\right) + [\![T_{xx}]\!] + [\![T_{xy}]\!]\left(-\frac{\partial \xi}{\partial y}\right) + \gamma \frac{\partial^2 \xi}{\partial y^2} = 0 \tag{12}$$

By virtue of the forsight in writing the equilibrium pressure, Eq. 2, the equilibrium parts of Eq. 12 balance out. The perturbation part requires that

$$-\frac{\rho g}{j\omega}\hat{v}_x^e - \hat{S}_{xx}^e + \epsilon_o E_o \hat{e}_x^d - \frac{\gamma k^2}{j\omega}\hat{v}_x^e = 0 \tag{13}$$

Prob. 8.16.3 (cont.)

With i=y, (the shear component of the stress balance) Eq. 11 requires that

$$[\![S_{yx}]\!] + [\![S_{yy}]\!]_o \left(-\frac{\partial \xi}{\partial y}\right) + [\![T_{yx}]\!] + [\![T_{yy}]\!]_o \left(-\frac{\partial \xi}{\partial y}\right) = 0 \tag{14}$$

Observe that the equilibrium quantities $[\![S_{yy}]\!]_o = -\frac{1}{2}\epsilon_o E_o^2$ and $[\![T_{yy}]\!] = -\frac{1}{2}\epsilon_o E_o^2$ so that this expression reduces to

$$-\hat{S}^e_{yx} - \frac{\epsilon_o E_o^2 k}{\omega}\hat{v}^e_x + jk\epsilon_o E_o \hat{\Phi}^d = 0 \tag{15}$$

The combination of the bulk and boundary conditions, Eqs. 3-6,8,10,13 and 15, comprise eight equations in the unknowns $(\hat{e}^d_x, \hat{e}^e_x, \hat{\Phi}^d, \hat{\Phi}^e, \hat{S}^e_{xx}, \hat{S}^e_{yx}, \hat{v}^e_x, \hat{v}^e_y)$. The dispersion equation will now be determined in two steps. First, consider the "electrical" relations. With the use of Eqs. 3 and 4, Eqs. 8 and 10 become

$$\begin{bmatrix} j\omega k\epsilon_o \coth ka & j\omega\epsilon k\coth kb + \sigma k\coth kb \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \hat{\Phi}^d \\ \hat{\Phi}^e \end{bmatrix} = \begin{bmatrix} jk\epsilon_o E_o \hat{v}^e_y \\ \dfrac{E_o}{j\omega}\hat{v}^e_x \end{bmatrix} \tag{16}$$

From these two expressions, it follows that

$$\hat{\Phi}^d = \frac{j\epsilon_o E_o \hat{v}^e_y + \frac{E_o}{j\omega}\hat{v}^e_x(j\omega\epsilon\coth kb + \sigma\coth kb)}{j\omega(\epsilon_o\coth ka + \epsilon\coth kb) + \sigma\coth kb} \tag{17}$$

In terms of $\hat{\Phi}^d$, $\hat{e}^d_x$ is easily written using Eq. 3.

The remaining two boundary conditions, the stress balance conditions of Eqs. 13 and 15 can now be written in terms of $(\hat{v}^e_y, \hat{v}^e_x)$ alone.

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\begin{bmatrix} \dfrac{\hat{v}^e_x}{j\omega} \\ \hat{v}^e_y \end{bmatrix} = 0 \tag{18}$$

where

$$M_{11} = -j\omega\eta P_{11} - \rho g - \gamma k^2 + \frac{\epsilon_o E_o^2 k \coth ka \coth kb\,(j\omega\epsilon + \sigma)}{j\omega(\epsilon_o \coth ka + \epsilon \coth kb) + \sigma \coth kb}$$

$$M_{12} = -\eta P_{13} + \frac{j\,\epsilon_o^2 E_o^2 k \coth ka}{j\omega(\epsilon_o \coth ka + \epsilon \coth kb) + \sigma \coth kb}$$

$$M_{21} = -j\omega\eta P_{31} - j\epsilon_o E_o^2 k + \frac{j\,k\epsilon_o E_o^2 (j\omega\epsilon + \sigma)\coth kb}{j\omega(\epsilon_o \coth ka + \epsilon \coth kb) + \sigma \coth kb}$$

$$M_{22} = -\eta P_{33} - \frac{k\,\epsilon_o^2 E_o^2}{j\omega(\epsilon_o \coth ka + \epsilon \coth kb) + \sigma \coth kb}$$

The dispersion equation follows from Eq. 18 as

$$\underline{M}_{11}\underline{M}_{22} - \underline{M}_{12}\underline{M}_{21} = 0 \tag{19}$$

Here, it is convenient to normalize variables such that

$$\underline{\omega} = \frac{\omega b \eta}{\gamma} \;;\; \underline{a} = \frac{a}{b} \;;\; \underline{P}_{ij} = b P_{ij}$$
$$\underline{\rho} = \frac{\rho g b^2}{\gamma} \;;\; \underline{U} = \frac{b\epsilon_o E_o^2}{\gamma} \;;\; \frac{\omega\epsilon}{\sigma} = \underline{\omega}\tau \tag{20}$$
$$\underline{k} = kb \;;\; \tau = \frac{\gamma}{b\eta}\frac{\epsilon}{\sigma}$$

and to define

$$C = \frac{\epsilon_o}{\epsilon}\coth \underline{k}\underline{a} + \coth \underline{k} \;;\; R = \coth \underline{k} \;;\; S = \coth \underline{k}\underline{a} \tag{21}$$

so that in Eq. 19,

$$\underline{M}_{11} = \frac{b^2}{\gamma}M_{11} = -\underline{P}_{11}\, j\underline{\omega} - \underline{\rho} - \underline{k}^2 + \frac{\underline{k}\underline{U} R S (j\underline{\omega}\tau + 1)}{j\underline{\omega}\tau C + R}$$

$$\underline{M}_{12} = \frac{b}{\eta}M_{12} = -\underline{P}_{13} + j\frac{\frac{\epsilon_o}{\epsilon}\tau \underline{U}\,\underline{k}\, S}{j\underline{\omega}\tau C + R} \tag{22}$$

$$\underline{M}_{21} = \frac{b^2}{\gamma}M_{21} = -\underline{P}_{31}\, j\underline{\omega} - j\underline{U}\underline{k} + j\frac{\underline{k}\underline{U}(j\underline{\omega}\tau + 1)R}{j\underline{\omega}\tau C + R}$$

$$\underline{M}_{22} = \frac{b}{\eta}M_{22} = -\underline{P}_{33} - \frac{\underline{k}\underline{U}\frac{\epsilon_o}{\epsilon}\tau}{j\underline{\omega}\tau C + R}$$

Prob. 8.16.3 (cont.)

If viscous stresses dominate those due to inertia, the P_{ij} in these expressions are independent of frequency. In the following, this approximation of low-Reynolds number flow is understood. (Note that the dispersion equation can be used if inertial effects are included simply by using Eq. 7.19.13 to define the P_{ij}. However, there is then a complex dependence of these terms on the frequency, reflecting the fact that viscous diffusion occurs on time scales of interest.)

With the use of Eqs. 22, Eq. 19 becomes

$$\left\{(j\underline{\omega}\, r\, C + R)(-j\underline{\omega}\,\underline{P}_{11} - \rho - \underline{k}^2) + \underline{k}\,\underline{U}\, R\, S(j\underline{\omega}\, r + 1)\right\}\left\{-\underline{P}_{33}(j\underline{\omega}\, r\, C + R) - \underline{k}\,\underline{U}\,\frac{\epsilon_o}{\epsilon}\, r\right\}$$
$$+\left\{\underline{P}_{13}(j\underline{\omega}\, r\, C + R) - j\,\frac{\epsilon_o}{\epsilon}\, r\, \underline{U}\,\underline{k}\, S\right\}\left\{-(\underline{P}_{31}\, j\underline{\omega} + j\,\underline{U}\,\underline{k})(j\underline{\omega}\, r\, C + R) + j\,\underline{k}\,\underline{U}(j\underline{\omega}\, r + 1)R\right\} = 0 \qquad (23)$$

That this dispersion equation is in general cubic in $j\omega$ reflects the coupling it represents of the gravity-capillary-electrostatic waves, shear waves and the charge relaxation phenomena (the third root).

Consider the limit where charge relaxation is complete on time scales of interest. Then the interface behaves as an equipotential, $r \rightarrow 0$, and Eq. 23 reduces to

$$j\underline{\omega} = \frac{(\underline{k}\,\underline{U}\, S - \rho - \underline{k}^2)\,\underline{P}_{33}}{(\underline{P}_{11}\underline{P}_{33} - \underline{P}_{13}\underline{P}_{31})} \qquad (24)$$

That there is only one mode is to be expected. Charge relaxation has been eliminated (is instantaneous) and because there is no tangential electric field on the interface, the shear mode has as well. Because damping dominates inertia, the gravity-capillary-electrostatic wave is over damped, or grows as a pure exponential. The factor

$$\frac{\underline{P}_{33}}{\underline{P}_{11}\underline{P}_{33} - \underline{P}_{13}\underline{P}_{31}} \equiv f(\underline{k}) = \frac{\left(\frac{1}{4}\sinh 2\underline{k} - \frac{\underline{k}}{2}\right)\left(\sinh^2\underline{k} - \underline{k}^2\right)}{\underline{k}\left(\frac{1}{4}\sinh^2 2\underline{k} - \underline{k}^2 - \underline{k}^4\right)} \qquad (25)$$

is positive, so the interface is unstable if

$$\underline{U} > (\rho + \underline{k}^2)/S\underline{k} \qquad (26)$$

Prob. 8.16.3 (cont.)

In the opposite extreme, where the liquid is sufficiently insulating that charge relaxation is negligible so that $r >> 1$, Eq. 23 reduces to a quadratic expression ($P_{13} = -P_{31}$).

$$a(j\omega)^2 + b(j\omega) + c = 0$$

$$a \equiv \underline{P}_{11}\underline{P}_{33} + \underline{P}_{13}^2;\; b = \left[(\underline{\rho} + \underline{k}^2)P_{33} + \underline{U}\underline{k}\left(\frac{\underline{P}_{11}}{C}\frac{\epsilon_0}{\epsilon} - \frac{RS}{C}P_{33}\right) - 2j\frac{\underline{k}\underline{U}P_{13}S}{C}\frac{\epsilon_0}{\epsilon}\right];\; c = \frac{\underline{k}\underline{U}}{C}\frac{\epsilon_0}{\epsilon}\left[\underline{\rho} + \underline{k}^2 - \underline{U}\underline{k}S\right] \tag{27}$$

The roots of this expression represent the gravity-capillary-electrostatic and shear modes. In this limit of a relatively insulating layer, there are electrical shear stresses on the interface. In fact these dominate in the transport of the surface charge.

To find the general solution of Eq. 23, it is necessary to write it as a cubic in $j\omega$.

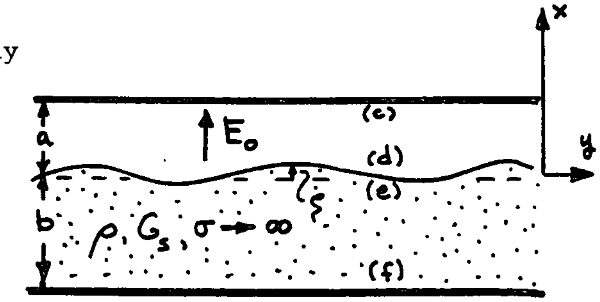
$$(j\omega)^3 + P'(j\omega)^2 + Q'(j\omega) + R' = 0 \tag{28}$$

$$P' = \Big\{2\underline{P}_{11}\underline{P}_{33}rCR + \underline{P}_{33}r^2C\left[C(\underline{\rho} + \underline{k}^2) - \underline{k}\underline{U}RS\right] + r^2C\underline{P}_{11}\underline{k}\underline{U}\frac{\epsilon_0}{\epsilon} + \underline{P}_{13}rC\left[2\underline{P}_{13}R - j\underline{k}\underline{U}r\frac{2\epsilon_0}{\epsilon}S\right]\Big\}\Big/r^2C^2(\underline{P}_{11}\underline{P}_{33} + \underline{P}_{13}^2)$$

$$Q' = \Big\{\underline{P}_{33}rC\left[(\underline{\rho} + \underline{k}^2)R - \underline{k}URS\right] + \left[R\underline{P}_{11} + rC(\underline{\rho} + \underline{k}^2) - r\underline{k}\underline{U}RS\right]\left[\underline{k}\underline{U}\frac{\epsilon_0}{\epsilon}r + \underline{P}_{33}R\right] + \left[\underline{P}_{13}R - j\frac{\epsilon_0}{\epsilon}r\underline{U}\underline{k}S\right]\left[\underline{P}_{13}R - j\underline{k}r\underline{U}\frac{\epsilon_0}{\epsilon}S\right]\Big\}\Big/r^2C^2(\underline{P}_{11}\underline{P}_{33} + \underline{P}_{13}^2)$$

$$R' = \Big\{\left[\underline{k}\underline{U}\frac{\epsilon_0}{\epsilon}r + \underline{P}_{33}R\right]\left[(\underline{\rho} + \underline{k}^2)R - \underline{k}\underline{U}RS\right]\Big\}\Big/r^2C^2(\underline{P}_{11}\underline{P}_{33} + \underline{P}_{13}^2)$$

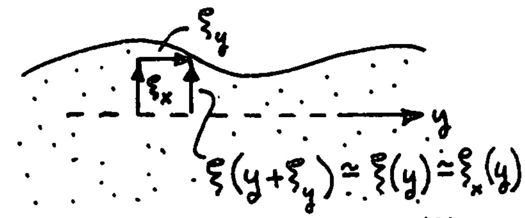
Prob 8.16.4 Because the solid is relatively conducting compared to the gas above, the equilibrium electric field is simply



$$\bar{E} = \begin{cases} E_o \bar{i}_x & x > 0 \\ 0 & x < 0 \end{cases} \quad (1)$$

In the solid, the equations of motion are

$$\rho \frac{\partial^2 \bar{\xi}}{\partial t^2} = -\nabla p + G_s \nabla^2 \bar{\xi} - \rho g \bar{i}_x ; \ \nabla \cdot \bar{\xi} = 0 \quad (2)$$



where

$$S_{ij} = -p + G_s \left(\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i} \right) \quad (3)$$

It follows from Eq. 2b that

$$\frac{\partial \xi_x}{\partial x} = 0 \Rightarrow \xi_x = \text{const} = 0 \quad (4)$$

so that the static x component of the force equation reduces to

$$\frac{\partial p}{\partial x} = G_s \frac{\partial^2 \xi_x}{\partial x^2} - \rho g \Rightarrow p = \begin{cases} \Pi_a & ; x > 0 \\ \Pi_b - \rho g x & ; x < 0 \end{cases} \quad (5)$$

This expression, together with the condition that the interface be in stress equilibrium, determines the equilibrium stress distribution

$$S_{xx} = -p = \begin{cases} -\Pi_a & ; x > 0 \\ \rho g x - \Pi_a + \frac{1}{2}\epsilon_o E_o^2 & ; x < 0 \end{cases} \quad (6)$$

In the gas above, the perturbation fields are represented by Laplace's equation, and hence the transfer relations (a) of Table 2.16.1

$$\begin{bmatrix} \hat{e}_x^c \\ \hat{e}_x^d \end{bmatrix} = k \begin{bmatrix} -\coth ka & \frac{1}{\sinh ka} \\ -\frac{1}{\sinh ka} & \coth ka \end{bmatrix} \begin{bmatrix} \hat{\Phi}^c \\ \hat{\Phi}^d \end{bmatrix} \quad (7)$$

Perturbation deformations in the solid are described by the analogue transfer relations

$$\begin{bmatrix} \hat{S}_{xx}^e \\ \hat{S}_{xx}^f \\ \hat{S}_{yx}^e \\ \hat{S}_{yx}^f \end{bmatrix} = G_s [P_{ij}] \begin{bmatrix} \hat{\xi}_x^e \\ \hat{\xi}_x^f \\ \hat{\xi}_y^e \\ \hat{\xi}_y^f \end{bmatrix} \quad \text{where } \gamma \equiv \sqrt{k^2 - \frac{\omega^2 \rho}{G_s}} \quad (8)$$

The interface is described in Eulerian coordinates by $\xi(y,t)$ with this variable related to the deformation of the interface as suggested by the figure.

Prob. 8.16.4 (cont.) Boundary conditions on the fields in the gas recognize that the electrode and the interface are each equipotentials.

$$\hat{\Phi}^c = 0 \tag{9}$$

$$\left[\bar{n} \times \bar{E}\right]_{x=\xi} = 0 \Rightarrow \hat{\Phi}^d = E_o \hat{\xi}^e_x \tag{10}$$

Stress equilibrium for the interface is in general represented by

$$[\![S_{ij}]\!] n_j + [\![T_{ij}]\!] n_j = 0 \tag{11}$$

where i is either x or y. To linear terms, the x component requires that

$$-\hat{S}^e_{xx} + \epsilon_o E_o \hat{e}^d_x - \rho g \hat{\xi}^e_x = 0 \tag{12}$$

where the equilibrium part balances out by virtue of the static equilibrium, Eq. 5. The shear component of Eq. 11, i=y, becomes

$$\left(S^d_{yx} - S^e_{yx}\right) + \left(S^d_{yy} - S^e_{yy}\right)\left(-\frac{\partial \xi}{\partial y}\right) + \left(T^d_{yx} - T^e_{yx}\right) + \left(T^d_{yy} - T^e_{yy}\right)\left(-\frac{\partial \xi}{\partial y}\right) = 0 \tag{13}$$

Because there is no electrical shear stress on the interface, a fact represented by Eq. 10, this expression reduces to

$$\hat{S}^e_{yx} = 0 \tag{14}$$

In addition, the rigid bottom requires that

$$\hat{\xi}^f_x = 0 \;;\; \hat{\xi}^f_y = 0 \tag{15}$$

The dispersion equation is now found by writing Eqs. 12 and 14 in terms of $(\hat{\xi}^e_x, \hat{\xi}^e_y)$. To this end, Eq. 8a is substituted for $\hat{S}^e_{xx}$ using Eqs. 15 and $\hat{e}^d_x$ is substituted using Eq. 7b evaluated using Eqs. 9 and 10. This is the first of the two expressions

$$\begin{bmatrix} -G_s P_{11} + \epsilon_o k \coth ka \, E_o^2 - \rho g & -G_s P_{13} \\ -G_s P_{31} & -G_s P_{33} \end{bmatrix} \begin{bmatrix} \hat{\xi}^e_x \\ \hat{\xi}^e_y \end{bmatrix} = 0 \tag{16}$$

Prob. 8.16.4 (cont.)

The second expression is Eq. 14 evaluated using Eq. 8c for $\hat{S}^e_{yx}$ with Eqs. 15.

It follows from Eq. 16 that the desired dispersion equation is

$$P_{11}P_{33} - P_{33}\frac{\epsilon_o E_o^2 k \coth ka - \rho g}{G_s} - P_{13}P_{31} = 0 \qquad (17)$$

where in general, P_{ij} are defined with Eq. 7.19.13 (γ defined with Eq. 8). In the limit where $k^2 \gg \omega^2\rho/G_s$, the P_{ij} become those defined with Eq. 7.20.6.

With the assumption that perturbations having a given wavenumber, k, become unstable by passing into the right half $j\omega$ plane through the origin, it is possible to interprete the roots of Eq. 17 in the limit $\omega \to 0$ as giving the value of $\epsilon_o E_o^2/G_s$ required for instability.

$$\frac{P_{11}P_{33} - P_{13}P_{31}}{P_{33}} = \frac{\epsilon_o E_o^2 k}{G_s}\coth ka - \frac{\rho g}{G_s} \qquad (18)$$

In particular, this expression becomes

$$\frac{\epsilon_o E_o^2 k \coth ka}{G_s} - \frac{\rho g}{G_s} = \frac{\left\{\left[\frac{1}{4}\sinh(2kb) + \frac{kb}{2}\right]\left[\frac{1}{4}\sinh 2kb - \frac{kb}{2}\right] - \frac{1}{4}(kb)^4\right\}}{\left[\frac{1}{4}\sinh(2kb) - \frac{kb}{2}\right]\left[\sinh^2 kb - (kb)^2\right]} \qquad (19)$$

so that the function on the right depends on kb and a/b. In general, a graphical solution would give the most critical value of kb. Here, the short-wave limit of Eq. 19 is taken, where it becomes

$$\epsilon_o E_o^2 = G_s/4 \qquad (20)$$

Problem 8.18.1 For the linear distribution of charge density, the equation is $q = q_e + Dq_e x$. Thus, the upper uniform charge density must have value of $(3d/4)q$ while the lower one must have magnitude of $(d/4)q$. Evaluation gives

$$q_a = q_e + \frac{3}{4} D q_e d \quad ; \quad q_b = q_e + \frac{1}{4} D q_e d \tag{1}$$

The associated equilibrium electric field follows from Gauss' Law and the condition that the potential at x=0 is V_o.

$$E_x = \begin{cases} E_o + \frac{q_a}{\epsilon_o}\left(x - \frac{d}{2}\right) ; & x > \frac{d}{2} \\ E_o + \frac{q_b}{\epsilon_o}\left(x - \frac{d}{2}\right) ; & x < \frac{d}{2} \end{cases} \tag{2}$$

and the condition that the potential be V_o at x=0 and be 0 at x=d.

$$V_o = \int_0^d E_x \, dx = E_o d + (q_a - q_b)\frac{d^2}{8\epsilon_o} \tag{3}$$

With the use of Eqs. 1, this expression becomes

$$E_o = \frac{V_o}{d} - \frac{d^2}{16\epsilon_o} D q_e \tag{4}$$

Similar to Eqs. 1 are those for the mass densities in the layer model.

$$\rho_a = \rho_m + \frac{3}{4} D\rho_m d \quad ; \quad \rho_b = \rho_m + \frac{1}{4} D\rho_m d \tag{5}$$

For the two layer model, the dispersion equation is Eq. 8.14.25, which evaluated using Eqs. 1, 4 and 5, becomes

$$\frac{\omega^2}{k}\rho_m\left(2 + \frac{D\rho_m}{\rho_m} d\right)\coth\left(\frac{kd}{2}\right) = \frac{1}{2}\left[\frac{V_o}{d} D q_e - g D\rho_m\right] d + \frac{(Dq_e)^2 d^3}{\epsilon_o}\left[\frac{1}{8kd\coth\left(\frac{kd}{2}\right)} - \frac{1}{32}\right] \tag{6}$$

In terms of the normalization given with Eq. 8.18.2, this expression becomes

$$\frac{\omega^2}{k}\left(2 + \frac{D\rho_m}{\rho_m}\right)\coth\frac{k}{2} = \frac{1}{2}\left[\frac{\frac{V_o}{d} D q_e - g D\rho_m}{\frac{|V_o|}{d}|Dq_e|}\right] + S\left[\frac{1}{8k\coth\left(\frac{k}{2}\right)} - \frac{1}{32}\right]\frac{Dq_e}{|Dq_e|} \tag{7}$$

With the numbers $Dq_e/|Dq_e| = 1$, $V_o/|V_o| = 1$, $k = 1$, $D\rho_m = 0$ and $S = 1$, Eq. 7 gives $\omega = 0.349$. The weak gradient approximation represented by Eq.

Prob. 8.18.1(cont.)

8.18.10 gives for comparison $\omega = 0.303$ while the numerical result representing the "exact" model, Fig. 8.18.2, gives a frequency that is somewhat higher than the weak gradient result but still lower than the layer model result, about 0.31. The layer model is clearly useful for estimating the frequency or growth rate of the dominant mode.

In the long-wave limit, $k \ll 1$, the weak-gradient imposed field result, Eq. 8.18.10, becomes

$$\omega^2 \to \frac{k^2 \mathcal{N}}{\pi^2} \tag{8}$$

In the same approximation it is appropriate to set S=0 in Eq. 7, which becomes

$$\omega^2 \to \frac{k^2 \mathcal{N}}{8} \tag{9}$$

where $D\rho_m \to 0$. Thus the layer model gives a frequency that is $\pi/\sqrt{8} = 1.11$ times that of the imposed-field weak gradient model.

In the short-wave limit, $k \gg 1$, the layer model predicts that the frequency increases with $\sqrt{k}$. This is in contrast to the dependence typified by Fig. 8.18.4 at short wavelengths with a smoothly inhomogeneous layer. This inadequacy of the layer model is to be expected, because it presumes that the structure of the discontinuity between layers is always sharp no matter how fine the scale of the surface perturbation. In fact, at short enough wavelengths, systems of miscible fluids will have an interface that is smoothly inhomogeneous because of molecular diffusion.

To describe higher order modes in the smoothly inhomogeneous system for wavenumbers that are not extremely short, more layers should be used. Presumably, for each interface, there is an additional pair of modes introduced. Of course, the modes are not identified with a single interface but rather involve the self-consistent deformation of all interfaces. The situation is formally similar to that introduced in Sec. 5.15.

Problem 8.18.2 The basic equations for the magnetizable but insulating inhomogeneous fluid are

$$\rho\left(\frac{\partial \bar{v}}{\partial t} + \bar{v}\cdot\nabla\bar{v}\right) = -\nabla p - \rho g\,\bar{i}_x - \frac{1}{2}H^2\nabla\mu \tag{1}$$

$$\nabla\cdot\bar{v} = 0 \tag{2}$$

$$\nabla\cdot\mu\bar{h} = 0 \tag{3}$$

$$\nabla\times\bar{h} = 0 \tag{4}$$

$$\frac{D\mu}{Dt} = 0 \tag{5}$$

$$\frac{D\rho}{Dt} = 0 \tag{6}$$

where $\bar{H} = H_s(x)\bar{i}_z + \bar{h}$.

In view of Eq. 4, $\bar{h} = -\nabla\psi$. This means that $\hat{h}_z = jk_z\hat{\psi}$ and for the present purposes it is more convenient to use $\hat{h}_z$ as a scalar "potential"

$$\hat{h}_x = -\frac{1}{jk_z}D\hat{h}_z \;;\; \hat{h}_y = \frac{k_y}{k_z}\hat{h}_z \tag{7}$$

With the definitions $\mu = \mu_s(x) + \mu'$ and $\rho = \rho_s(x) + \rho'$, Eqs. 5 and 6 link the perturbations in properties to the fluid displacement

$$\hat{\mu} = -\frac{\hat{v}_x D\mu_s}{j\omega} \;;\; \hat{\rho} = -\frac{\hat{v}_x D\rho}{j\omega} \tag{8}$$

Thus, with the use of Eq. 8a and Eqs. 7, the linearized version of Eq. 3 is

$$D(\mu_s D\hat{h}_z) = k^2\mu_s\hat{h}_z + j\frac{k_z^2}{\omega}H_s(D\mu_s)\hat{v}_x \;;\; k^2 \equiv k_y^2 + k_z^2 \tag{9}$$

and this represents the magnetic field, given the mechanical deformation.

To represent the mechanics, Eq. 2 is written in terms of complex amplitudes.

$$D\hat{v}_x = jk_y\hat{v}_y + jk_z\hat{v}_z \tag{1}$$

and, with the use of Eq. 8b, the x component of Eq. 1 is written in the linearized form

$$[\omega^2\rho_s + gD\rho_s + \tfrac{1}{2}H_s^2 D^2\mu_s]\hat{v}_x + \tfrac{1}{2}H_s^2(D\mu_s)D\hat{v}_x - j\omega H_s(D\mu_s)\hat{h}_z = j\omega D\hat{p} \tag{11}$$

Prob. 8.18.2(cont.)

Similarly, the y and z components of Eq. 1 become

$$j\omega\rho_s\hat{v}_y = jk_y\hat{p} - \frac{1}{2}\frac{k_y}{\omega}H_s^2(D\mu_s)\hat{v}_x \tag{12}$$

$$j\omega\rho_s\hat{v}_z = jk_z\hat{p} - \frac{1}{2}\frac{k_z}{\omega}H_s^2(D\mu_s)\hat{v}_x \tag{13}$$

With the objective of making $\hat{v}_x$ a scalar function representing the mechanics, these last two expressions are solved for $\hat{v}_y$ anf $\hat{v}_z$ and substituted into Eq. 10.

$$\omega\rho_s D\hat{v}_x = jk^2\hat{p} - \frac{1}{2}\frac{k^2}{\omega}H_s^2(D\mu_s)\hat{v}_x \tag{14}$$

This expression is then solved for $\hat{p}$, and the derivative taken with respect to x. This derivative can then be used to eliminate the pressure from Eq. 11.

$$D[\rho_s(D\hat{v}_x)] - k^2\left[\rho_s - \frac{\mathcal{N}}{\omega^2}\right]\hat{v}_x + j\frac{k^2 H_s(D\mu_s)}{\omega}\hat{h}_z = 0 \tag{15}$$

$$\mathcal{N} \equiv -gD\rho_s + \frac{1}{2}(D\mu_s)D(H_s^2) \cong -gD\rho_s$$

Equations 9 and 15 comprise the desired relations.

In an imposed field approximation where $H_s = H_o$ = constant and the properties have the profiles $\rho_s = \rho_m \exp\beta x$ and $\mu_s = \mu_o \exp\beta x$, Eqs. 9 and 15 become

$$\left[L + \frac{k^2\mathcal{N}}{\rho_s\omega^2}\right]\hat{v}_x + \left[\frac{jk^2 H_o\beta\mu_m}{\rho_m\omega}\right]\hat{h}_z = 0 \tag{16}$$

$$[L]\hat{h}_z + \left[\frac{jk_z^2 H_o\beta}{\omega}\right]\hat{v}_x = 0 \tag{17}$$

where $L \equiv D^2 + \beta D - k^2$

For these constant coefficient equations, solutions take the form $\exp\gamma x$ and $L \to \gamma^2 + \beta\gamma - k^2$. From Eqs. 16 and 17 it follows that

$$L^2 + \frac{k^2\mathcal{N}}{\rho_s\omega^2}L + \frac{k^2k_z^2}{\omega^2}\frac{H_o^2\beta^2\mu_m}{\rho_m} = 0 \tag{18}$$

Prob. 8.18.2(cont.)

Solution for L results in

$$L = a \pm b\,;\; a \equiv \frac{g\beta k^2}{2\omega^2}\;;\; b \equiv \left[\left(\frac{g\beta k^2}{2\omega^2}\right)^2 - \left(\frac{k k_z}{\omega}\frac{H_o\beta}{\sqrt{\rho_m/\mu_m}}\right)^2\right]^{1/2} \tag{19}$$

From the definition of L, the γ's representing the x dependence follow as

$$\gamma = -\frac{\beta}{2} \pm c_{\pm}\;;\; c_{\pm} \equiv \left[\left(\frac{\beta}{2}\right)^2 + k^2 + a \pm b\right]^{\frac{1}{2}} \tag{20}$$

In terms of these γ's,

$$\hat{v}_x = e^{-\frac{\beta}{2}x}\left[\hat{A}_1 e^{c_+x} + \hat{A}_2 e^{-c_+x} + \hat{A}_3 e^{c_-x} + \hat{A}_4 e^{-c_-x}\right] \tag{21}$$

The corresponding $\hat{h}_z$ is written in terms of these same coefficients with the help of Eq. 17

$$\hat{h}_z = -j\frac{k_z^2 H_o\beta}{\omega}\left[\frac{\hat{A}_1 e^{c_+x}}{a+b} + \frac{\hat{A}_2 e^{-c_+x}}{a+b} + \frac{\hat{A}_3 e^{c_-x}}{a-b} + \frac{\hat{A}_4 e^{-c_-x}}{a-b}\right]e^{-\frac{\beta}{2}x} \tag{22}$$

Thus, the four boundary conditions require that

$$\begin{bmatrix} e^{c_+\ell} & e^{-c_+\ell} & e^{c_-\ell} & e^{-c_-\ell} \\ 1 & 1 & 1 & 1 \\ \frac{e^{c_+\ell}}{a+b} & \frac{e^{-c_+\ell}}{a+b} & \frac{e^{c_-\ell}}{a-b} & \frac{e^{-c_-\ell}}{a-b} \\ \frac{1}{a+b} & \frac{1}{a+b} & \frac{1}{a-b} & \frac{1}{a-b} \end{bmatrix}\begin{bmatrix}\hat{A}_1\\ \hat{A}_2\\ \hat{A}_3\\ \hat{A}_4\end{bmatrix} = 0 \tag{23}$$

This determinant is easily reduced by first subtracting the second and fourth columns from the first and third respectively and then expanding by minors.

$$\sinh(c_+\ell)\sinh(c_-\ell)\frac{2b}{a^2-b^2} = 0 \tag{24}$$

Thus, eigenmodes are $c_+\ell = jn\pi$ and $c_-\ell = jn\pi$. The eigenfrequencies follow from Eqs. 19 and 20.

$$\omega_n^2 = \frac{k^2 k_z^2 H_o^2\beta^2\mu_m}{K_n^4\rho_m} - \frac{g\beta k^2}{K_n^2}\;;\; K_n \equiv \left(\frac{n\pi}{\ell}\right)^2 + \left(\frac{\beta}{2}\right)^2 + k^2 \tag{25}$$

For perturbations with peaks and valleys running perpendicular to the imposed fields, the magnetic field stiffens the fluid. Internal electromechanical waves

Prob. 8.18.2(cont.)

propagate along the lines of magnetic field intensity. If the fluid were confined between parallel plates in the x-z planes, so that the fluid were indeed forced to undergo only two dimensional motions, the field could be used to balance a heavy fluid on top of a light one.... to prevent the gravitational form of Rayleigh-Taylor instability. However, for perturbations with hills and valleys running parallel to the imposed field, the magnetic field remains undisturbed, and there is no magnetic restoring force to prevent the instability. The role of the magnetic field, here in the context of an internal coupling, is similar to that for the hydromagnetic system described in Sec. 8.12 where interchange modes of instability for a surface coupled system were found.

The electric polarization analogue to this configuration might be as shown in Fig. 8.11.1, but with a smooth distribution of ϵ and ρ in the x direction.

<u>Problem 8.18.3</u> Starting with Eqs. 9 and 15 from Prob. 8.18.2, multiply the first by $\hat{h}_z^*$ and integrate from 0 to ℓ .

$$\int_0^\ell \hat{h}_z^* D(\mu_s D\hat{h}_z)dx - \int_0^\ell k^2\mu_s \hat{h}_z\hat{h}_z^* dx - j\int_0^\ell \frac{k_z^2}{\omega}(D\mu_s)H_s\hat{v}_x\hat{h}_z^* dx = 0 \tag{1}$$

Integration of the first term by parts and use of the boundary conditions on $\hat{h}_z$ gives integrals on the left that are positive definite.

$$-\int_0^\ell \mu_s(D\hat{h}_z)(D\hat{h}_z)^* dx - k^2\int_0^\ell \mu_s\hat{h}_z\hat{h}_z^* dx - j\frac{k_z^2}{\omega}\int_0^\ell (D\mu_s)H_s\hat{v}_x\hat{h}_z^* dx = 0 \tag{2}$$

In summary

$$I_1 = -j\frac{k_z^2}{\omega}\hat{I}_4^* \; ; \; I_1 \equiv \int_0^\ell [\mu_s|D\hat{h}_z|^2 + k^2\mu_s|\hat{h}_z|^2]dx, \; I_4 \equiv \int_0^\ell H_s(D\mu_s)\hat{v}_x^*\hat{h}_z dx \tag{3}$$

Now, multiply Eq. 15 from Prob. 8.18.2 by $\hat{v}_x^*$ and integrate.

$$\int_0^\ell \hat{v}_x^* D\rho_s(D\hat{v}_x)dx - k^2\int_0^\ell \rho_s\hat{v}_x\hat{v}_x^* dx + \frac{k^2}{\omega^2}\int_0^\ell N\hat{v}_x\hat{v}_x^* dx + j\frac{k^2}{\omega}\int_0^\ell H_s D\mu_s\hat{v}_x^*\hat{h}_z dx = 0 \tag{4}$$

Prob. 8.18.3(cont.)

Integration of the first term by parts and the boundary conditions on $\hat{v}_x$ gives

$$-\int_0^l \rho_k D\hat{v}_x D\hat{v}_x^* dx - k^2 \int_0^l \rho_k \hat{v}_x \hat{v}_x^* dx + \frac{k^2}{\omega^2}\int_0^l \mathcal{N}\hat{v}_x \hat{v}_x^* dx + j\frac{k^2}{\omega}\int_0^l H_p(D\mu_k)\hat{v}_x^* \hat{h}_z dx = 0 \quad (5)$$

and this expression takes the form

$$I_2 - \frac{I_3}{\omega^2} = j\frac{k^2}{\omega}\hat{I}_4 \; ; I_2 \equiv \int_0^l (\rho_k |D\hat{v}_x|^2 + k^2 \rho_k |\hat{v}_x|^2) dx \; ; I_3 = \int_0^l k^2 \mathcal{N} |\hat{v}_x|^2 dx \quad (6)$$

Multiplication of Eq. 3 by Eq. 6 results in yet another positive definite quantity

$$I_1 I_2 - \frac{I_1 I_3}{\omega^2} = \frac{k^2 k_z^2}{\omega^2} |I_4|^2 \quad (7)$$

and this expression can be solved for the frequency

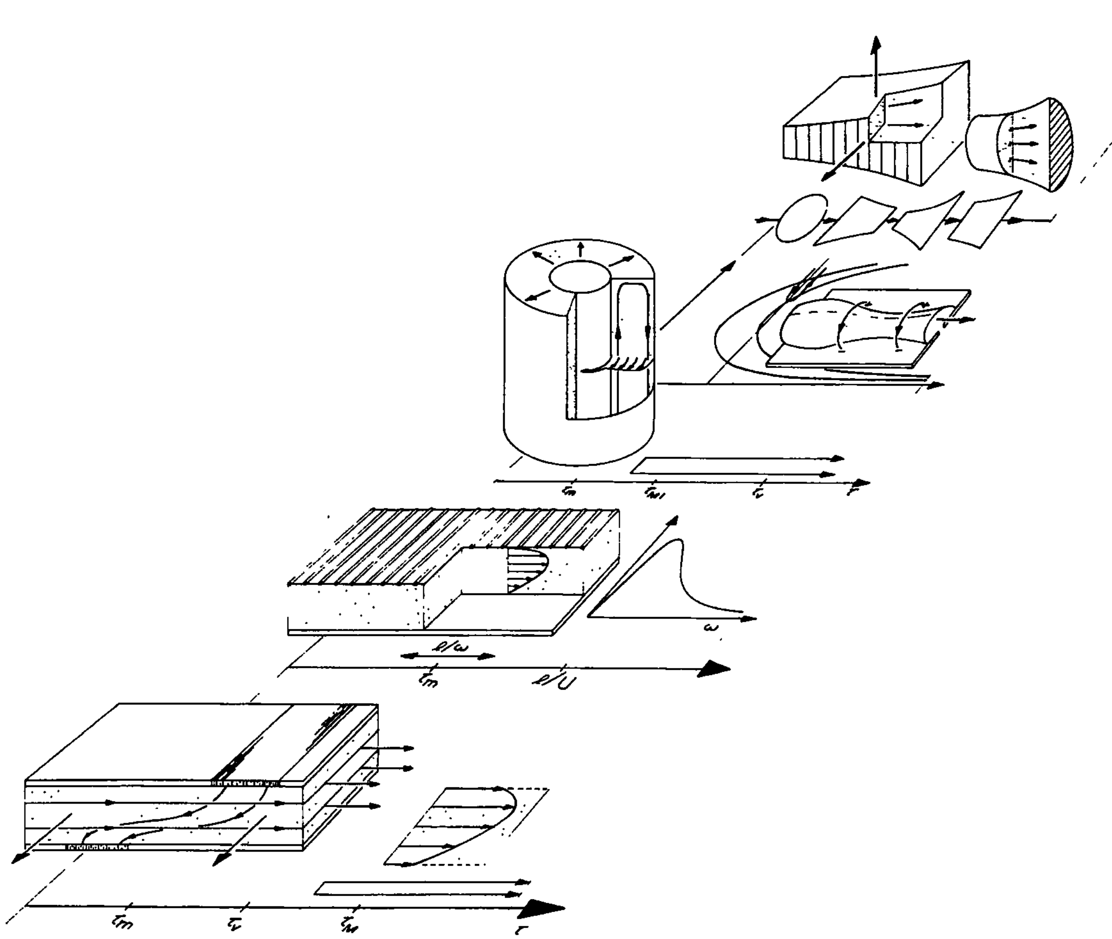
$$\omega^2 = \frac{k^2 k_z^2 |\hat{I}_4|^2 + I_1 I_3}{I_1 I_2} \quad (8)$$

Because the terms on the right are real, it follows that either the eigenfrequencies are real or they represent modes that grow and decay without oscillation. Thus, the search for eigenfrequencies in the general case can be restricted to the real and imaginary axes of the s plane.

Note that a sufficient condition for stability is $\mathcal{N} > 0$, because that insures that I_3 is positive definite.

9

Electromechanical Flows



Prob. 9.3.1 (a) With $\partial p'/\partial y = 0$ and $T_{xy}=0$, Eq. (a) reduces to $v = v^\beta + (v^\alpha - v^\beta)\left(\frac{x}{\Delta}\right)$. Thus, the velocity profile is seen to be linear in x. (b) With $v^\beta = v^\alpha = 0$ and $T_{yx}=0$, Eq. (a) becomes

$$v(x) = \frac{\Delta^2}{2\eta}\frac{\partial p'}{\partial y}\left[\left(\frac{x}{\Delta}\right)-1\right]\frac{x}{\Delta}$$

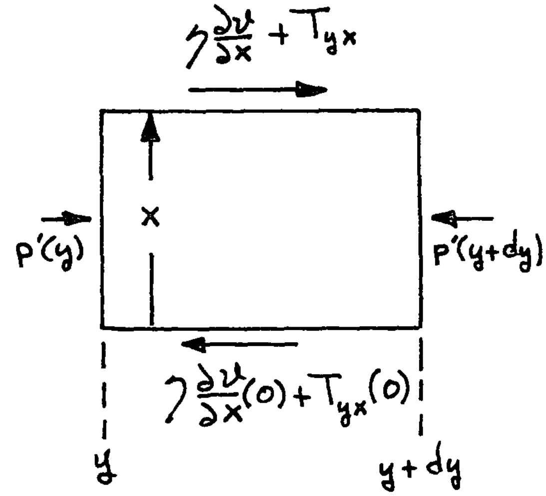
and the velocity profile is seen to be parabolic. The peak velocity is at the center of the channel, where it is $-(\Delta^2/8\eta)\partial p'/\partial y$. The volume rate of flow follows as

$$Q_v = w\Delta\int_0^1 v\, d\left(\frac{x}{\Delta}\right) = \frac{w\Delta^3}{2\eta}\frac{\partial p'}{\partial y}\left[\frac{1}{3}\left(\frac{x}{\Delta}\right)^3 - \frac{1}{2}\left(\frac{x}{\Delta}\right)^2\right]_0^1 = \frac{-w\Delta^3}{12\eta}\frac{\partial p'}{\partial y}$$

Hence, the desired relation of volume rate of flow and the difference between outlet pressure and inlet pressure, Δp, is

$$Q_v = \frac{-w\Delta^3}{12\eta}\left(\frac{\Delta p}{l}\right)$$

Prob. 9.3.2 The control volume is as shown with hybrid pressure p' acting on the longitudinal surfaces (which have height x) and shear stresses acting on transverse surface. With the assumption that these surface stresses represent all of the forces (that there is no acceleration), the force equilibrium is represented by



$$[p'(y+dy) - p'(y)]x = \left(\eta\frac{\partial v}{\partial x} + T_{yx}\right)dy - \left(\eta\frac{\partial v}{\partial x}(0) + T_{yx}(0)\right)dy$$

Divided by dy, this expression becomes Eq. (5)

Prob. 9.3.3 Unlike the other fully developed flows in Table 9.3.1, this one involves an acceleration. The Navier-Stokes equation is

$$(\bar{v}\cdot\nabla)\bar{v} + \nabla p = \nabla(\rho\bar{g}\cdot\bar{r}) + \eta\nabla^2\bar{v} + \nabla\cdot\bar{\bar{T}} \tag{1}$$

With $\bar{v} = v(r)\,\bar{i}_\theta$, continuity is automatically satisfied, $\nabla\cdot\bar{v} = 0$. The radial component of Eq. 1 is

$$-\frac{v^2}{r} + \frac{\partial p}{\partial r} = \frac{\partial}{\partial r}(\rho\bar{g}\cdot\bar{r}) + F_r(r) \tag{2}$$

It is always possible to find a scalar $\mathcal{E}(r)$ such that $F_r = -\partial\mathcal{E}/\partial r$ and to define a scalar $T(r)$ such that $T = -\int(v^2/r)\,dr$. Then, Eq. (2) reduces to

$$\frac{\partial p'}{\partial r} = 0\ ;\ p' \equiv p + T(r) + \mathcal{E}(r) - \rho\bar{g}\cdot\bar{r} \tag{3}$$

The θ component of Eq. (1) is best written so that the viscous shear stress is evident. Thus, the viscous term is written as the divergence of the viscous stress tensor, so that the θ component of Eq. (1) becomes

$$\frac{1}{r}\frac{\partial p'}{\partial\theta} = \frac{\partial}{\partial r}\left(T_{r\theta} + T^v_{r\theta}\right) + \frac{2}{r}\left(T_{r\theta} + T^v_{r\theta}\right) \tag{4}$$

where

$$T^v_{r\theta} = \eta r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) \tag{5}$$

Multiplication of Eq. (4) by r^2 makes it possible to write the right hand side as a perfect differential.

$$r\frac{\partial p'}{\partial\theta} = \frac{\partial}{\partial r}\left[r^2\left(T_{r\theta} + T^v_{r\theta}\right)\right] \tag{6}$$

Then, because the flow is reentrant, $\partial p'/\partial\theta = 0$ and Eq. (6) can be integrated.

$$r^2\left[T_{r\theta} + \eta r\frac{d}{dr}\left(\frac{v}{r}\right)\right] = C \tag{7}$$

a second integration of Eq. (7) divided by r^3 gives

$$\int_\beta^r \frac{T_{r\theta}}{r}\,dr + \eta\left(\frac{v}{r} - \frac{v^\beta}{\beta}\right) = \int_\beta^r \frac{C}{r^3}\,dr = -\frac{C}{2}\left(\frac{1}{r^2} - \frac{1}{\beta^2}\right) \tag{8}$$

Prob. 9.3.3 (cont.)

The coefficient C is determined in terms of the velocity v^α on the outer surface by evaluating Eq. (8) on the outer boundary and solving for C.

$$C = \frac{2}{\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)}\left[\int_\beta^\alpha \frac{T_{r\theta}}{r}\,dr + \eta\left(\frac{v^\alpha}{\alpha} - \frac{v^\beta}{\beta}\right)\right] \tag{9}$$

This can now be introduced into Eq. (8) to give the desired velocity distribution, Eq. (b) of Table 9.2.1.

Prob. 9.3.4 With $T_{r\theta}=0$, Eq. (b) of Table 9.2.1 becomes

$$v = \frac{1}{\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)}\left[v^\alpha\left(\frac{r}{\beta} - \frac{\beta}{r}\right) + v^\beta\left(\frac{\alpha}{r} - \frac{r}{\alpha}\right)\right] \tag{1}$$

The viscous stress follows as

$$T^v_{r\theta} = \eta r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) = \frac{\eta r}{\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)}\left(\frac{2v^\alpha\beta}{r^3} - \frac{2v^\beta\alpha}{r^3}\right) \tag{2}$$

Substituting $v^\alpha = \alpha\Omega_a$ and $v^\beta = \beta\Omega_b$, at the inner surface where $r = \beta$ this becomes

$$T^v_{r\theta} = \frac{-2\eta\alpha}{\beta\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)}(\Omega_b - \Omega_a) \tag{3}$$

The torque on the inner cylinder is its area multiplied by the lever-arm β and the stress $T_{r\theta}$.

$$T = (2\pi w\beta)\beta\left(T^v_{r\theta}\right)^\beta = \frac{-4\pi\beta^2 w\eta\alpha}{\beta\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)}(\Omega_b - \Omega_a) \tag{4}$$

Note that in the limit where the outer cylinder is far away, this becomes

$$T = -4\pi\beta^2 w\eta(\Omega_b - \Omega_a) \tag{5}$$

(b) Expand the term multiplying v^α in Eq. (1) letting $r = \beta + r'$, $r' \ll \beta$ so that $r^{-1} \cong (1/\beta - r'/\beta^2)$. In the term multiplying v^β, expand $r = \alpha - r''$ so that $r^{-1} = (1/\alpha + r''/\alpha^2)$. Thus, Eq. (1) becomes

$$v \simeq \frac{\alpha\beta}{(\alpha-\beta)(\alpha+\beta)}\left[v^\alpha\frac{2r'}{\beta} + v^\beta\frac{2r''}{\alpha}\right] \tag{6}$$

Prob. 9.3.4(cont.)

The term out in front becomes approximately $\alpha/(\alpha-\beta)2$.

Thus, with the identification $r' \to x$, $r'' \to \Delta - x$ and $\alpha-\beta \to \Delta$ the velocity profile becomes

$$v = v^{\alpha}\left(\frac{x}{\Delta}\right) + v^{\beta}\left(1 - \frac{x}{\Delta}\right) \tag{7}$$

which is the plane Couette flow profile (Prob. 9.2.1).

<u>Prob. 9.3.5</u> With the assumption $\bar{v} = v(r)\bar{i}_z$, continuity is automatically satisfied and the radial component of the Navier Stokes equation becomes

$$\frac{\partial p}{\partial r} = \frac{\partial}{\partial r}(\rho \bar{g}\cdot\bar{r}) + F_r(r) \ ; \ F_r = -\frac{d\mathcal{E}}{dr} \tag{1}$$

so that the radial force density is balanced by the pressure in such a way that p' is independent of r, where $p' \equiv p - \rho\bar{g}\cdot\bar{r} + \mathcal{E}$.

Multiplied by r, the longitudinal component of the Navier Stokes equation is

$$r\frac{\partial p'}{\partial z} = \frac{\partial}{\partial r}(rT_{zr}) + \eta\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) \tag{2}$$

This expression is integrated to obtain

$$\frac{1}{2}\frac{\partial p'}{\partial z}(r^2-\beta^2) = rT_{zr} - \beta T_{zr}^{\beta} + \eta\left[r\frac{\partial v}{\partial r} - \beta\left(\frac{\partial v}{\partial r}\right)^{\beta}\right] \tag{3}$$

A second integration of this expression multiplied by r leads to the velocity v(r)

$$\frac{1}{2}\frac{\partial p'}{\partial z}\left[\frac{1}{2}(r^2-\beta^2) - \beta^2 \ln\left(\frac{r}{\beta}\right)\right] = \int_{\beta}^{r} T_{zr}\,dr - \beta T_{zr}^{\beta}\ln\left(\frac{r}{\beta}\right) + \eta(v - v^{\beta}) - \eta\beta\left(\frac{\partial v}{\partial r}\right)^{\beta}\ln\left(\frac{r}{\beta}\right)$$

in terms of the constant $(\partial v/\partial r)^{\beta}$. To replace this constant with the velocity evaluated on the outer boundary, Eq. (4) is evaluated at the outer boundary, r = α , where $v = v^{\alpha}$ and that expression solved for $(\partial v/\partial r)^{\beta}$. Substitution of the resulting expression into Eq. (4) gives an expression that can be solved for the velocity profile in terms of v^{α} and v^{β}, Eq. (c) of Table 9.2.1.

Prob. 9.3.6 This problem is probably more easily solved directly than by taking the limit of Eq. (c). However, it is instructive to take the limit. Note that $T_{zr}=0$, $v^\alpha=0$ and $\alpha=R$. But, so long as v^β is finite, the term $\beta^2\ln(r/\beta)/\ln(\alpha/\beta)$ goes to zero as $\beta\to 0$. Moreover,

$$\lim_{\beta\to 0}\alpha^2\ln(r/\beta)/\ln(\alpha/\beta)=\lim_{\beta\to 0}\alpha^2\frac{[\ln(r)-\ln(\beta)]}{[\ln(\alpha)-\ln(\beta)]}=\alpha^2$$

so that the required circular Couette flow has a parabola as its profile

$$v=\frac{1}{4\eta}\frac{\partial p'}{\partial z}(r^2-R^2)$$

(b) The volume rate of flow follows from Eq. (2)

$$Q_v=\int_0^R v\,2\pi r\,dr=-\frac{\pi}{8\eta}R^4\frac{dp'}{dz}=-\frac{\pi}{8\eta}R^4\frac{\Delta p}{\ell}$$

where Δp is the pressure at the outlet minus that at the inlet.

Prob. 9.4.1 Equation 5.14.11 gives the surface force density in the form

$$\langle T_z\rangle_z=c(\epsilon_a\sigma_b-\epsilon_b\sigma_a)\frac{S_E}{1+S_E^2}\equiv T_o \tag{1}$$

Thus, the interface tends to move in the positive y direction if the upper region (the one nearest the electrode) is insulating and the lower one is filled with semi-insulating liquid and if S_E is greater than zero, which it is if the wave travels in the y direction and the interface moves at a phase velocity less than that of the wave.

For purposes of the fluid mechanics analysis, the coordinate origin for x is moved to the bottom of the tank. Then, Eq. (a) of Table 9.3.1 is applicable with $v^\beta=0$ and $v^\alpha=U$ (the unknown surface velocity). There are no internal force densities in the y direction, so $T_{yx}=0$. In this expression, there are two unknowns, U and $\partial p'/\partial y$. These are determined

Prob. 9.4.1 (cont.)

by the stress balance at the interface, which requires that

$$\eta \frac{\partial v_y}{\partial x}\bigg|_{x=0} = T_o \tag{2}$$

and the condition that mass be conserved.

$$\int_0^b v_y \, dx = 0 \tag{3}$$

These require that

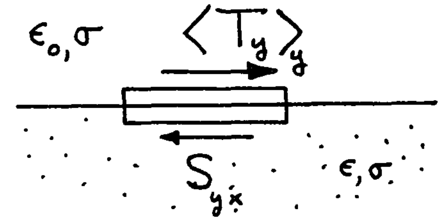
$$\begin{bmatrix} \frac{\eta}{b} & \frac{b}{2} \\ \frac{b}{2} & -\frac{b^3}{12\eta} \end{bmatrix} \begin{bmatrix} U \\ \frac{\partial p'}{\partial y} \end{bmatrix} = \begin{bmatrix} T_o \\ 0 \end{bmatrix} \tag{4}$$

and it follows that $U = bT_o/4\eta$ and $\partial p'/\partial y = 3T_o/2b$

so that the required velocity profile, Eq. (a) of Table 9.3.1 is

$$v_y = \frac{bT_o}{4\eta} \frac{x}{b}\left(3\frac{x}{b} - 2\right) \tag{5}$$

<u>Prob. 9.4.2</u> The time average electric surface force density is found by adapting Eq. 5.14.11. That configuration models the upper region and the infinite half space if it is turned upside down and $z \to y$, $a \to a$, $\epsilon_a \to \epsilon$, $\epsilon_b \to \epsilon_o$, $\sigma_a \to \sigma$, $\sigma_b \to 0$ and $b \to \infty$. Then,



$$\langle T_y \rangle_y = -\frac{1}{2}\epsilon |k\hat{V}_o|^2 K \epsilon_o \sigma \frac{S_E}{1+S_E^2} \tag{1}$$

where

$$S_E \equiv \omega \tau_E \left(1 - \frac{kU}{\omega}\right)$$

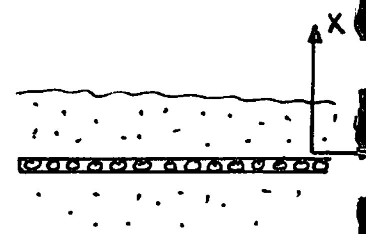
$$\tau_E \equiv \frac{\epsilon \coth ka + \epsilon_o}{\sigma \coth ka} \qquad (k > 0)$$

$$K = \left\{\sinh^2 ka [\epsilon \coth ka + \epsilon_o][\sigma \coth ka]\right\}^{-1}$$

Note that for $|\omega| > |kU|$, the electric surface force density is negative.

Prob. 9.4.2 (cont.)

With x defined as shown to the right, Eq. (a) of Table 9.3.1 is adapted to the flow in the upper section by setting $v^\alpha = U$, $v^\beta = 0$, $\Delta \to a$ and $T_{yx} = 0$ so that

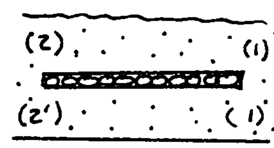


$$v(x) = U\frac{x}{a} + \frac{a^2}{2\eta}\frac{\partial p'}{\partial y}\left[\left(\frac{x}{a}\right)^2 - \frac{x}{a}\right] \tag{2}$$

From this, the viscous shear stress follows as

$$S_{yx} = \eta\frac{\partial v}{\partial x} = \frac{\eta U}{a} + \frac{a}{2}\frac{\partial p'}{\partial y}\left(\frac{2x}{a} - 1\right) \tag{3}$$

Thus, shear stress equilibrium at the interface requires that $(\partial p'/\partial y \equiv (p^1 - p^2)/\ell)$



$$\langle T_y\rangle_y = S_{yx}(x=a) = \frac{\eta U}{a} + \frac{a}{2}\frac{(p^1-p^2)}{\ell} \tag{4}$$

Thus,

$$U = \frac{a}{\eta}\langle T_y\rangle_y - \frac{a^2}{2\eta}\frac{(p^1-p^2)}{\ell} \tag{5}$$

and Eq. 2 becomes

$$v(x) = \left[\frac{a}{\eta}\langle T_y\rangle_y - \frac{a^2}{2\eta}\frac{(p^1-p^2)}{\ell}\right]\frac{x}{a} + \frac{a^2}{2\eta}\frac{p^1-p^2}{\ell}\left[\left(\frac{x}{a}\right)^2 - \frac{x}{a}\right] \tag{6}$$

It is the volume rate of flow that is in common to the upper and lower regions. For the upper region

$$Q_v \equiv \int_0^a v(x)\,dx = \frac{a^2}{2\eta}\langle T_y\rangle_y - \frac{(p^1-p^2)}{\ell}\frac{a^3}{3\eta} \tag{7}$$

In the lower region, where $\Delta \to b$, $v^\beta = v^\alpha = 0$ $T_{yx} = 0$, Eq. (a) of Table 9.3.1 becomes ($p^1 \approx p^{1'}$ and $p^2 \approx p^{2'}$)

$$v = \frac{b^2}{2\eta}\frac{(p^1-p^2)}{\ell}\left[\left(\frac{x}{b}\right)^2 - \frac{x}{b}\right] \tag{8}$$

Thus, in the lower region, the volume rate of flow is

Prob. 9.4.2 (cont.)

$$Q_v = \int_0^b v(x)dx = \frac{-b^3}{12\eta}\frac{(p^1-p^2)}{\ell} \tag{9}$$

Because Q_v in the upper and lower sections must sum to zero, it follows from Eqs. 7 and 9 that

$$\frac{p^1-p^2}{\ell} = \frac{\frac{a^2}{2\eta}\langle T_y\rangle_y}{\frac{a^3}{3\eta}+\frac{b^3}{12\eta}} = \frac{6a^2\langle T_y\rangle_y}{4a^3+b^3} \tag{10}$$

This expression is then substituted into Eq. 5 to obtain the surface velocity, U.

$$U = \frac{a}{\eta}\left[\frac{3a^3+b^3}{4a^3+b^3}\right]\langle T_y\rangle_y \tag{11}$$

Note that because $\langle T_y\rangle_y$ is negative (if the imposed traveling wave of potential travels to the right with a velocity greater than that of the fluid in that same direction) the actual velocity of the interface is to the left, as illustrated in Fig. 9.4.2b.

Prob. 9.4.3 It is assumed that the magnetic skin depth is very short compared to the depth b of the liquid. Thus, it is appropriate to model the electromechanical coupling by a surface force density acting at the interface of the liquid. First, what is the magnetic field distribution under the assumption that $v_y \ll \omega/k$, so that there is no effect of the liquid motion on the field? In the air gap, Eqs. (a) of Table 6.5.1 with σ =0 show that

$$\begin{bmatrix}\hat{H}_x^a\\ \hat{H}_x^b\end{bmatrix} = -j\begin{bmatrix}-\coth ka & \frac{1}{\sinh ka}\\ \frac{-1}{\sinh ka} & \coth ka\end{bmatrix}\begin{bmatrix}\hat{H}_y^a\\ \hat{H}_y^b\end{bmatrix} \tag{1}$$

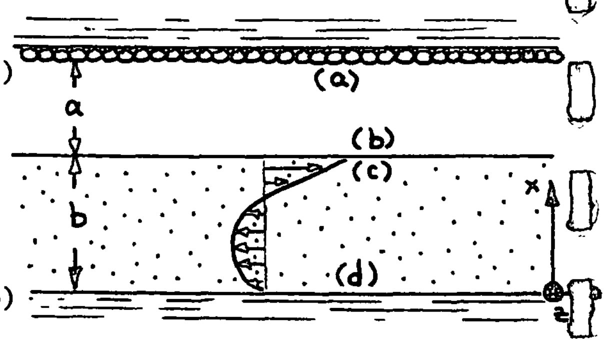
while in the liquid, Eq. 6.8.5 becomes

$$\hat{H}_x^c = \frac{1}{2}(1+j)k\delta\,\hat{H}_y^c \tag{2}$$

Boundary conditions are

$$\hat{H}_y^a = -\hat{K}_o,\ \mu_o\hat{H}_x^b = \mu\hat{H}_x^c,\ \hat{H}_y^b = \hat{H}_y^c \tag{3}$$

Prob. 9.4.3 (cont.)



Thus, it follows that

$$\hat{H}_y^b = -j\hat{K}_o \Big/ \left\{\sinh ka\left[\tfrac{1}{2}k\delta\frac{\mu}{\mu_o} + j\left(\coth ka + \tfrac{1}{2}k\delta\frac{\mu}{\mu_o}\right)\right]\right\} \tag{4}$$

It follows from Eq. 6.8.10 that the time-average surface force density is

$$\langle T_y \rangle_t = \frac{1}{4}\mu k\delta\left|\hat{H}_y^b\right|^2 \tag{5}$$

Under the assumption that the interface remains flat, shear stress balance at the interface requires that

$$\eta\left[\frac{\partial v_y}{\partial x}\right]^{(c)} = \frac{1}{4}\mu k\delta \frac{|\hat{K}_o|^2}{\sinh^2 ka\left\{\left(\frac{1}{2}k\delta\frac{\mu}{\mu_o}\right)^2 + \left(\coth ka + \frac{1}{2}k\delta\frac{\mu}{\mu_o}\right)^2\right\}} \tag{6}$$

The fully developed flow, Eq. (a) from Table 9.3.1, is used with the bulk shear stress set equal to zero and $v^d=0$. That there is no net volume rate of flow is represented by

$$v_y^c = \frac{b^2}{6\eta}\frac{\partial p}{\partial y} \tag{7}$$

So, in terms of the "to be determined" surface velocity, the profile is

$$v_y = 3\left(\frac{x}{b} - \frac{2}{3}\right)\left(\frac{x}{b}\right)v^c \tag{8}$$

The surface velocity can now be determined by using this expression to evaluate the shear stress balance of Eq. 6.

$$\eta\left[\frac{\partial v_y}{\partial x}\right]^c = \frac{4\eta}{b}v^c \tag{9}$$

Thus, the required surface velocity is

$$v^c = \frac{\mu k\delta b}{16\eta}\,\frac{|\hat{K}_o|^2}{\sinh^2 ka\left\{\left(\frac{1}{2}k\delta\frac{\mu}{\mu_o}\right)^2 + \left(\coth ka + \frac{1}{2}k\delta\frac{\mu}{\mu_o}\right)^2\right\}} \tag{10}$$

Note that $k\delta\mu/\mu_o << 1$, this expression is closely approximated by

$$v^c = \frac{\mu k\delta b}{16\eta}\,\frac{|\hat{K}_o|^2}{\cosh^2 ka} \tag{11}$$

Prob. 9.4.3 (cont.)

This result could have been obtained more simply by approximating $H_x^b \approx 0$ in Eq. 1 and ignoring Eq. 2. That is, the fields in the gap could be approximated as being those for a perfectly conducting fluid.

Prob. 9.4.4 This problem is the same as Problem 9.4.3 except that the uniform magnetic surface force density is given by Eq. 8 from Solution 6.9.2. Thus, shear stress equilibrium for the interface requires that

$$\eta\left[\frac{\partial v_y}{\partial x}\right]^c = \frac{\mu_o}{4}\left|\hat{H}_o\right|^2 \frac{\delta}{a} S \tag{1}$$

Using the velocity profile, Eq. 8 from Solution 9.4.3, to evaluate Eq. 1 results in

$$v_c = \frac{\mu_o b}{16\eta}\left|\hat{H}_o\right|^2 \frac{\delta}{a} S \tag{2}$$

Prob. 9.5.1 With the skin depth short compared to both the layer thickness and the wavelength, the magnetic fields are related by Eqs. 6.8.5. In the configuration of Table 9.3.1, the origin of the exponential decay is the upper surface, so the solution is translated to $x=\Delta$ and written as

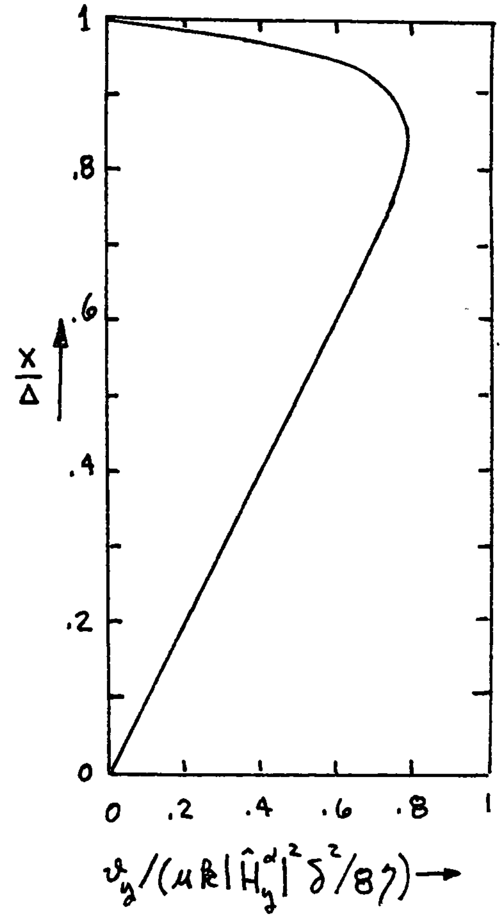
$$\hat{H}_y = \hat{H}_y^d e^{(1+j)(x-\Delta)/\delta} \; ; \; \hat{B}_x^d = \tfrac{1}{2}(1-j)k\mu\delta \hat{H}_y^d \tag{1}$$

It follows that the time-average magnetic shear stress is

$$T_{yx} = \tfrac{1}{2}\mathrm{Re}\,\hat{B}_x^d (\hat{H}_y^d)^* = \tfrac{1}{4}k\mu\delta\left|\hat{H}_y^d\right|^2 e^{2(x-\Delta)/\delta} \tag{2}$$

This distribution can now be substituted into Eq. (a) of Table 9.3.1 to obtain the given velocity profile. (b) For $\delta/\Delta = 0.1$, the magnetically induced part of this profile is as sketched in the figure.

Prob. 9.5.1 (cont.)



Prob. 9.5.2 Boundary conditions at the inner and outer wall are

$$\hat{H}_\theta^a = -\hat{K}_o \; ; \; \hat{H}_\theta^b = 0 \tag{1}$$

Thus, from Eq. b of Table 6.5.1, the complex amplitudes of the vector potential are

$$\hat{A}^a = -\mu F_m(b,a,\gamma)\hat{K}_o \; ; \; \hat{A}^b = -\mu G_m(b,a,\gamma)\hat{K}_o \tag{2}$$

In terms of these amplitudes, the distribution of $\hat{A}(r)$ is given by Eq. 6.5.10. In turn, the magnetic field components needed to evaluate the shear stress are now determined.

$$\hat{H}_\theta = -\frac{1}{\mu}\frac{d\hat{A}}{dr} = -\frac{j\gamma}{\mu}\left\{\hat{A}^a\frac{[H_m(j\gamma b)J_m'(j\gamma r) - J_m(j\gamma b)H_m'(j\gamma r)]}{[H_m(j\gamma b)J_m(j\gamma a) - J_m(j\gamma b)H_m'(j\gamma a)]} + \hat{A}^b\frac{[J_m(j\gamma a)H_m'(j\gamma r) - H_m(j\gamma a)J_m'(j\gamma r)]}{[H_m(j\gamma b)J_m(j\gamma a) - H_m(j\gamma a)J_m(j\gamma b)]}\right\} \tag{3}$$

$$\hat{B}_r = -\frac{jm}{r}\hat{A} \tag{4}$$

Thus,

$$T_{\theta r} = \frac{1}{2}\mathrm{Re}\,\hat{B}_r\hat{H}_\theta^* \tag{5}$$

and the velocity profile given by Eq. b of Table 9.3.1 can be evaluated.

Prob. 9.5.2 (cont.)

Because there are rigid walls at r=a and r=b, v^{α} =0 and v^{β}=0.

$$\bar{v} = \bar{i}_\theta v \,;\; v = a\frac{\left(\frac{r}{b}-\frac{b}{r}\right)}{\left(\frac{a}{b}-\frac{b}{a}\right)\eta}\int_b^a \frac{T_{\theta r}}{r}dr - \frac{r}{\eta}\int_b^r \frac{T_{\theta r}}{r}dr \tag{6}$$

The evaluation of these integrals is conveniently carried out numerically, as is the determination of the volume rate of flow Q_v. For a length ℓ in the z direction,

$$Q_v = \ell\int_b^a v\,dr \tag{7}$$

Prob. 9.5.3 With the no slip boundary conditions on the flow, v^a =0 and v^b =0, Eq. (c) of Table 9.3.1 gives the velocity profile as

$$v(r) = \frac{1}{4\eta}\frac{\partial p'}{\partial z}\left[(r^2-b^2)-(a^2-b^2)\frac{\ln\left(\frac{r}{b}\right)}{\ln\left(\frac{a}{b}\right)}\right] - \frac{1}{\eta}\int_b^r T_{zr}\,dr + \frac{\ln\left(\frac{r}{b}\right)}{\eta\ln\left(\frac{a}{b}\right)}\int_b^a T_{zr}\,dr \tag{1}$$

To evaluate this expression, it is necessary to determine the magnetic stress distribution. To this end, Eq. 6.5.15 gives

$$\hat{\Lambda} = \hat{A}^a\frac{\left[H_1(j\gamma b)\,r\,J_1(j\gamma r) - J_1(j\gamma b)\,r\,H_1(j\gamma r)\right]}{\left[H_1(j\gamma b)J_1(j\gamma a) - J_1(j\gamma b)H_1(j\gamma a)\right]}$$
$$+\hat{A}^b\frac{\left[J_1(j\gamma a)\,r\,H_1(j\gamma r) - H_1(j\gamma a)\,r\,J_1(j\gamma r)\right]}{\left[J_1(j\gamma a)H_1(j\gamma b) - H_1(j\gamma a)J_1(j\gamma b)\right]} \tag{2}$$

where

$$\hat{B}_r = \frac{jk}{r}\hat{\Lambda}$$

and because $H_z = (1/r)\partial\Lambda/\partial r$, H_z follows as

Prob. 9.5.3 (cont.)

$$H_z = \frac{j\gamma}{\mu}\left\{ \hat{A}^a \frac{[H_1(j\gamma b)J_0(j\gamma r) - J_1(j\gamma b)H_0(j\gamma r)]}{[H_1(j\gamma b)J_1(j\gamma a) - J_1(j\gamma b)H_1(j\gamma a)]} \right. \tag{4}$$

$$\left. + \hat{A}^b \frac{[J_1(j\gamma a)H_0(j\gamma r) - H_1(j\gamma a)J_0(j\gamma r)]}{[J_1(j\gamma a)H_1(j\gamma b) - H_1(j\gamma a)J_1(j\gamma b)]} \right\}$$

Here, Eq. 2.16.26d has been used to simplify the expressions.

Boundary conditions consistent with the excitation and infinitely permeable inner and outer regions are

$$\hat{H}_z^a = \hat{K}_o \quad ; \quad \hat{H}_z^b = 0 \tag{5}$$

Thus, the transfer relations f of Table 6.5.1 give the complex amplitudes needed to evaluate Eqs. (3) and (4).

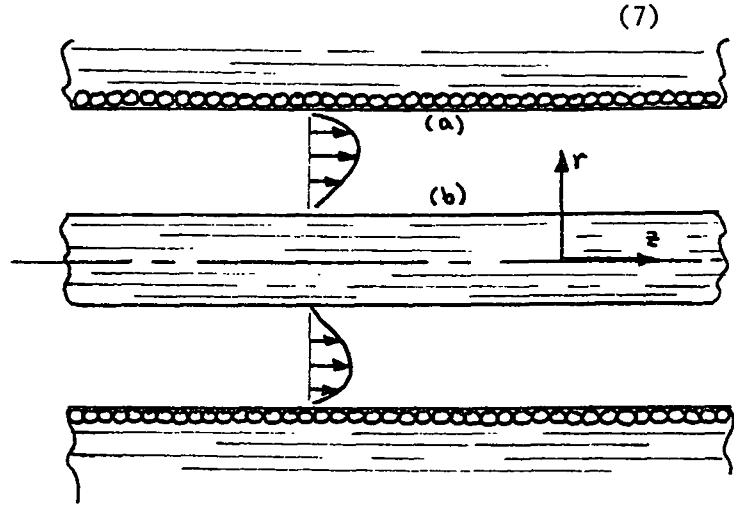
$$\hat{A}^a = \frac{\hat{\Lambda}^a}{a} = -\frac{\mu}{\gamma^2} f_o(b,a,\gamma)\hat{K}_o \; ; \; \hat{A}^b = \frac{\hat{\Lambda}^b}{b} = -\frac{\mu}{\gamma^2} g_o(b,a,\gamma)\hat{K}_o \tag{6}$$

and the required magnetic shear stress follows as

$$T_{zr} = \frac{1}{2}\,\mathrm{Re}\,\hat{B}_r \hat{H}_z \tag{7}$$

The volume rate of flow is related to the axial pressure gradient and magnetic pressure $\mu_o K_o^2$ by integrating Eq. (1).

$$Q_v = \int_b^a v_r \, 2\pi r \, dr \tag{8}$$



<u>Prob. 9.6.1</u> The stress tensor consistent with the force density $F_o\bar{i}_y$ is $T_{yx}=F_o x$. Then, Eq. (a) of Table 9.3.1 with $v^\alpha = 0$ and $v^\beta = 0$, as well as $\partial p'/\partial y = 0$, reflecting the fact that the flow is reentrant, gives the

Prob. 9.6.1 (cont.)

velocity profile

$$v = -\frac{1}{\eta}\int_0^x F_o x'dx' + \frac{x}{\eta\Delta}\int_0^\Delta F_o x'dx' = \frac{F_o\Delta^2}{2\eta}\frac{x}{\Delta}\left(1-\frac{x}{\Delta}\right) \tag{1}$$

For the transient solution, the appropriate plane flow equation is

$$\rho\frac{\partial v}{\partial t} = F_o + \eta\frac{\partial^2 v}{\partial x^2} \tag{2}$$

The particular solution given by Eq. 1 can be subtracted from the total solution with the result that Eq. 2 becomes

$$\rho\frac{\partial v_h}{\partial t} - \eta\frac{\partial^2 v_h}{\partial x^2} = 0 \tag{3}$$

Solutions to this expression of the form $\hat{V}_n(x)\exp s_n t$ must satisfy the equation

$$\frac{d^2\hat{V}_n}{dx^2} + \gamma_n^2\hat{V}_n = 0 \qquad \text{where} \quad \gamma_n^2 \equiv -\frac{\rho s_n}{\eta} \tag{4}$$

The particular solution already satisfies the boundary conditions. So must the homogeneous solution. Thus, to satisfy boundary conditions $v(0,t)=0,\ v(\Delta,t)=0$

$$v_h = \sum_{n=1}^{\infty} \mathrm{Re}\,\hat{V}_n \sin(\gamma_n x)\, e^{s_n t} \quad ; \quad \gamma_n = \frac{n\pi}{\Delta} \tag{5}$$

To satisfy the initial conditions, $v_y(x,0) = v_{part} + v_h(x,0) = 0$ and so

$$\sum_{n=1}^{\infty} \mathrm{Re}\,\hat{V}_n \sin(\gamma_n x)\, e^{s_n t} = -\frac{F_o\Delta^2}{2\eta}\frac{x}{\Delta}\left(1-\frac{x}{\Delta}\right) \tag{6}$$

Multiplication by sin $(m\pi x/\Delta)$ and integration from x=0 to x= Δ serves to evaluate the Fourier coefficients. Thus, the transient solution is

$$v(x,t) = \frac{F_o\Delta^2}{2\eta}\left\{\left(\frac{x}{\Delta}\right)\left(1-\frac{x}{\Delta}\right) - 4\sum_{n=1}^{\infty}\left(\frac{1-\cos n\pi}{(n\pi)^3}\right)\sin\left(\frac{n\pi x}{\Delta}\right)e^{-\left[\frac{\eta}{\rho}\left(\frac{n\pi}{\Delta}\right)^2\right]t}\right\} \tag{7}$$

Although it is the viscous diffusion time that determines how long is required for the fully developed flow to be established, the viscosity is

Prob. 9.6.1 (cont.)

not involved in determining how quickly the bulk of the fluid will respond. Because the force is distributed throughout the bulk, it is the fluid inertia that determines the degree to which the fluid will in general respond. This can be seen by taking the limit of Eq. 7 where times are short compared to the viscous diffusion time and the exponential can be approximated by the first two terms in the series expansion. Then, for $(\eta/\rho)(\pi/\Delta)^2 t << 1$,

$$v(x,t) \rightarrow \left[\frac{2F_o}{\rho}\sum_{n=1}^{\infty}\left(\frac{1-\cos n\pi}{n\pi}\right)\sin\frac{n\pi x}{\Delta}\right]t \tag{8}$$

which is what would be expected by simply equating the mass times acceleration of the fluid to the applied force.

<u>Prob. 9.6.2</u> The general procedure for finding the temporal transient outlined with Prob. 9.6.2 makes clear what is required here. If the profile is to remain invariant, then the fully developed flow must have the same profile as the transient or homogeneous part at any instant. The homogeneous response takes the form of Eq. 5 from the solution to Prob. 9.6.1. For the fully developed flow to have the same profile requires

$$v_{fd} = \frac{F_n}{\eta}\left(\frac{\Delta}{n\pi}\right)^2 \sin\left(\frac{n\pi}{\Delta}x\right) \tag{1}$$

where the coefficient has been adjusted so that the steady force equation is satisfied with the force density given by

$$F_o = F_n \sin\left(\frac{n\pi}{\Delta}x\right) \tag{2}$$

The velocity temporal transient is then the sum of the fully developed and the homogeneous solutions, with the coefficient in front of the latter adjusted to make v(x,0)=0.

Prob. 9.6.2 (cont.)

$$v = \frac{F_n}{\eta}\left(\frac{\Delta}{n\pi}\right)^2 \sin\frac{n\pi}{\Delta}x\left(1 - e^{s_n t}\right);\ s_n \equiv -\frac{\eta}{\rho}\left(\frac{n\pi}{\Delta}\right)^2 \qquad (3)$$

Thus, if the force distribution is the same as any one of the eigenmodes, the resulting velocity profile will remain invariant.

<u>Prob. 9.7.1</u> The boundary layer equations again take the similarity form of Eqs. 17. However, the boundary conditions are

$$v_x(0,y) = 0 \Rightarrow f(0)=0;\ v_y(0,y)=U \Rightarrow g(0)=-2;\ v_y(\infty,y)=0 \Rightarrow g(\infty)\to 0 \qquad (1)$$

where U now denotes the velocity in the y direction adjacent to the plate. The resulting distributions of f, g and h are shown in Fig. P9.7.1. The condition as $\xi\to\infty$ is obtained by iterating with h(0) to obtain h(0) = Thus, the viscous shear stress at the boundary is (Eq. 19)

$$S_{yx}(0,y) = \frac{1}{4}U\eta\sqrt{\frac{\rho U}{\eta y}}\ h(0) = \qquad (2)$$

and it follows that the total force on a length L of the plate is

$$f_y = w\int_0^L S_{yx}(0,y)\,dy = \frac{h(0)}{2}wU\sqrt{\eta\rho UL} \qquad (3)$$

Prob. 9.7.2 What is expected is that the similarity parameter, ξ, is essentially

$$\sqrt{\frac{\tau_v}{\tau_t}} = \sqrt{\frac{\rho x^2}{\eta \tau_t}} \qquad (1)$$

where τ_t is the time required for a fluid element at the interface to reach the position y. Because the interfacial velocity is not uniform, this time must be found. In Eulerian coordinates, the interfacial velocity is given by Eq. 9.7.28.

$$v_y = K y^{1/3}; \quad K \equiv \left(\frac{T_o^2}{\rho\eta}\right)^{1/3} 1.296 \qquad (2)$$

(2)

For a particle having the position y, it follows that

$$\frac{dy}{dt} = K y^{1/3} \Rightarrow \frac{dy}{y^{1/3}} = K\, dt \qquad (3)$$

and integration gives

$$\int_0^y y^{-1/3} dy = K \int_0^{\tau_t} dt \Rightarrow \tau_t = \frac{3}{2} y^{2/3} / K \qquad (4)$$

Substitution into Eq. 1 then gives

$$\sqrt{\frac{\tau_v}{\tau_t}} = \sqrt{\frac{2(1.296)}{3}} \left(\frac{T_o \rho}{\eta^2}\right)^{1/3} \times y^{-\frac{1}{3}} \qquad (5)$$

In the definition of the similarity parameter, Eq. 25, the numerical factor has been set equal to unity.

<u>Prob. 9.7.3</u> Similarity parameter and function are assumed to take the forms given by Eq. 23. The stress equilibrium at the interface, $S_{yx}(x=0) = -T(y)$, requires that

$$-\frac{T_o}{a^k} y^k = -\eta c_1^2 c_2 y^{m+2n} f'' \qquad (1)$$

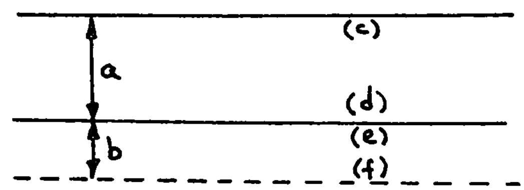
so that m+2n=k and $\eta c_1^2 c_2 = -T_o/a^k$. Substitution into Eq. 14 shows that for the similarity solution to be valid, 2m+2n-1 = m+3n or m=n+1. Thus, it follows that $n=(k-1)/3$ and $m=(k+2)/3$. If $(\eta/\rho)(c_1/c_2) = -1$, the boundary layer equation then reduces to

$$f''' - \frac{(2k+1)}{3}(f')^2 + m f f'' = 0 \qquad (2)$$

which is equivalent to the given system of first order equations. The only boundary condition that appears to be different from those of Eq. 27 is on the interfacial shear stress. However, with the parameters as defined, Eq. 1 reduces to simply $h(0)=-1$.

<u>Prob. 9.7.4</u> (a) In the liquid volume, the potential must satisfy Laplace's equation, which it does. It also satisfies the boundary condition imposed on the potential by the lower electrodes. At the upper interface, the electric field is $\bar{E} = V_b y/b^2$, which satisfies the condition that there be no normal electric field (and hence current density) at the interface. (b) With the given potential at x=a, the x directed electric field is the potential difference divided by the spacing: $E_x = [-V_b y^2/2b^2 + V_a y^2 2b^2]/a$. Thus, the surface force density is $T = \epsilon_o E_x E_y = (\epsilon_o V_a/2b^4 a)(V_a - V_b) y^3$. (c) With the identification $T_o/a^k \to (\epsilon_o V_a/2b^4 a)(V_a - V_b)$ and k=3, the surface force density takes the form assumed in Prob. 9.7.3.

Prob. 9.8.1 First, determine the electric fields and hence the surface force density. The applied potential can be written in the complex notation as $\Phi^f = Re\,\frac{V_o}{2}\left(e^{-j\beta y}+e^{j\beta y}\right)$ so that the desired standing wave solution is the superposition of two traveling wave solutions with amplitudes $\tilde{\Phi}^f_{\pm} = V_o/2$. Boundary conditions are



$$\tilde{J}_x^c = 0\ ,\ \tilde{\Phi}^d = \tilde{\Phi}^e\ ,\ \sigma_a \tilde{E}_x^d = \sigma_b \tilde{E}_x^e\ ,\ \tilde{\Phi}^f = \frac{V_o}{2} \tag{1}$$

And bulk transfer relations are (Eqs. (a), Table 2.16.1)

$$\begin{bmatrix} \tilde{E}_x^c \\ \tilde{E}_x^d \end{bmatrix} = \beta \begin{bmatrix} -\coth\beta a & \frac{1}{\sinh\beta a} \\ \frac{-1}{\sinh\beta a} & \coth\beta a \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^c \\ \tilde{\Phi}^d \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} \tilde{E}_x^e \\ \tilde{E}_x^f \end{bmatrix} = \beta \begin{bmatrix} -\coth\beta b & \frac{1}{\sinh\beta b} \\ \frac{-1}{\sinh\beta b} & \coth\beta b \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^e \\ \tilde{\Phi}^f \end{bmatrix} \tag{3}$$

It follows that

$$\tilde{\Phi}^e = \frac{V_o\,\sigma_b}{2\sinh\beta b\,(\sigma_a \tanh\beta a + \sigma_b \coth\beta b)} \tag{4}$$

where then

$$\tilde{E}_z^e = j\beta\tilde{\Phi}^e\ ;\ \epsilon_a \tilde{E}_x^d = \frac{\epsilon_a\beta\tilde{\Phi}^e}{\coth\beta a}\ ;\ \epsilon_b \tilde{E}_x^e = \frac{\epsilon_b\beta\tilde{\Phi}^e}{\coth\beta a}\,\frac{\sigma_a}{\sigma_b} \tag{5}$$

Now, observe that $\tilde{E}_x$ and $\tilde{\Phi}^e$ are real and even in β while E_z is imaginary and odd in β. Thus, the surface force density reduces to

$$T_y = -(\epsilon_a \tilde{E}^d_{x+} - \epsilon_b \tilde{E}^d_{x+})\,j\,\tilde{E}^e_{z+}\ 2\sin 2\beta y \tag{6}$$

and evaluation gives $T_y = T_o \sin 2\beta y$

$$T_o \equiv \frac{\beta^2 V_o^2\,\sigma_b\,(\epsilon_a\sigma_b - \epsilon_b\sigma_a)}{2\sinh^2\beta b\,(\sigma_a \tanh\beta a + \sigma_b\coth\beta b)^2 \coth\beta a} \tag{7}$$

Prob. 9.8.1 (cont.)

The mechanical boundary conditions consistent with the assumption that gravity holds the interface flat are

$$\tilde{v}_x^c = 0,\ \tilde{v}_y^c = 0,\ \tilde{v}_x^d = 0,\ \tilde{v}_x^e = 0,\ \tilde{v}_y^d = \tilde{v}_y^e,\ \tilde{v}_x^f = 0,\ \tilde{v}_y^f = 0 \tag{8}$$

Stress equilibrium for the interface requires that

$$T_y + S_{yx}^d - S_{yx}^e = 0 \tag{9}$$

In terms of the complex amplitudes, this requires

$$\pm j\frac{T_o}{2} + \tilde{S}_{yx\pm}^d - S_{yx\pm}^e = 0 \tag{10}$$

With the use of the transfer relations from Sec. 7.20 for cellular creep flow, Eqs. 7.20.6, this expression becomes

$$\pm j\frac{T_o}{2} + (\eta_a P_{44}^a - \eta_b P_{33}^b)\tilde{v}_{y\pm}^d = 0 \tag{11}$$

and it follows that the velocity complex amplitudes are

$$\tilde{v}_{y\pm}^d = \mp j\frac{T_o}{2(\eta_a P_{44}^a - \eta_b P_{33}^b)} \tag{12}$$

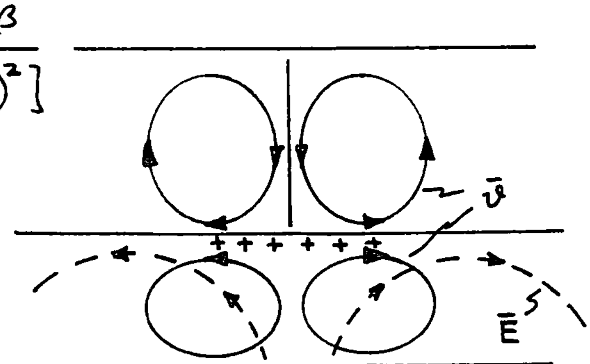
The actual interfacial velocity can now be stated

$$v_y = \mathrm{Re}\left(\tilde{v}_{y+}e^{-j\beta y} + \tilde{v}_{y-}e^{j\beta y}\right) = \frac{T_o \sin 2\beta y}{-\eta_a P_{44}^a + \eta_b P_{33}^b} \tag{13}$$

where, from Eq. 7.20.6,

$$P_{44}^a = -\frac{\left[\frac{1}{4}\sinh 4\beta a - \beta a\right]8\beta}{\left[\sinh^2 2\beta a - (2\beta a)^2\right]}$$

$$P_{33}^b = \frac{\left[\frac{1}{4}\sinh 4\beta b - \beta b\right]8\beta}{\left[\sinh^2 2\beta b - (2\beta b)^2\right]}$$



Note that P_{44} and P_{33} are positive. Thus, the coefficient of $\sin 2\beta y$ is positive and circulations are as sketched and as would be expected in view of the sign of σ_f and E_z at the interface.

Prob. 9.8.1 (cont.)

Charge conservation, including the effect of charge convection at the interface, is represented by the boundary condition

$$\sigma_a E_x^d - \sigma_b E_x^e + \frac{\partial}{\partial y}\left[(\epsilon_a E_x^d - \epsilon_b E_x^e)\, v_y\right] = 0 \tag{14}$$

The convection term will be negligible if

$$\frac{\epsilon}{\sigma} v_y \beta << 1 \tag{15}$$

where ϵ/σ is the longest time constant formed from ϵ_a, ϵ_b and σ_a, σ_b. (A more careful comparison of terms would give a more specific combination of ϵ's and σ's in forming this time constant.) The velocity is itself a function of three lengths, $2\pi/\beta$, a and b. With the assumption that βa and βb are of the order of unity, the velocity given by Eqs. 13 and 7 is typically $\epsilon(\beta V_o)^2/\eta\beta$ and it follows that Eq. 15 takes the form of a condition on the ratio of the charge relaxation time to the electroviscous time.

$$\frac{\epsilon}{\sigma} \Big/ \frac{\eta}{(\beta V_o)^2 \epsilon} << 1 \tag{16}$$

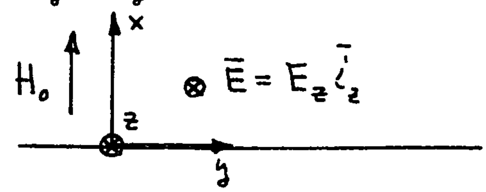
Effects of inertia are negligible if the inertial and viscous force densities bare the relationship

$$|\rho \bar{v}\cdot\nabla\bar{v}| << \eta|\nabla^2 \bar{v}| \Rightarrow \frac{\rho v_y}{\beta \eta} << 1 \tag{17}$$

With the velocity again taken as being of the order of $\epsilon(\beta V_o)^2/\eta\beta$, this condition results in the requirement that the ratio of the viscous diffusion time to the electroviscous time be small.

$$\frac{\rho}{\eta\beta^2} \Big/ \frac{\eta}{(\beta V_o)^2 \epsilon} << 1 \tag{18}$$

Prob. 9.9.1 The flow is fully developed, so $\bar{v} = v_y(x)\bar{i}_y$. Thus, inertial terms in the Navier-Stoke's equation are absent. The x and y components of that equation therefore become



$$\frac{\partial p}{\partial x} = 0 \tag{1}$$

$$\frac{\partial p}{\partial y} = \eta \frac{\partial^2 v_y}{\partial x^2} + \sigma\,(E_z - v_y \mu_o H_o)\mu_o H_o \tag{2}$$

Because E_z is independent of x, this expression is written in the form

$$\frac{d^2 v_y}{dx^2} - \frac{\sigma}{\eta}(\mu_o H_o)^2 v_y = \left(\frac{\partial p}{\partial y} - \sigma \mu_o H_o\right)\frac{1}{\eta} \tag{3}$$

so that what is on the right is independent of x. Solutions to this expression that are appropriate for the infinite half space are exponentials. The growing exponential is excluded, so the homogeneous solution is $\exp -\gamma x$ where $\gamma \equiv \mu_o H_o \sqrt{\frac{\sigma}{\eta}}$. The particular solution is $\left(-\frac{\partial p}{\partial y} + \sigma \mu_o H_o E_z\right)/\sigma(\mu_o H_o)^2$. The combination of these that makes v_y=0 at the wall where x=0 is

$$v_y = \left(\sigma \mu_o H_o E_z - \frac{\partial p}{\partial y}\right)\frac{(1 - e^{-\gamma x})}{\sigma(\mu_o H_o)^2} \tag{4}$$

Thus, the boundary layer has a thickness that is approximately γ^{-1}.

Prob. 9.14.1 There is no electromechanical coupling, so $\mathcal{E}$ =0 and Eq. 3 becomes p= $-\rho g(x-\xi)$. Thus, Eq. 5 becomes $p + \rho g x = \rho g \xi$ and in turn Eq. 4 is

$$\rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y}\right) + \rho g \frac{\partial \xi}{\partial y} = 0 \tag{1}$$

Because $A = \xi - \Xi$, Eq. 9 is

$$\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial y}\left[(\xi - \Xi)v\right] = 0 \tag{2}$$

In the steady state, Eq. 2 shows that

$$(\xi - \Xi)v = \xi_\infty v_\infty \tag{3}$$

while Eq. 1 gives

$$\frac{d}{dy}\left(\frac{1}{2}\rho v^2 + \rho g \xi\right) = 0 \Rightarrow \frac{1}{2}\rho v^2 + \rho g \xi = \frac{1}{2}\rho v_\infty^2 + \rho g \xi_\infty \tag{4}$$

Combined, these expressions show that

$$H \equiv \frac{1}{2}\rho \frac{\xi_\infty^2 v_\infty^2}{(\xi - \Xi)^2} + \rho g \xi = \frac{1}{2}\rho v_\infty^2 + \rho g \xi_\infty \tag{5}$$

The plot of this function with the bottom elevation Ξ (y) as a parameter is

Prob. 9.14.1 (cont.)

shown in the figure. The Flow conditions establish the vertical line along which the transition must evolve. Given the bottom elevation and hence the particular curve,

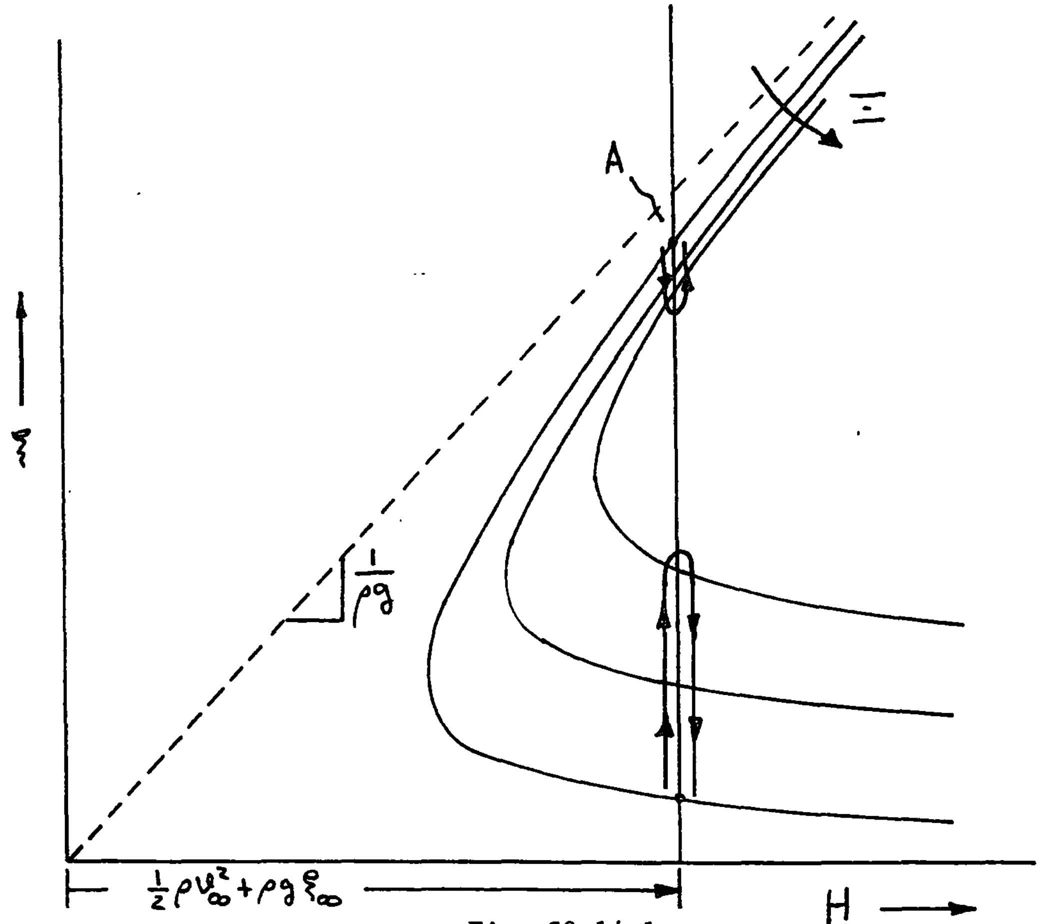


Fig. S9.14.1

the local depth follows from the intersection with the vertical line. If the flow is initiated above the minimum in H(ξ), the flow enters subcritical, whereas if it enters below the minimum ($\xi < \xi_c$), the flow enters supercritical. This can be seen by evaluating

$$\frac{dH}{d\xi} = 0 \Rightarrow (\xi_c - \Xi)^3 = \frac{\xi_\infty^2 v_\infty^2}{g} \tag{8}$$

and observing that the critical depth in the figure comes at

$$\frac{(\xi_c - \Xi)}{\xi_\infty} = \frac{v_\infty}{\sqrt{g(\xi_c - \Xi)}} \tag{9}$$

Consider three types of conservative transitions caused by having a positive bump in the bottom. For a flow initiated at A, the depth decreases where Ξ increases and then returns to its entrance value, as shown in Fig. 2a. For flow entering with depth at B, the reverse is true. The depth increases where the bump occurs. These situations are distinguished

Prob. 9.14.1 (cont.)

by what the entrance depth is relative to the critical depth, given by Eq. (9). If the entrance depth ξ_a is greater than critical, $(\xi_c - \Xi)$, then it follows from Eq. 9 that the entrance velocity, $\mathcal{V}_\infty$, is less than the gravity wave velocity $\sqrt{g(\xi_c - \Xi)}$ for the critical depth. A third possibility is that a flow initiated at A reaches the point of tangency between the vertical line and the head curve. Eqs. 3 and 8 combine to show that

$$v = \sqrt{g(\xi - \Xi)} \tag{10}$$

Then, critical conditions prevail at the peak of the bump and the flow can continue into the subcritical regime, as sketched in Fig. 2c. A similar super-subcritical transition is also possible. (See Rouse, H., Elementary Mechanics of Fluids, John Wiley & Sons, N.Y. (1946), p. 139.

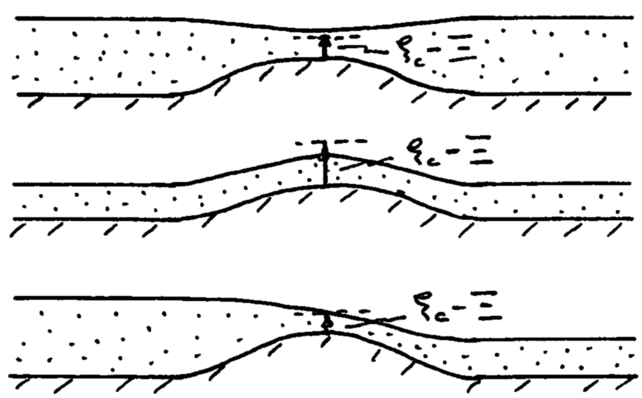


Fig. S9.14.2

Prob. 9.14.2 The normalized mass conservation and momentum equations are

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{1}$$

$$\left(\frac{d}{\ell}\right)^2\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y}\right) + \frac{\partial p}{\partial x} = -1 \tag{2}$$

$$\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + \frac{\partial p}{\partial y} = 0 \tag{3}$$

Thus, to zero order in $(d/\ell)^2$, the vertical force equation reduces to a static equilibrium; $p = -\rho g(x - \xi)$. The remaining two expressions then comprise the fundamental equations. Observe that these expressions in themselves do not require that $v_y = v_y(y,t)$. In fact, the quasi-one-dimensional model allows rotational flows. However, if it is specified that the flow is irrotational to begin with, then it follows from Kelvin's Theorem on vorticity that the flow remains irrotational. This is a result of the expressions above, but is best seen in general. The condition of irrota-

Prob. 9.14.2 (cont.)

tionality in dimensionless form is

$$\left(\frac{d}{\ell}\right)^2 \frac{\partial v_x}{\partial y} = \frac{\partial v_y}{\partial x} \tag{4}$$

and hence the quasi-one-dimensional space-rate expansion, to zero order, requires that $v_y = v_y(y,t)$. Thus, Eqs. 1 and 2 become the fundamental laws for the quasi-one-dimensional model

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{5}$$

$$\frac{\partial v_y}{\partial t} + v_y \frac{\partial v_y}{\partial y} + \frac{\partial p}{\partial y} = 0 \tag{6}$$

with the requirements that p is determined by the transverse static equilibrium and $v_y = v_y(y,t)$.

Prob. 9.14.3 With gravity ignored, the pressure is uniform over the liquid cross-section. This means that it is the same pressure that appears in the normal stress balance for each of the interfaces.

$$\frac{1}{2}(\epsilon-\epsilon_o)\left(\frac{V_a}{\beta\xi_b}\right)^2 = -p = \frac{1}{2}(\epsilon-\epsilon_o)\left(\frac{V_a}{\alpha\xi_a}\right)^2 \tag{1}$$

It follows that the interfacial positions are related.

$$\beta\xi_b = \alpha\xi_a \tag{2}$$

Within a constant associated with the fluid in the neighborhood of the origin, the cross-sectional area is then

$$A = \pi\xi_a^2\left(\frac{\alpha}{2\pi}\right) + \pi\xi_b^2\left(\frac{\beta}{2\pi}\right) = \frac{\alpha}{2}\left(1+\frac{\alpha}{\beta}\right)\xi_a^2 \tag{3}$$

or essentially represented by the variable ξ_a^2. Mass conservation, Eq. 9.13.9, gives

$$\frac{\partial \xi_a^2}{\partial t} + \frac{\partial}{\partial z}\left(v\xi_a^2\right) = 0 \tag{4}$$

Because the pressure is uniform throughout, Eq. 9.13.3 is simply the force balance equation for the interface (either one).

$$p = -\frac{1}{2}(\epsilon-\epsilon_o)\frac{V_a^2}{\alpha^2\xi_a^2} \tag{5}$$

Thus, the force equation, Eq. 9.13.4, becomes the second equation of motion.

$$\rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z}\right) + \frac{1}{2}(\epsilon-\epsilon_o)\frac{V_a^2}{\alpha^2}\frac{1}{(\xi_a^2)^2}\frac{\partial^2\xi_a^2}{\partial z^2} = 0 \tag{6}$$

Prob. 9.16.1 Substitution of Eq. 8 for T in Eq. 2 gives

$$c_p T_o \left(\frac{v A}{v_o A_o}\right)^{1-\gamma} + \frac{1}{2} v^2 = c_p T_o + \frac{1}{2} v_o^2 \tag{1}$$

Manipulation then results in

$$\left(\frac{v}{v_o}\right)\left(\frac{A}{A_o}\right) = \left\{ 1 + \frac{v_o^2}{2 c_p T_o}\left[1 - \left(\frac{v}{v_o}\right)^2\right]\right\}^{\frac{1}{1-\gamma}} \tag{2}$$

Note that

$$\frac{v_o^2}{2 c_p T_o} = \frac{\gamma R v_o^2}{2 c_p \gamma R T_o} = \frac{\gamma R}{2 c_p} M_o^2 = \frac{\frac{c_p}{c_v} R}{2 c_p} M_o^2 = (\gamma - 1)\frac{M_o^2}{2} \tag{3}$$

where use has been made of the relations $\gamma \equiv c_p/c_v$ and $R = c_p - c_v$ and it follows that Eq. 2 is

$$\left(\frac{v}{v_o}\right)\left(\frac{A}{A_o}\right) = \left\{ 1 + (\gamma - 1)\frac{M_o^2}{2}\left[1 - \left(\frac{v}{v_o}\right)^2\right]\right\}^{\frac{1}{1-\gamma}} \tag{4}$$

so that the required relation, Eq. 9, results.

Prob. 9.16.2 The derivative of Eq. 9.16.9 that is required to be zero is

$$\begin{aligned}\frac{d(A/A_o)}{d(v/v_o)} = &-\left(\frac{v_o}{v}\right)^2\left\{ 1 + (\gamma - 1)\frac{M_o^2}{2}\left[1 - \left(\frac{v}{v_o}\right)^2\right]\right\}^{1/(1-\gamma)} \\ &+ M_o^2 \left\{ 1 + (\gamma - 1)\frac{M_o^2}{2}\left[1 - \left(\frac{v}{v_o}\right)^2\right]\right\}^{\frac{1}{1-\gamma} - 1}\end{aligned} \tag{1}$$

This expression can be factored and written as

$$\begin{aligned}\frac{d(A/A_o)}{d(v/v_o)} = &\left(\frac{v_o}{v}\right)^2\left\{ 1 + (\gamma - 1)\frac{M_o^2}{2}\left[1 - \left(\frac{v}{v_o}\right)^2\right]\right\}^{\frac{1}{1-\gamma}} \cdot \\ &\left\{-1 + \left(\frac{v}{v_o}\right)^2 M_o^2 \left[1 + (\gamma - 1)\frac{M_o^2}{2}\left[1 - \left(\frac{v}{v_o}\right)^2\right]\right]^{-1}\right\}\end{aligned} \tag{2}$$

By definition, the Mach number is

$$M \equiv \frac{v}{\sqrt{\gamma R T}} \; ; \; M_o \equiv \frac{v_o}{\sqrt{\gamma R T_o}} \tag{3}$$

Thus,

$$\frac{M}{M_o} = \frac{v}{v_o}\sqrt{\frac{T_o}{T}} \tag{4}$$

Through the use of Eq. 9.16.8, this expression becomes

$$\frac{M}{M_o} = \left(\frac{v}{v_o}\right)\left\{ 1 + (\gamma - 1)\frac{M_o^2}{2}\left[1 - \left(\frac{v}{v_o}\right)^2\right]\right\}^{-\frac{1}{2}} \tag{5}$$

Substitution of the quantity on the left for the group on the right as it

Prob. 9.16.2 (cont.)

appears in Eq. 2 reduces the latter expression to

$$\frac{d(A/A_o)}{d(v/v_o)} = \left(\frac{v_o}{v}\right)^2 \left\{ 1 + (\gamma - 1)\frac{M_o^2}{2}\left[1 - \left(\frac{v}{v_o}\right)^2\right]\right\}^{\frac{1}{1-\gamma}} \left(-1 + M^2\right) \tag{6}$$

Thus, the derivative is zero at M = 1.

<u>Prob. 9.16.3</u> Eqs. (c) and (e) require that

$$\frac{d}{dz}(c_p T) = -\frac{d}{dz}\left(\frac{1}{2}v^2\right) \tag{1}$$

so that the force equation becomes

$$\frac{dp}{dz} = -\rho v \frac{dv}{dz} = -\rho \frac{d}{dz}\left(\frac{1}{2}v^2\right) = \rho c_p \frac{dT}{dz} \tag{2}$$

In view of the mechanical equation of state, Eq. (d), this relation becomes

$$\frac{dp}{dz} = \rho \frac{d}{dz}\left(\frac{c_p p}{\rho R}\right) = \frac{1}{1-\gamma}\rho \frac{d}{dz}\left(\frac{p}{\rho}\right) = \frac{1}{1-\gamma}\left(\frac{dp}{dz} - \frac{p}{\rho}\frac{d\rho}{dz}\right) \tag{3}$$

With the respective derivatives placed on opposite sides of the equation, this expression becomes

$$\gamma \frac{dp}{p} = \frac{d\rho}{\rho} \tag{4}$$

and hence integration results in the desired isentropic equation of state.

$$\frac{p}{p_o} = \left(\frac{\rho}{\rho_o}\right)^{-\gamma} \tag{5}$$

<u>Prob. 9.17.1</u> Equations (a)-(e) of Table 9.15.1 with F and EJ provided by Eqs.5, 7 and 8 are the starting relations

$$\rho v A = \rho_o v_o A_o \tag{1}$$

$$\rho v \frac{dv}{dz} + \frac{dp}{dz} = -\sigma v B^2 (1-K) \tag{2}$$

$$\rho \frac{d}{dz}\left(c_p T + \tfrac{1}{2} v^2\right) = -\sigma v B^2 (1-K) K \tag{3}$$

$$p = \rho R T \tag{4}$$

That the Mach number remains constant requires that

$$v^2/\gamma R T = M_o^2 \tag{4}$$

and differentiation of this relation shows that

$$2 v\, dv = \gamma R M_o^2\, dT \tag{5}$$

Substitute for ρ in Eqs. 2 and 3 using Eq. 4. Then multiply Eq. 2 by -K and add to Eq. 3 to obtain

$$\frac{p}{RT}(1-K)\frac{d}{dz}\left(\tfrac{1}{2}v^2\right) - K\frac{dp}{dz} + \frac{c_p p}{RT}\frac{dT}{dz} = 0 \tag{6}$$

In view of the constraint from Eq. 5, the first term can be expressed as a function of T

$$\frac{p}{RT}\left[c_p + (1-K)\frac{\gamma R M_o^2}{2}\right]\frac{dT}{dz} - K\frac{dp}{dz} = 0 \tag{7}$$

Then, division by p and rearrangement gives

$$\alpha \frac{dT}{T} = \frac{dp}{p} \tag{8}$$

where

$$\alpha \equiv \frac{\gamma}{K(\gamma-1)}\left[1 - \tfrac{1}{2}(\gamma-1)M_o^2(K-1)\right]$$

Hence,

$$\frac{p}{p_o} = \left(\frac{T}{T_o}\right)^{\alpha} \tag{9}$$

In turn, it follows from Eq. 4 that

$$\frac{\rho}{\rho_o} = \left(\frac{T}{T_o}\right)^{\alpha-1} \tag{10}$$

Prob. 9.17.1 (cont.)

The velocity is already determined as a function of T by Eq. 4.

$$\frac{v}{v_o} = \sqrt{\frac{T}{T_o}} \tag{11}$$

Finally, the area follows from Eq. 1 and these last two relations.

$$\frac{A}{A_o} = \left(\frac{T_o}{T}\right)^{\alpha-\frac{1}{2}} \tag{12}$$

The key to now finding all of the variables is T(z), which is now found by substituting Eq. 11 into the energy equation, Eq. 3

$$\left(c_p T_o + \frac{v_o^2}{2}\right)\left(\frac{T}{T_o}\right)^{\alpha-3/2}\frac{d}{dz}\left(\frac{T}{T_o}\right) = -\sigma\frac{v_o}{\rho_o}B^2(1-K)K \tag{13}$$

This expression can be integrated to provide the temperature evolution with z.

$$\frac{T}{T_o} = \left\{1 - \left[\frac{\sigma B^2(1-K)K(\alpha-\frac{1}{2})\,v_o}{(c_p T_o + \frac{1}{2}v_o^2)\rho_o}\right]z\right\}^{1/(\alpha-\frac{1}{2})} \tag{14}$$

Given this expression for T(z), the other variables follow from Eqs. 9-12.

The specific entropy is also now evaluated. Equation 7.23.12 is evaluated using Eqs. 9 and 10 to obtain

$$s_T = s_T^o + c_v[\alpha - \gamma(\alpha-1)]\ln\left[\frac{T}{T_o}\right] \tag{15}$$

Note that $\quad c_v[\alpha - \gamma(\alpha-1)] = c_p - \alpha R.$

Prob. 9.17.2 First arrange the conservation equations as given.

Conservation of mass, Eq. (a) of Table 9.15.1, is

$$\frac{d}{dz}(\rho v A) = A\left(\rho\frac{dv}{dz} + v\frac{d\rho}{dz}\right) + \rho v\frac{dA}{dz} = 0 \tag{1}$$

Conservation of momentum is Eq. (b) of that table with F given by Eq. 9.17.4.

$$\rho v\frac{dv}{dz} + \frac{dp}{dz} = \sigma B(E + vB) \tag{2}$$

Conservation of energy is Eq. (c), JE expressed using Eq. 9.17.5.

$$\rho v\frac{d}{dz}\left(c_p T + \frac{1}{2}v^2\right) = -\sigma E(E + vB) \tag{3}$$

Because $\gamma = c_p/c_v$, $R = c_p - c_v$, and $M^2 = v^2/\gamma RT$, this expression becomes

$$\rho v^3\frac{1}{v}\frac{dv}{dz} + \rho v c_p\frac{dT}{dz} = \rho v^3\frac{1}{v}\frac{dv}{dz} + \frac{\rho v^3}{M^2(\gamma-1)}\frac{1}{T}\frac{dT}{dz} = \sigma E(E+vB) \tag{4}$$

The mechanical equation of state becomes

$$p = \rho RT \rightarrow \frac{1}{p}\frac{dp}{dz} + \frac{1}{\rho}\frac{d\rho}{dz} + \frac{1}{T}\frac{dT}{dz} = 0 \tag{5}$$

Finally, from the definition of M^2,

$$\frac{d}{dz}M^2 = \frac{d}{dz}\left(\frac{v^2}{\gamma RT}\right) \Rightarrow \frac{M^{2\prime}}{M^2} + \frac{T'}{T} - \frac{2v'}{v} = 0 \tag{6}$$

Arranged in matrix form, Eqs. 1,2,4,5 and 6 are the expression summarized in the problem statement.

The matrix is inverted by using Cramer's rule. As a check in carrying out this inversion, the determinant of the matrix is

$$Det = (1 - M^2)\frac{M^2\gamma^2 p^4}{\gamma - 1} \tag{7}$$

Integration of this system of first order equations is straightforward if conditions at the inlet are given. (Numerical integration can be carried out using standard packages such as the Fortran IV IMSL Integration Package DEVREK.) As suggested by the discussion in Sec. 9.16, whether the flow is "super-critical" or "sub-critical" will play a role in determining cause and effect and hence in establishing the appropriate boundary conditions. When the channel is fitted into a system, it is in general necessary to meet conditions at the downstream end. This could be done by using one or more of the upstream conditions as interation variables. This technique is familiar from the integration of boundary layer equations in Sec. 9.7.

Prob. 9.17.2 (cont.)

If the channel is to be designed to have a given distribution of one of the variables on the left, with the channel area to be so determined, these expressions should be rewritten with that variable on the right and A'/A on the left. For example, if the mach number is a given function of z, then the last expression can be solved for A'/A as a function of $(M^2)'/M^2$, $\sigma B(E+vB)$ and $\sigma E(E+vB)$. The other expressions can be written in terms of these same variables by substituting for A'/A with this expression.

Prob. 9.17.3 From Prob. 9.17.2, $A'=0$, reduces the transition equations to $[J=\sigma(E+vB)]$.

$$\begin{bmatrix} \frac{\rho'}{\rho} \\ \frac{p'}{p} \\ \frac{v'}{v} \\ \frac{T'}{T} \\ \frac{(M^2)'}{M^2} \end{bmatrix} = \frac{1}{1-M^2} \begin{bmatrix} -\frac{1}{p} & -\frac{(\gamma-1)}{\gamma v p} \\ -\frac{1}{p}[1+M^2(\gamma-1)] & -\frac{M^2(\gamma-1)}{pv} \\ \frac{1}{p} & \frac{\gamma-1}{\gamma v p} \\ \frac{-M^2(\gamma-1)}{p} & \frac{-(\gamma-1)(\gamma M^2-1)}{\gamma p v} \\ \frac{M^2(\gamma-1)+2}{p} & \frac{(\gamma M^2+1)(\gamma-1)}{\gamma p v} \end{bmatrix} \begin{bmatrix} JB \\ EJ \end{bmatrix} \tag{1}$$

(a) For subsonic and supersonic generator operation, $M^2 \lessgtr 1$ and $JB>0$ while $EJ<0$. Eq. 1a gives $[J=\sigma(E+vB)]$.

$$\frac{\rho'}{\rho} = \frac{-J}{1-M^2}\left[\frac{1}{pv}\right]\left[Bv + \frac{(\gamma-1)E}{\gamma}\right] = \frac{-1}{(1-M^2)pv}\left(\frac{J^2}{\sigma} - \frac{EJ}{\gamma}\right) \lessgtr 0 \tag{2}$$

Eq. 1b can be written as

$$\frac{p'}{p} = \frac{-J}{(1-M^2)pv}\left[vB + M^2(\gamma-1)(vB+E)\right] = \frac{-1}{(1-M^2)pv}\left[vJB + (\gamma-1)\frac{M^2J^2}{\sigma}\right] \lessgtr 0 \tag{3}$$

Prob. 9.17.3 (cont.)

Except for sign, Eq. 1c is the same as Eq. 1a, so

$$\frac{v'}{v} \gtrless 0 \tag{4}$$

Eq. 1d is

$$\begin{aligned}\frac{T'}{T} &= \frac{-(\gamma-1)J}{pv(1-M^2)}\left[M^2 v B + \frac{(\gamma M^2-1)}{\gamma}E\right] \\ &= \frac{-(\gamma-1)}{pv(1-M^2)}\left(\frac{M^2 J^2}{\sigma} - \frac{EJ}{\gamma}\right) \lessgtr 0\end{aligned} \tag{5}$$

and finally, Eq. 1e is

$$\begin{aligned}\frac{(M^2)'}{M^2} &= \frac{J}{(1-M^2)pv}\left\{Bv\left[M^2(\gamma-1)+2\right] + E\frac{(\gamma M^2+1)(\gamma-1)}{\gamma}\right\} \\ &= \frac{J}{(1-M^2)pv}\left\{(Bv+E)(\gamma-1)M^2 + 2Bv + E\frac{(\gamma-1)}{\gamma}\right\} \\ &= \frac{J}{(1-M^2)pv}\left\{(Bv+E)(\gamma-1)M^2 + 2(Bv+E) - \frac{E(\gamma+1)}{\gamma}\right\} \\ &= \frac{1}{(1-M^2)pv}\left\{\frac{J^2}{\sigma}\left[(\gamma-1)M^2+2\right] - \frac{JE(\gamma+1)}{\gamma}\right\} \gtrless 0\end{aligned} \tag{6}$$

With $JB > 0$, the force is retarding the flow and it "might be expected" that the gas would slow down and that the ma density would increase. What has been found is that for subsonic flow, the velocity increases while the mass density pressure and temperature decrease. From Eq. (6), it also follows that the Mach number decreases. That is, as the gas velocity goes up and the sonic velocity goes down (the temperature goes down) the critical sonic condition $M^2 = 1$ is approached.

For supersonic flow, all conditions are reversed. The velocity decreases with increasing z while the pressure, density and temperature

Prob. 9.17.3 (cont.)

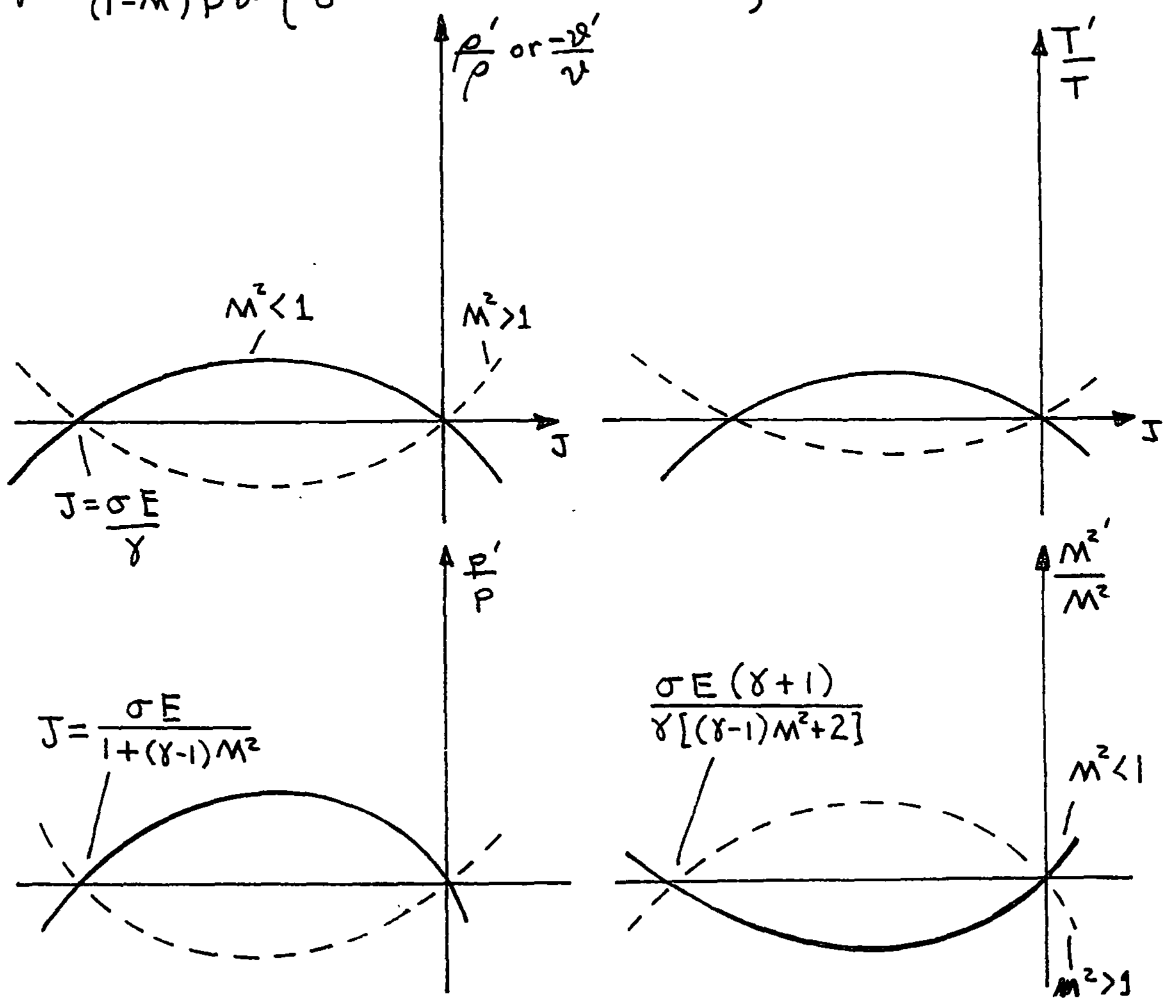
increase. However, because the Mach number is now decreasing with increasing z, the flow again approaches the critical sonic condition.

(b) In the "accelerator" mode, $EJ > 0$ and $JB < 0$. For the discussion, take B as positive so that $J < 0$, which means that

$$E + \mathscr{v}B < 0 \Rightarrow E < -\mathscr{v}B \tag{7}$$

Note that this means that EJ is automatically greater than zero. Note that this leaves unclear the signs of the right-hand sides of Eqs. 2-6. Consider a section of the channel where the voltage is uniformly distributed with z. Then E is constant and the dependence on J of the right-hand sides of Eqs. 2-6 can be sketched as shown in the figure. In sketching Eq. 3, it is necessary to recognize that $\mathscr{v}B = \frac{J}{\sigma} - E$ so that Eq. 3 is also

$$\frac{p'}{p} = \frac{-1}{(1-M^2)\,p\mathscr{v}}\left\{\frac{J^2}{\sigma}\left[1 + (\gamma-1)M^2\right] - JE\right\} \tag{8}$$



Prob. 9.17.3 (cont.)

By way of illustrating the significance of these sketches, consider the dependence of T'/T on $\mathcal{J}$. If at some location in the duct $\frac{\sigma E}{\gamma M^2} < \mathcal{J} < 0$, then the temperature is increasing with z if the flow is subsonic and decreasing if it is supersonic. The opposite is true if $\mathcal{J} < \sigma E/\gamma M^2$. (Remember that E is negative.)

<u>Prob. 9.18.1</u> The mechanical equation of state is Eq. (d) of Table 9.15.1

$$p = \rho R T \tag{1}$$

The objective is now to eliminate ϑ, Φ and p from Eqs. 9.18.21 and 9.18.22. Substitution of the former into the latter gives

$$c_p \frac{dT}{dz} - \frac{1}{\rho}\frac{dp}{dz} = 0 \tag{2}$$

Now, with T eliminated by use of Eq. 1, this becomes

$$\frac{c_p}{R}\frac{d}{dz}\left(\frac{p}{\rho}\right) - \frac{1}{\rho}\frac{dp}{dz} = 0 \tag{3}$$

Because $R = c_p - c_v$, (Eq. 7.22.13) and $\gamma \equiv c_p/c_v$ so $c_p/R = \gamma/(\gamma-1)$, it follows that Eq. 3 can be written as

$$\frac{dp}{p} = \gamma \frac{d\rho}{\rho} \tag{4}$$

Integration from the "d" state to the state of interest gives the first of the desired expressions

$$\ln\left(\frac{p}{p_d}\right) = \gamma \ln\left(\frac{\rho}{\rho_d}\right) \Rightarrow \frac{p}{p_d} = \left(\frac{\rho}{\rho_d}\right)^{\gamma} \tag{5}$$

The second relation is simply a statement of Eq. 1 divided by p_d on the left and $\rho_d R T_d$ on the right.

Prob. 9.18.2 Because the channel is designed to make the temperature constant, it follows from the mechanical equation of state (Eq. 9.18.13) that

$$p = \rho R T \Rightarrow \frac{p}{p_d} = \frac{\rho}{\rho_d} \tag{1}$$

At the same time, it has been shown that the transition is adiabatic, so Eq. 9.18.23 holds.

$$\frac{p}{p_d} = \left(\frac{\rho}{\rho_d}\right)^{\gamma} ; \gamma \neq 1 \tag{2}$$

Thus, it follows that both the temperature and mass density must also be constant

$$p = p_d \quad ; \quad \rho = \rho_d \tag{3}$$

In turn, Eq. 9.18.10, which expresses mass conservation, becomes

$$v A = v_d A_d \tag{4}$$

and Eq. 9.18.20 can be used to show that the charge density is constant

$$\rho_f = \frac{I}{A_d v_d} \tag{5}$$

So, with the relation $E = -d\Phi/dz$, Eq. 9.18.9 is (Gauss' Law)

$$-\frac{d}{dz}\left(A \frac{d\Phi}{dz}\right) = \frac{A}{\epsilon_o} \frac{I}{A_d v_d} \tag{6}$$

In view of the isothermal condition, Eq. 9.18.22 requires that

$$\frac{1}{2} v^2 + \frac{I}{\rho_d A_d v_d} \Phi = \frac{1}{2} v_d^2 + \frac{I}{\rho_d A_d v_d} \Phi_d \tag{7}$$

The required relation of the velocity to the area is gotten from Eq. 3.

$$v = v_d \frac{A_d}{A} \tag{8}$$

and substitution of this relation into Eq. 7 gives the required expression for Φ in terms of the area.

$$\Phi = \frac{1}{2} v_d^2 \left(1 - \frac{A_d^2}{A^2}\right) \frac{\rho_d A_d v_d}{I} + \Phi_d \tag{9}$$

Substitution of this expression into Eq. 5 gives the differential equation for the area dependence on z that must be used to secure a constant temperature.

$$\frac{d}{dz^2} A^{-1} - k^2 A = 0 \quad ; \quad k^2 \equiv \frac{\rho_{fd} I \rho_d^2}{\epsilon_o (A_d \rho_d v_d)^3} \tag{10}$$

Prob. 9.18.2 (cont.)

Multiplication of Eq. 10 by dA^{-1}/dt results in an expression that can be written as

$$\frac{d}{dz}\left[\frac{1}{2}\left(\frac{dA^{-1}}{dt}\right)^2 - k^2 \ln A^{-1}\right] = 0 \tag{11}$$

(Note that this approach is motivated by a similar one taken in dealing with potential-well motions.) To evaluate the constant of integration for Eq. 10, note from the derivative of Eq. 9 that E is proportional to dA^{-1}/dz

$$E = \frac{-v_d^3 \rho_d A_d^3}{I} A^{-1} \frac{dA^{-1}}{dz} \tag{12}$$

Thus, conditions at the outlet are

$$A = A_d \;;\; \frac{dA^{-1}}{dz} = 0 \quad \text{at} \quad z = \ell \tag{13}$$

and Eq. 12 becomes

$$\frac{1}{2}\left(\frac{dA^{-1}}{dz}\right)^2 - k^2 \ln A^{-1} = -k^2 \ln A_d^{-1} \tag{14}$$

The second integration proceeds by writing Eq. 14 as

$$\frac{dA^{-1}}{dz} = \pm\sqrt{2k^2 \ln\left(\frac{A^{-1}}{A_d}\right)} \tag{15}$$

and introducing as a new parameter

$$x^2 = \ln\left(\frac{A^{-1}}{A_d^{-1}}\right) \Rightarrow d(A^{-1}) = 2\,x\,A_d^{-1} e^{x^2} dx \tag{16}$$

Then, Eq. 15 is

$$\pm\frac{\sqrt{2}}{k A_d}\int_x^0 e^{x^2} dx = \int_z^{\ell} dz = \ell - z \tag{17}$$

This expression can be written as (choosing the - sign)

$$F(x)\, e^{x^2} = (\ell - z)\left(\frac{\rho_{fd}^2}{2\epsilon_o \rho_d v_d^2}\right)^{1/2} \tag{18}$$

Prob. 9.18.2 (cont.)

where

$$F(x) = e^{-x^2}\int_0^x e^{x^2}dx \tag{19}$$

and Eq. 5 has been used to write $\rho_{fd} = I/A_d v_d$.

Eq. 12 and Eq. 14 evaluated at the entrance give

$$\left(\frac{I}{v_d^3 \rho_d A_d^3}\right)^2 \frac{1}{2}E_o^2 A_o^2 = k^2 \ln\left(\frac{A_d}{A_o}\right) \tag{20}$$

while from Eq. 4 $v_o A_o = v_d A_d$. Because $\rho_d = \rho_o$ this expression therefore becomes the desired one.

$$\frac{1}{2}\frac{\epsilon E_o^2}{\rho_o v_o^2} = \ln\left(\frac{A_d}{A_o}\right) \tag{21}$$

Finally, the terminal voltage follows from Eq. 9 as

$$V = \Phi_d - \Phi_o = \frac{1}{2}v_d^2\left[\left(\frac{A_d}{A_o}\right)^2 - 1\right]\frac{\rho_d A_d v_d}{I} \tag{22}$$

Thus, the electrical power out is

$$VI = \frac{1}{2}v_d^3 \rho_d A_d\left[\left(\frac{A_d}{A_o}\right)^2 - 1\right] \tag{23}$$

The area ratio A_d/A_o follows from Eq. 20 and can be substituted into Eq. 22, written using the facts that $\rho_d = \rho_o$, $v_d = v_o A_o/A_d$ as $[r \equiv (\epsilon E_o^2/2)/(\rho_o v_o^2/2)$ so that $(A_d/A_o)^2 = \exp r]$,

$$VI = v_o A_o \frac{1}{2}\rho_o v_o^2\left(\frac{A_o}{A_d}\right)^2\left[\left(\frac{A_d}{A_o}\right)^2 - 1\right] = v_o A_o \frac{\left(\frac{1}{2}\epsilon_o E_o^2\right)}{r}\left(1 - e^{-r}\right) \tag{24}$$

Thus, it is clear that the maximum power that can be extracted $\left(\frac{1}{2}\epsilon_o E_o^2 \rightarrow \infty\right)$ is the kinetic power $v_o A_o\left(\frac{1}{2}\rho_o v_o^2\right)$.

Prob. 9.18.3 With the understanding that the duct geometry is given, so that ξ'/ξ is known, the electrical relations are, Eq. 9.18.8

$$\frac{d}{dz}\left[\rho_f \pi \xi^2 (bE + v) + 2\pi\sigma_s \xi E\right] = 0 \tag{1}$$

or with primes indicating derivatives,

$$\rho_f\xi^2(bE' + v') + \rho_f 2\xi\xi'(bE+v) + \rho_f'\xi^2(bE+v) + 2\sigma_s\xi E' + 2\sigma_s\xi'E = 0 \tag{2}$$

Eq. 9.18.9

$$\frac{d}{dz}(\xi^2 E) + \frac{2\sigma_s}{\rho_f b}\frac{d}{dz}(\xi E) = \frac{\rho_f \xi^2}{\epsilon_0} \tag{3}$$

which is

$$\xi^2 E' + 2\xi\xi' E + \frac{2\sigma_s}{\rho_f b}\xi' E + \frac{2\sigma_s}{\rho_f b}\xi E' - \frac{\rho_f \xi^2}{\epsilon_0} = 0 \tag{4}$$

The mechanical relations are

$$\frac{d}{dz}(\rho v \xi^2) = 0 \tag{5}$$

which can be written as

$$\rho v 2\xi\xi' + \rho v'\xi^2 + \rho' v \xi^2 = 0 \tag{6}$$

Eq. 9.18.11

$$\rho v v' + p' - \rho_f E = 0 \tag{7}$$

Eq. 9.18.12

$$\rho v c_p T' + \rho v^2 v' - \rho_f E(bE + v) - \frac{2\sigma_s E^2}{\xi} = 0 \tag{8}$$

and Eq. 9.18.13

$$p' - \rho R T' - \rho' R T = 0 \tag{9}$$

Although redundant, the Mach relation is

$$M^2 = \frac{v^2}{\gamma R T} \tag{10}$$

which is equivalent to

$$M^{2\prime} - \frac{2 v v'}{\gamma R T} + \frac{v^2 T'}{\gamma R T^2} = 0 \tag{11}$$

With the definition

$$Q \equiv \left[-2\left(1 + \frac{\sigma_s}{\xi\rho_f b}\right)\frac{\xi'}{\xi} + \frac{\rho_f}{\epsilon_0 E}\right] \Big/ \left[1 + \frac{2\sigma_s}{\xi\rho_f b}\right] \tag{12}$$

Eqs. 6,7,8,2,4,9 and 11 are respectively written in the orderly form

Prob. 9.18.3 (cont.)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ \gamma M^2 & 1 & 0 & 0 & 0 & 0 & 0 \\ M^2(\gamma-1) & 0 & 1 & 0 & 0 & 0 & 0 \\ v & 0 & 0 & v & \left(b+\frac{2\sigma_s}{\rho_f \xi}\right)E & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v'/v \\ p'/p \\ T'/T \\ \left(\frac{bE}{v}+1\right)\frac{\rho_f'}{\rho_f} \\ E'/E \\ \rho'/\rho \\ M^{2\prime}/M^2 \end{bmatrix} = \begin{bmatrix} -2\xi'/\xi \\ \rho_f E/p \\ \frac{\rho_f E(\gamma-1)}{p v\ \gamma}\left[\left(b+\frac{2\sigma_s}{\rho_f \xi}\right)E+v\right] \\ -2\left(bE+v+\frac{\sigma_s E}{\rho_f \xi}\right)\frac{\xi'}{\xi} \\ Q \\ 0 \\ 0 \end{bmatrix} \tag{13}$$

In the inversion of these equations, the determinant of the coefficients is

$$\text{Det} = (M^2-1)(-v) \tag{14}$$

Thus, the required relations are

$$\begin{bmatrix} v'/v \\ p'/p \\ T'/T \\ \left(\frac{bE}{v}+1\right)\frac{\rho_f'}{\rho_f} \\ E'/E \\ \rho'/\rho \\ M'^2/M^2 \end{bmatrix} = \frac{1}{M^2-1}\left[A_{ij}\right] \begin{bmatrix} -2\xi'/\xi \\ \rho_f E/p \\ \frac{\rho_f E}{p v}\left(\frac{\gamma-1}{\gamma}\right)\left(\left(b+\frac{2\sigma_s}{\rho_f \xi}\right)E+v\right) \\ -2\left(bE+v+\frac{\sigma_s E}{\xi \rho_f}\right)\left(\frac{\xi'}{\xi}\right) \\ Q \end{bmatrix} \tag{15}$$

$$A_{ij} \equiv \begin{bmatrix} -1 & 1 & -1 & 0 & 0 \\ M^2\gamma & [-M^2(\gamma-1)-1] & M^2\gamma & 0 & 0 \\ M^2(\gamma-1) & -M^2(\gamma-1) & (M^2\gamma-1) & 0 & 0 \\ 1 & -1 & 1 & \frac{(1-M^2)}{v} & (1-M^2)\frac{E}{v}\left(b+\frac{2\sigma_s}{\rho_f \xi}\right) \\ 0 & 0 & 0 & 0 & M^2-1 \\ M^2 & -1 & 1 & 0 & 0 \\ -[M^2(\gamma-1)+2] & [M^2(\gamma-1)+2] & -(M^2\gamma+1) & 0 & 0 \end{bmatrix}$$

Prob. 9.18.4 In the limit of no convection, the appropriate laws represent Gauss, charge conservation and the terminal current. These are Eqs. 9.18.8, 9.18.9 and 9.18.10.

$$\frac{d}{dz}\left(\rho_f b E \pi \xi^2 + 2\pi \sigma_s \xi E\right) = 0 \tag{1}$$

$$\frac{d}{dz}(\xi^2 E) + \frac{2\sigma_s}{\rho_f b}\frac{d}{dz}(\xi E) = \frac{\rho_f \xi^2}{\epsilon_o} \tag{2}$$

$$I = \rho_f b E_o \pi \xi^2 + 2\pi \xi \sigma_s E_o = \rho_{fo} b E_o \pi \xi_o^2 + 2\pi \xi_o \sigma_s E_o \tag{3}$$

This last expression serves to determine the entrance charge density, given the terminal current I.

$$\rho_{fo} = \frac{I - 2\pi \xi_o \sigma_s E_o}{b E_o \pi \xi_o^2} \tag{4}$$

Using this expression, it is possible to evaluate the integration constant needed to integrate Eq. 1. Thus, that expression shows that

$$\rho_f = \frac{I - 2\pi\sigma_s E_o \xi}{b E_o \pi \xi^2} \tag{5}$$

Substitution of this expression (of how the charge density thins out as the channel expands) into Eq. 2 gives a differential equation for the channel radius.

$$E_o \frac{d}{dz}\xi^2 + \frac{2\sigma_s E_o^2 \pi \xi^2}{(I - 2\pi\sigma_s E_o \xi)}\frac{d\xi}{dz} = \frac{I - 2\pi\sigma_s E_o \xi}{\epsilon_o b E_o^2 \pi} \tag{6}$$

This expression can be written so as to make it clear that it can be integrated.

$$\int_1^{\underline{\xi}} \left[\frac{2\underline{\xi}}{1 - \Sigma \underline{\xi}} + \frac{\Sigma \underline{\xi}^2}{(1 - \Sigma \underline{\xi})^2}\right] d\underline{\xi} = \underline{z} \tag{7}$$

where

$$\Sigma \equiv 2\pi\sigma_s E_o \xi_o / I \quad ; \quad \underline{\xi} \equiv \xi/\xi_o$$

$$\ell_o \equiv \epsilon_o b E_o^2 \pi \xi_o / I \quad ; \quad \underline{z} \equiv z/\ell_o$$

Thus, integration from the entrance, where $z = 0$ and $\xi = \xi_o$, gives

Prob. 9.18.4 (cont.)

$$\frac{1}{\Sigma^2}\left\{4\left[(1-\Sigma\xi)-(1-\Sigma)\right]-\frac{1}{2}\left[(1-\Sigma\xi)^2-(1-\Sigma)^2\right]\right. \\ \left. -3\left[\ln(1-\Sigma\xi)-\ln(1-\Sigma)\right]\right\}=z \tag{8}$$

Given a normalized radius $\underline{\xi}$, this expression can be used to find the associated normalized position $\underline{z}$, with the normalized wall conductivity, Σ, as a dimensionless parameter.

Prob. 9.19.1 It is clear from the energy equation, Eq. 9.16.2, that because the velocity decreases (as it by definition does in a diffuser), then the temperature must increase. The temperature is related to the pressure by the mechanical equation of state, Eq. (d) of Table 9.15.1.

$$p=\rho RT \Rightarrow \frac{p}{p_o}=\frac{\rho}{\rho_o}\frac{T}{T_o} \tag{1}$$

In the diffuser, the transition is also adiabatic, so Eq. 9.16.3 also applies

$$\frac{p}{p_o}=\left(\frac{\rho}{\rho_o}\right)^{\gamma} \tag{2}$$

These equations can be combined to eliminate the mass density.

$$\left(\frac{p}{p_o}\right)^{(\gamma-1)/\gamma}=\frac{T}{T_o} \tag{3}$$

Because $\gamma>1$, it follows that because the temperature increases, so does the pressure.

Prob. 9.19.2 The fundamental equation representing components in the cycle is Eq. 9.19.7

$$\int_V \bar{E}\cdot\bar{J}\,dV = \oint_S \rho\bar{v}\left(H_T + \tfrac{1}{2}\bar{v}\cdot\bar{v}\right)\cdot\bar{n}\,da \tag{1}$$

In the heat-exchanger the gas is raised in temperature and entropy as it passes from $i \rightarrow f$. Here, the electrical power input represented by the left side of Eq. 1 is replaced by a thermal power input. Thus, with the understanding that the vaporized water leaves the heat exchanger at f with negligible kinetic energy,

$$\frac{\text{thermal energy input/unit time}}{\text{mass/unit time}} = H_T^f - H_T^i \tag{2}$$

In representing the turbine, it is assumed that the vapor expansion that turns the thermal energy into kinetic energy occurs within the turbine and that the gas has negligible kinetic energy as it leaves the turbine

$$-\frac{\text{turbine power output}}{\text{mass/unit time}} = \frac{-VI}{A\rho v} = H_T^g - H_T^f \tag{3}$$

Heat rejected in the condensor, $g \rightarrow h$, is taken as lost. The power required to raise the pressure of the condensed liquid, from $h \rightarrow i$, is (assuming perfect pumping efficiency)

$$\frac{\text{pump power in}}{\text{mass/unit time}} = H_T^i - H_T^g \tag{4}$$

Combining these relations and recognizing that the electrical power output is η_g times the turbine shaft power gives

$$\frac{\text{electrical power output - pumping power}}{\text{thermal power in}} = \frac{\eta_g(-H_T^g + H_T^f) - (H_T^i - H_T^g)}{H_T^f - H_T^i} \tag{5}$$

Prob. 9.19.2 (cont.)

Now, let the heat input $i \rightarrow f$ be that rejected in $e \rightarrow a$ of the MHD or EHD system of Fig. 9.19.1.

To describe the combined systems, let $\mathring{M}_T$ and $\mathring{M}_S$ represent the mass rates of flow in the topping and steam cycles respectively. The efficiency of the overall system is then

$$\eta = \frac{\text{electrical power out of topping cycle - compressor power} + \text{electrical power out of steam cycle - pump power}}{\text{heat power into topping cycle}}$$

$$= \frac{\mathring{M}_T\left[(H_T^c - H_T^e) - (H_T^b - H_T^a)\right] + \mathring{M}_s\left[(H_T^f - H_T^g)\eta_g - (H_T^i - H_T^h)\right]}{\mathring{M}_T\,(H_T^c - H_T^b)} \tag{6}$$

Because the heat rejected by the topping cycle from $e \rightarrow a$ is equal to that into the steam cycle,

$$\mathring{M}_T(H_T^e - H_T^a) = \mathring{M}_s(H_T^f - H_T^i)$$
$$\Rightarrow \frac{\mathring{M}_s}{\mathring{M}_T} = \frac{H_T^e - H_T^a}{H_T^f - H_T^i} \tag{7}$$

and it follows that Eq. (6) can also be written as

$$\eta = \frac{\left[(H_T^c - H_T^e) - (H_T^b - H_T^a)\right] + \left[\dfrac{H_T^e - H_T^a}{H_T^f - H_T^i}\right]\left[(H_T^f - H_T^g)\eta_g - (H_T^i - H_T^h)\right]}{H_T^c - H_T^b} \tag{8}$$

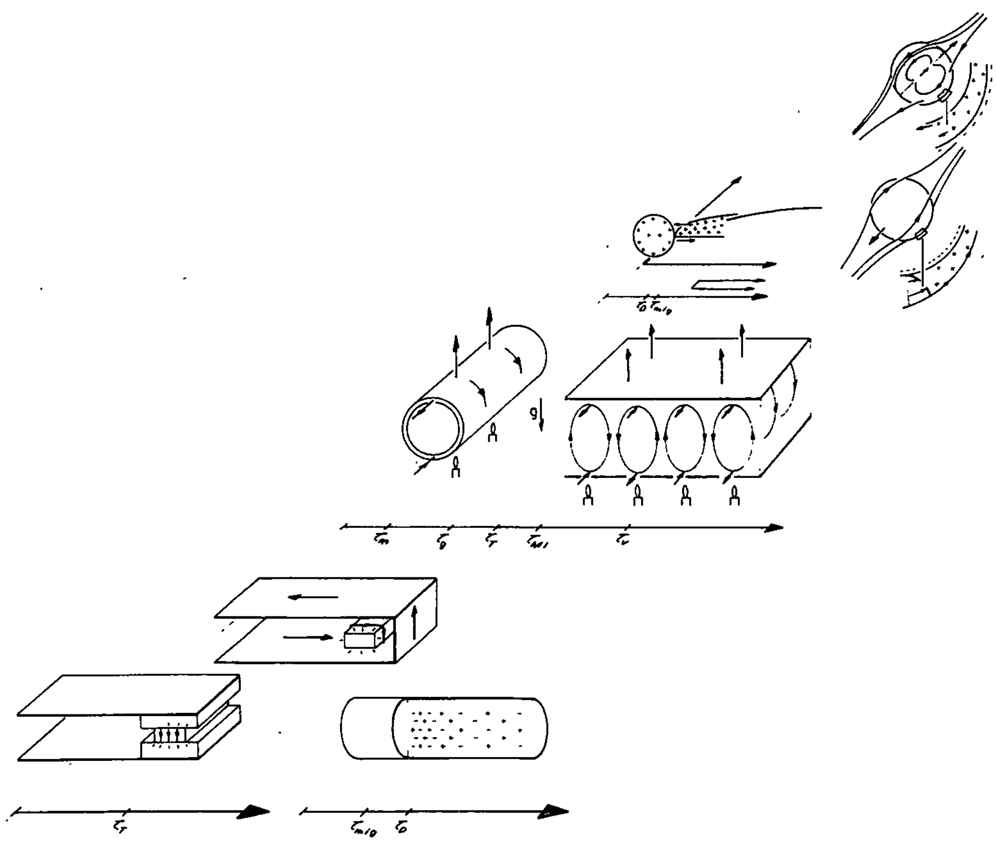
With the requirement that $\eta_g = 1$, and again using Eq. 7 to reintroduce $\mathring{M}_s / \mathring{M}_T$, Eq. 8 can be written as

$$\eta = \frac{\mathring{M}_T(H_T^c - H_T^b) - \mathring{M}_s(H_T^h - H_T^g)}{\mathring{M}_T(H_T^c - H_T^b)} \tag{9}$$

This efficiency expression takes the form of Eq. 9.19.13.

10

Electromechanics with Thermal and Molecular Diffusion



Prob. 10.2.1 (a) In one dimension, Eq. 10.2.2 is simply

$$\frac{d^2T}{dx^2} = -\frac{\Phi_d}{k_T} \tag{1}$$

The motion has no effect because $\bar{v}$ is perpendicular to the heat flux.

This expression is integrated twice from x=0 to an arbitrary location, x. Multiplied by $-k_T$, the constant from the first integration is the heat flux at x=0, T^β. The second integration has T^β as a constant of integration. Hence,

$$T = -\frac{1}{k_T}\int_0^x\int_0^{x'}\phi_d(x'')dx''dx' - \frac{T^\beta}{k_T}\Delta + T^\beta \tag{2}$$

Evaluation of this expression at x=0 where $T = T^\beta$ gives a relation that can be solved for T^β. Substitution of T^β back into Eq. 2, gives the desired temperature distribution.

$$T = -\frac{1}{k_T}\int_0^x\int_0^{x'}\phi_d(x'')dx''dx' + T^\beta - \frac{x}{\Delta}(T^\beta - T^\alpha) + \frac{x}{\Delta k_T}\int_0^\Delta\int_0^{x'}\phi_d(x'')dx''dx' \tag{3}$$

(b) The heat flux is gotten from Eq. 3 by evaluating

$$T' = -k_T\frac{dT}{dx} = \int_0^x\phi_d(x')dx' + \frac{k_T}{\Delta}(T^\beta - T^\alpha) - \frac{1}{\Delta}\int_0^\Delta\int_0^{x'}\phi_d(x'')dx''dx' \tag{4}$$

At the respective boundaries, this expression becomes

$$T'^\alpha = \int_0^\Delta\phi_d(x')dx' + \frac{k_T}{\Delta}(T^\beta - T^\alpha) - \frac{1}{\Delta}\int_0^\Delta\int_0^{x'}\phi_d(x'')dx''dx' \tag{5}$$

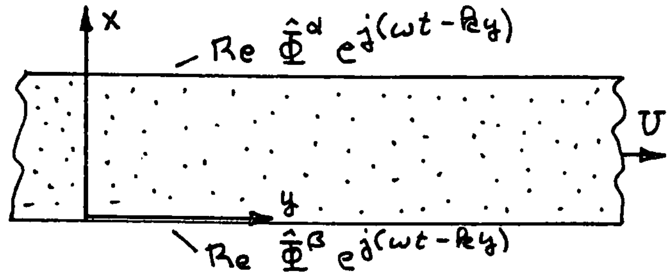
$$T'^\beta = \frac{k_T}{\Delta}(T^\beta - T^\alpha) - \frac{1}{\Delta}\int_0^x\int_0^{x'}\phi_d(x'')dx''dx' \tag{6}$$

Prob. 10.3.1 In Eq. 10.3.20, the transient heat flux at the surfaces is zero, so $\hat{T}'^\alpha = \hat{T}'^\beta = 0$.

$$\begin{bmatrix} -\coth\gamma_T\Delta & \frac{1}{\sinh\gamma_T\Delta} \\ \frac{-1}{\sinh\gamma_T\Delta} & \coth\gamma_T\Delta \end{bmatrix}\begin{bmatrix}\hat{T}^\alpha \\ \hat{T}^\beta\end{bmatrix} = \sum_{i=1}^{\infty}\frac{\left(\frac{i\pi}{\Delta}\right)\hat{\Phi}_i/k_T\gamma_T}{\left[\left(\frac{i\pi}{\Delta}\right)^2 + k_z^2 + j\frac{(\omega_z - k_zU)}{K_T}\right]}\begin{bmatrix}(-1)^i \\ 1\end{bmatrix} \tag{1}$$

These expressions are inverted to find the dynamic part of the surface temperatures.

$$\begin{bmatrix}\hat{T}^\alpha \\ \hat{T}^\beta\end{bmatrix} = \sum_{i=1}^{\infty}\frac{\left(\frac{i\pi}{\Delta}\right)\hat{\Phi}_i/k_T\gamma_T}{\left[\left(\frac{i\pi}{\Delta}\right)^2 + k_z^2 + j\frac{(\omega_z - k_zU)}{K_T}\right]}\begin{bmatrix}(-1)^i\coth\gamma_T\Delta & \frac{-1}{\sinh\gamma_T\Delta} \\ -\coth\gamma_T\Delta & \frac{(-1)^i}{\sinh\gamma_T\Delta}\end{bmatrix} \tag{2}$$



Prob. 10.3.2 (a) The EQS electrical dissipation density is

$$\phi_d = \sigma \bar{E}'\cdot\bar{E}' = \sigma \bar{E}\cdot\bar{E}$$

$$= \sigma\left[Re\,\hat{\bar{E}}(x)e^{j(\omega t - ky)}\right]^2 = \frac{\sigma}{4}\left[\hat{\bar{E}}\,e^{j(\omega t - ky)} - \hat{\bar{E}}^* e^{-j(\omega t - ky)}\right]^2$$

$$= \frac{1}{2}\sigma\left[\hat{\bar{E}}\,\hat{\bar{E}}^* - Re\,\hat{\bar{E}}\cdot\hat{\bar{E}}\,e^{j(\omega_2 t - k_2 y)}\right] \tag{1}$$

Thus, in Eq. 10.3.6

$$\phi_o = \frac{1}{2}\sigma\,\hat{\bar{E}}\cdot\hat{\bar{E}}^* \;;\; \hat{\Phi} = -\frac{1}{2}\sigma\,\hat{\bar{E}}^2 \tag{2}$$

The specific $\hat{\bar{E}}(x)$ follows from

$$\hat{\Phi}(x) = \hat{\Phi}^{\alpha}\frac{\sinh kx}{\sinh k\Delta} - \hat{\Phi}^{\beta}\frac{\sinh k(x-\Delta)}{\sinh k\Delta} \tag{3}$$

so that

$$\hat{\bar{E}} = -\frac{d\hat{\Phi}}{dx}\bar{i}_x + jk\hat{\Phi}\,\bar{i}_y$$

$$= \left[-k\hat{\Phi}^{\alpha}\frac{\cosh kx}{\sinh k\Delta} + k\hat{\Phi}^{\beta}\frac{\cosh k(x-\Delta)}{\sinh k\Delta}\right]\bar{i}_x \tag{4}$$

$$+jk\left[\hat{\Phi}^{\alpha}\frac{\sinh kx}{\sinh k\Delta} - \hat{\Phi}^{\beta}\frac{\sinh k(x-\Delta)}{\sinh k\Delta}\right]\bar{i}_y$$

Thus,

$$\Phi_o = \frac{1}{2}\frac{\sigma k^2}{\sinh^2 k\Delta}\Big\{\Big[\hat{\Phi}^{\alpha}(\hat{\Phi}^{\alpha})^*\cosh^2 kx - (\hat{\Phi}^{\alpha}\hat{\Phi}^{\beta *} + \hat{\Phi}^{\alpha *}\hat{\Phi}^{\beta})\cosh kx\cosh k(x-\Delta)$$
$$+ \hat{\Phi}^{\beta}\hat{\Phi}^{\beta *}\cosh^2 k(x-\Delta)\Big]$$
$$+\Big[\hat{\Phi}^{\alpha}\hat{\Phi}^{\alpha *}\sinh^2 kx - (\hat{\Phi}^{\alpha}\hat{\Phi}^{\beta *} + \hat{\Phi}^{\alpha *}\hat{\Phi}^{\beta})\sinh kx\sinh k(x-\Delta) \tag{5}$$
$$+ \hat{\Phi}^{\beta}\hat{\Phi}^{\beta *}\sinh k(x-\Delta)\Big]\Big\}$$

Prob. 10.5.1 Perturbation of Eqs. 16-18 with subscript o indicating the stationary state and time dependence, exp $\underline{s}t$, gives the relations

$$\begin{bmatrix} s+(1+f) & \Omega_o & T_{yo} \\ -\Omega_o & s+(1+f) & -T_{xo} \\ -R_a & 0 & \left(\frac{s}{P_T}+1\right) \end{bmatrix} \begin{bmatrix} T_x' \\ T_y' \\ \Omega' \end{bmatrix} = 0 \tag{1}$$

Thus, the characteristic equation for the natural frequencies is

$$\frac{s^3}{P_T} + s^2\left[\frac{2(1+f)}{P_T}+1\right] + s\left[2(1+f)+\frac{(1+f)^2}{P_T}+\frac{\Omega_o^2}{P_T}+R_aT_{yo}\right] + \left[(1+f)^2+\Omega_o^2+\Omega_oT_{xo}R_a+R_aT_{yo}(1+f)\right]=0 \tag{2}$$

To discover the conditions for incipience of overstability, note that it takes place as a root to Eq. 2 passes from the left to the right half s plane. Thus, at incipience, $\underline{s}=j\underline{\omega}$. Because the coefficients in Eq. 2 are real, it can then be divided into real and imaginary parts, each of which can be solved for the frequency. With the use of Eqs. 23, it then follows that

$$\omega^2 = P_T\left\{(1+f)+\frac{(1+f)^2}{P_T} + \left[R_a - \frac{(1+f)^2}{P_T}\right]\frac{f}{P_T}\right\}$$

$$\omega^2 = 2\left[R_a - \frac{(1+f)^2}{f}\right]f\Big/\left[\frac{2(1+f)}{P_T}+1\right]$$

The critical R_a is found by setting these expressions equal to each other. The associated frequency of oscillation then follows by substituting that critical R_a into either Eq. 3 or 4.

Prob. 10.5.2 With heating from the left, the thermal source term enters in the x component of the thermal equation rather than the y component. Written in terms of the rotor temperature, the torque equation is unaltered. Thus, in normalized form, the model is represented by

$$\frac{dT_x}{dt} = -\Omega T_y - T_x(1+f) - f \tag{1}$$

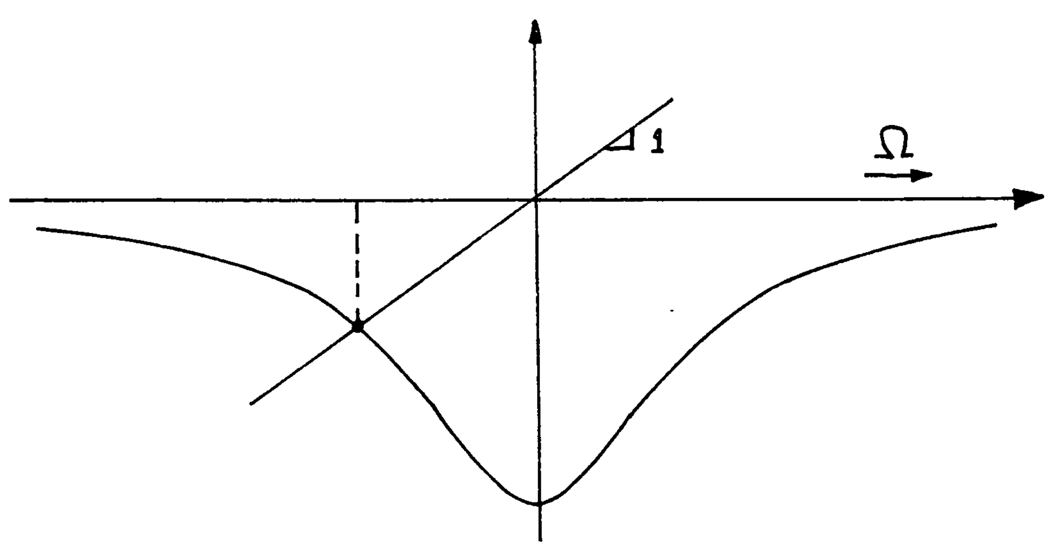
$$\frac{dT_y}{dt} = \Omega T_x - T_y(1+f) \tag{2}$$

$$\frac{1}{P_T}\frac{d\Omega}{dt} = -\Omega + R_a T_x \tag{3}$$

In the steady state, Eq. 2 gives T_y in terms of T_x and Ω, and this substituted into Eq. 1 gives T_x as a function of Ω. Finally, $T_x(\Omega)$, substituted into the torque equation, gives

$$\Omega = \frac{-f(1+f)R_a}{(1+f)^2 + \Omega^2} \tag{4}$$

The graphical solution to this expression is shown in Fig. P10.5.2. Note that for $T_e > 0$ and $d > 0$ the negative rotation is consistent with the left half of the rotor being heated and hence rising the right half being cooled and hence falling.



Prob. 10.6.1 (a) To prove the exchange of stabilities holds, multiply Eq. 8 by $\hat{v}_x^*$ and the complex conjugate of Eq. 9 by $\hat{T}$ and add. (The objective here is to obtain terms involving quadratic functions of $\hat{v}_x$ and $\hat{T}$ that can be manipulated into positive definite integrals.) Then, integrate over the normalized cross-section.

$$\int_0^1 \left\{ \hat{v}_x^* \left[\frac{s}{P_{TM}}(D^2 - k^2) + D^2 \right] \hat{v}_x + R_{am} k^2 \hat{T} \left[s^* - (D^2 - k^2) \right] \hat{T}^* \right\} dx = 0 \qquad (1)$$

The second-derivative terms in this expression are integrated by parts to obtain

$$\frac{s}{P_{TM}} \hat{v}_x^* D \hat{v}_x \Big]_0^1 - \int_0^1 |D\hat{v}_x|^2 \frac{s}{P_{TM}} dx - \frac{k^2 s}{P_{TM}} \int_0^1 |\hat{v}_x|^2 dx + \hat{v}_x^* D \hat{v}_x \Big|_0^1 - \int_0^1 |D\hat{v}_x|^2 dx$$
$$+ R_{am} k^2 \left\{ s^* \int_0^1 |\hat{T}|^2 dx - \hat{T} D \hat{T} \Big|_0^1 + \int_0^1 \left[|D\hat{T}|^2 + k^2 |\hat{T}|^2 \right] dx \right\} = 0 \qquad (2)$$

Boundary conditions eliminate the terms evaluated at the surfaces. With the definition of positive definite integrals

$$I_1 \equiv \int_0^1 |D\hat{v}_x|^2 dx \quad ; \quad I_3 = \int_0^1 |\hat{T}|^2 dx$$
$$I_2 \equiv \int_0^1 \left[k^2 |\hat{v}_x|^2 + |D\hat{v}_x|^2 \right] dx \quad ; \quad I_4 \equiv \int_0^1 \left[|D\hat{T}|^2 + k^2 |\hat{T}|^2 \right] dx \qquad (3)$$

The remaining terms in Eq. 2 reduce to

$$-\frac{s}{P_{TM}} I_2 - I_1 + s^* R_{am} k^2 I_3 + R_{am} k^2 I_4 = 0 \qquad (4)$$

Now, let $s = \alpha + j\omega$, where α and ω are real. Then, Eq. 4 divides into real and imaginary parts. The imaginary part is

$$\frac{\omega}{P_{TM}} I_1 + \omega R_{am} k^2 I_3 = 0 \qquad (5)$$

Prob. 10.6.1 (cont.)

It follows that if $R_{am} > 0$, then $\underline{\omega} = 0$. This is the desired proof. Note that if the heavy fluid is on the bottom ($R_{am} < 0$) the eigenfrequencies can be complex. This is evident from Eq. 17.

(b) Equations 8 and 9 show that with $\underline{s}=0$

$$\gamma^2(\gamma^2 - k^2) + R_{am}k^2 = 0 \qquad (\underline{6})$$

which has the four roots $\pm\gamma_a$, $\pm\gamma_b$ evaluated with $\underline{s}=0$. The steps to find the eigenvalues of R_{am} are now the same as used to deduce Eq. 15, except that $\underline{s}=0$ throughout. Note that Eq. 15 is unusually simple, in that in the section it is an equation for $\underline{\omega}$. It was only because of the simple nature of the boundary conditions that it could be solved for γ_a and γ_b directly. In any case, the γ's are the same here, $jn\pi$, and Eq. 6 can be evaluated to obtain the criticality condition, Eq. 18, for each of the modes.

<u>Prob. 10.6.2</u> Equation 10.6.14 takes the form

$$\begin{bmatrix} M_{ij} \end{bmatrix} \begin{bmatrix} \hat{T}_1 \\ \cdot \\ \cdot \\ \cdot \\ \hat{T}_4 \end{bmatrix} = \begin{bmatrix} \hat{T}^\alpha \\ \hat{T}^\beta \\ \hat{v}^\alpha \\ \hat{v}^\beta \end{bmatrix} \qquad (1)$$

In terms of these same coefficients $\hat{T}_1 \dots \hat{T}_4$, it follows from Eq. 10.6.10 that the normalized heat flux is

$$\hat{T}_x = -\sum_{n=1}^{4} \gamma_n \hat{T}_n e^{\gamma_n x} \qquad (2)$$

and from Eq. 11 that the normalized pressure is

$$\hat{p} = \sum_{n=1}^{4} B_n \hat{T}_n e^{\gamma_n x}$$

$$B_n \equiv \left\{ \frac{R_{am} P_{TM}}{\gamma_n} j\omega\left[j\omega - (\gamma_n^2 - k^2)\right]\right\} \hat{T}_n e^{\gamma_n x} \qquad (3)$$

Evaluation of these last two expressions at $\underline{x} = 1$ where $\hat{T}_x = \hat{T}_x^\alpha$ and $\hat{p} = \hat{p}^\alpha$ and at $\underline{x} = 0$ where $\hat{T}_x = \hat{T}_x^\beta$ and $\hat{p} = \hat{p}^\beta$ gives

Prob. 10.6.2 (cont.)

$$\begin{bmatrix} \hat{T}_x^{\alpha} \\ \hat{T}_x^{\beta} \\ \hat{p}^{\alpha} \\ \hat{p}^{\beta} \end{bmatrix} = \left[N_{ij}\right] \begin{bmatrix} \hat{T}_1 \\ \hat{T}_2 \\ \hat{T}_3 \\ \hat{T}_4 \end{bmatrix} \tag{4}$$

where (note that $B_1 = B_a \Rightarrow B_2 = -B_a$; $B_3 = B_b \Rightarrow B_4 = -B_B$.)

$$N_{ij} = \begin{bmatrix} -\gamma_a e^{\gamma_a} & \gamma_a e^{-\gamma_a} & -\gamma_b & \gamma_b \\ -\gamma_a & \gamma_a & -\gamma_b & \gamma_b \\ B_a e^{\gamma_a} & -B_a e^{-\gamma_a} & B_b e^{\gamma_b} & -B_b e^{-\gamma_b} \\ B_a & -B_a & B_b & -B_b \end{bmatrix} \tag{5}$$

Thus, the required transfer relations are

$$\begin{bmatrix} \hat{T}^{\alpha} \\ \hat{T}^{\beta} \\ \hat{p}^{\alpha} \\ \hat{p}^{\beta} \end{bmatrix} = \left[N_{ij}\right]\left[M_{ij}\right]^{-1} \begin{bmatrix} \hat{T}^{\alpha} \\ \hat{T}^{\beta} \\ \hat{v}^{\alpha} \\ \hat{v}^{\beta} \end{bmatrix} \tag{6}$$

So

$$C_{ij} = \left[N_{ij}\right]\left[M_{ij}\right]^{-1} \tag{7}$$

The matrix C_{ij} is therefore determined in two steps. First, Eq. 10.6.14 is inverted to obtain

Prob. 10.6.2 (cont.)

$$M_{ij}^{-1} = \left[4(b-a)\sinh\gamma_a \sinh\gamma_b\right]^{-1} \tag{8}$$

$$\begin{bmatrix} 2b\sinh\gamma_b & -2b\sinh\gamma_b e^{-\gamma_a} & -2\sinh\gamma_b & 2\sinh\gamma_b e^{-\gamma_a} \\ -2b\sinh\gamma_b & 2b\sinh\gamma_b e^{\gamma_a} & 2\sinh\gamma_b & -2\sinh\gamma_b e^{\gamma_a} \\ -2a\sinh\gamma_a & 2a\sinh\gamma_a e^{-\gamma_b} & 2\sinh\gamma_a & -2\sinh\gamma_a e^{-\gamma_b} \\ 2a\sinh\gamma_a & -2a\sinh\gamma_a e^{\gamma_b} & -2\sinh\gamma_a & 2\sinh\gamma_a e^{\gamma_b} \end{bmatrix}$$

Finally, Eq. 7 is evaluated using Eqs. 5 and 8.

$$C_{ij} = \left[(b-a)\sinh\gamma_a \sinh\gamma_b\right]^{-1}\left[C'_{ij}\right]$$

where

$$[C'_{ij}] =$$

$$\begin{bmatrix} [a\gamma_b\sinh\gamma_a\cosh\gamma_b - b\gamma_a\sinh\gamma_b\cosh\gamma_a] & [\gamma_a b\sinh\gamma_b - \gamma_b a\sinh\gamma_a] & [\gamma_a\sinh\gamma_b\cosh\gamma_a - \gamma_b\sinh\gamma_a\cosh\gamma_b] & [\gamma_b\sinh\gamma_a - \gamma_a\sinh\gamma_b] \\ [a\gamma_b\sinh\gamma_a - b\gamma_a\sinh\gamma_b] & [b\gamma_a\sinh\gamma_b\cosh\gamma_a - a\gamma_b\sinh\gamma_a\cosh\gamma_b] & [\gamma_a\sinh\gamma_b - \gamma_b\sinh\gamma_a] & [\gamma_b\sinh\gamma_a\cosh\gamma_b - \gamma_a\sinh\gamma_b\cosh\gamma_a] \\ [bB_a\sinh\gamma_b\cosh\gamma_a - aB_b\sinh\gamma_a\cosh\gamma_b] & [-bB_a\sinh\gamma_b + aB_b\sinh\gamma_a] & [B_b\sinh\gamma_a\cosh\gamma_b - B_a\sinh\gamma_b\cosh\gamma_a] & [B_a\sinh\gamma_b - B_b\sinh\gamma_a] \\ [bB_a\sinh\gamma_b - aB_b\sinh\gamma_a] & [aB_b\sinh\gamma_a\cosh\gamma_b - bB_a\cosh\gamma_a\sinh\gamma_b] & [-B_a\sinh\gamma_b + B_b\sinh\gamma_a] & [B_a\sinh\gamma_b\cosh\gamma_a - B_b\sinh\gamma_a\cosh\gamma_b] \end{bmatrix}$$

Prob. 10.6.3 (a) To the force equation, Eq. 4, is added the viscous force density, $\eta\nabla^2\bar{v}$. Operating on this with [-curl(curl)], then adds to Eq. 7, $\eta\nabla^4\hat{v}_x$. In terms of complex amplitudes, the result is

$$\left[\eta(D^2-k^2)^2-j\omega\rho(D^2-k^2)-\sigma(\mu_o H_o)^2D^2\right]\hat{v}_x=-\alpha\rho_o g k^2\hat{T} \tag{1}$$

Normalized as suggested, this results in the first of the two given equations. The second is the thermal equation, Eq. 3, unaltered but normalized.

(b) The two equations in (v_x,T) make it possible to determine the six possible solutions $\exp\gamma x$.

$$\left[(\gamma^2-k^2)^2-\frac{j\omega}{P_T}(\gamma^2-k^2)-\frac{T_m}{T_{MV}}\gamma^2\right]\left[(\gamma^2-k^2)-j\omega\right]+R_a=0 \tag{2}$$

The six roots comprise the solution

$$\hat{T}=\sum_{k=1}^{6}T_k\,e^{\gamma_k x} \tag{3}$$

The velocity follows from the second of the given equations

$$\hat{v}_x=\sum_{k=1}^{6}\left[j\omega-(\gamma_k^2-k^2)\right]T_k\,e^{\gamma_k x} \tag{4}$$

To find the transfer relations, the pressure is gotten from the x component of the force equation, which becomes

$$D\hat{p}=\left[-j\omega+P_T(D^2-k^2)\right]\hat{v}_x+R_a P_T\hat{T} \tag{5}$$

Thus, in terms of the six coefficients,

$$\hat{p}=\sum_{k=1}^{6}\left\{\left[-j\omega+P_T(\gamma_k^2-k^2)\right]\left[j\omega-(\gamma^2-k^2)\right]+R_a P_T\right\}\frac{T_k}{\gamma_k}e^{\gamma_k x} \tag{6}$$

For two-dimensional motions where $v_z=0$, the continuity equation suffices to find $\hat{v}_y$ in terms of $\hat{v}_x$. Hence,

$$\hat{v}_y=\frac{1}{jk_y}D\hat{v}_x \tag{7}$$

Prob. 10.6.3 (cont.)

From Eqs. 6 and 7, the stress components can be written as

$$\hat{S}_x = -\hat{p} + 2\eta D\hat{v}_x \tag{8}$$

$$\hat{S}_y = \eta\left(D\hat{v}_y - jk_y\hat{v}_x\right) \tag{9}$$

and the thermal flux is similarly written in terms of the amplitudes T_k.

$$\hat{T}'_x = -k_T D\hat{T} \tag{10}$$

These last three relations, respectively evaluated at the α and β surfaces provide the stresses and thermal fluxes in terms of the T_k's.

$$\begin{bmatrix} \hat{S}_x^\alpha \\ \hat{S}_x^\beta \\ \hat{S}_y^\alpha \\ \hat{S}_y^\beta \\ \hat{T}_x^\alpha \\ \hat{T}_x^\beta \end{bmatrix} = \left[A_{ij}\right] \begin{bmatrix} \hat{T}_1 \\ \hat{T}_2 \\ \hat{T}_3 \\ \hat{T}_4 \\ \hat{T}_5 \\ \hat{T}_6 \end{bmatrix} \tag{11}$$

By evaluating Eqs. 3, 4 and 7 at the respective surfaces, relations are obtained

$$\begin{bmatrix} \hat{v}_x^\alpha \\ \hat{v}_x^\beta \\ \hat{v}_y^\alpha \\ \hat{v}_y^\beta \\ \hat{T}^\alpha \\ \hat{T}^\beta \end{bmatrix} = \left[B_{ij}\right] \begin{bmatrix} \hat{T}_1 \\ \hat{T}_2 \\ \hat{T}_3 \\ \hat{T}_4 \\ \hat{T}_5 \\ \hat{T}_6 \end{bmatrix} \tag{12}$$

Inversion of these relations gives the amplitudes T_k in terms of the velocities and temperature. Hence,

$$\begin{bmatrix} \hat{S}_x^\alpha \\ \hat{S}_x^\beta \\ \hat{S}_y^\alpha \\ \hat{S}_y^\beta \\ \hat{T}_x^\alpha \\ \hat{T}_x^\beta \end{bmatrix} = [A][B]^{-1} \begin{bmatrix} \hat{v}_x^\alpha \\ \hat{v}_x^\beta \\ \hat{v}_y^\alpha \\ \hat{v}_y^\beta \\ \hat{T}^\alpha \\ \hat{T}^\beta \end{bmatrix} \tag{13}$$

Prob. 10.7.1 (a) The imposed electric field follows from Gauss' integral law and the requirement that the integral of $\bar{E}$ from r=R to r=a be V.

$$\bar{E} = \frac{\lambda \bar{i}_r}{2\pi \epsilon_o r} \quad ; \quad \lambda = \frac{V\, 2\pi\epsilon_o}{\ln\left(\frac{a}{R}\right)} \tag{1}$$

The voltage V can be constrained, or the cylinder allowed to charge up, in which case the cylinder potential relative to that at r=a is V. Conservation of ions in the quasi-stationary state is Eq. 10.7.4 expressed in cylindrical coordinates.

$$\frac{1}{r}\frac{d}{dr} r\left(\frac{b\lambda\rho}{2\pi\epsilon_o r} - K_+ \frac{d\rho}{dr}\right) = 0 \tag{2}$$

One integration, with the constant evaluated in terms of the current i to the cylinder, gives

$$2\pi r K_+ \frac{d\rho}{dr} - \frac{b\lambda}{\epsilon_o}\rho = i \tag{3}$$

The particular solution is $-\epsilon_o i/b\lambda$, while the homogeneous solution follows from

$$\int \frac{d\rho}{\rho} = \frac{b\lambda}{2\pi\epsilon_o K_+}\int\frac{dr}{r} \tag{4}$$

Thus, with the homogeneous solution weighted to make $\rho(a) = \rho_o$, the charge density distribution is the sum of the homogeneous and particular solutions,

$$\rho = \left(\rho_o + \frac{\epsilon_o i}{b\lambda}\right)\left(\frac{r}{a}\right)^{f\lambda} - \frac{\epsilon_o i}{b\lambda} \tag{5}$$

where f = $q/2\pi\epsilon_o kT$.

(b) The current follows from requiring that at the surface of the cylinder, r=R, the charge density vanish.

$$i = \frac{\rho_o b}{\epsilon_o}\lambda \frac{1}{\left[\left(\frac{a}{R}\right)^{f\lambda} - 1\right]} \tag{6}$$

With the voltage imposed, this expression is completed by using Eq. 1b.

Prob. 10.7.1 (cont.)

(c) With the cylinder free to charge up, the charging rate is determined by

$$i = \frac{d\lambda}{dt} \tag{7}$$

This expression can be integrated by writing it in the form

$$\int_0^t \frac{\rho_o b}{\epsilon_o}\,dt = \int_0^\lambda \frac{\left[\left(\frac{a}{R}\right)^{f\lambda} - 1\right]}{\lambda}\,d\lambda \tag{8}$$

By defining $g \equiv \ln(a/R)^f$ this becomes

$$t\left(\frac{\rho_o b}{\epsilon_o}\right) = \int_0^{g\lambda} \frac{[e^{g\lambda} - 1]}{g\lambda}\,dg\lambda = \frac{g\lambda}{1!} + \frac{g^2\lambda^2}{2\cdot 2!} + \frac{g^3\lambda^3}{3\cdot 3!} + \cdots \tag{9}$$

By defining $\lambda_D \equiv \frac{1}{g} = (q/2\pi\epsilon_o kT)/\ln(a/R)$ this takes the normalized form

$$\underline{t} = \frac{\underline{\lambda}}{1!} + \frac{\underline{\lambda}^2}{2\cdot 2!} + \frac{\underline{\lambda}^3}{3\cdot 3!} + \cdots \tag{10}$$

where

$$\underline{t} = t/\tau_e \quad ; \quad \tau_e \equiv \epsilon_o/\rho_o b$$

$$\underline{\lambda} = \lambda/\lambda_D$$

Prob. 10.7.2 Because there is no equilibrium current in the x direction,

$$\rho b E - K_+ \frac{d\rho}{dx} = 0 \tag{1}$$

For the unipolar charge distribution, Gauss' law requires that

$$\frac{d\epsilon E}{dx} = \rho \tag{2}$$

Substitution for ρ using Eq. 2 in Eq. 1 gives an expression that can be integrated once by writing it in the form

$$\frac{d}{dx}\left(\frac{1}{2} b E^2 - K_+ \frac{dE}{dx}\right) = 0 \tag{3}$$

As $x \rightarrow \infty$, $E \rightarrow 0$ and there is no charge density, so $dE/dx \rightarrow 0$. Thus, the quantity in brackets in Eq. 3 is zero, and a further integration can be performed

$$\int_E^{E_o} \frac{dE}{E^2} = \frac{1}{2}\frac{b}{K_+}\int_x^0 dx \tag{4}$$

Prob. 10.7.2 (cont.)

It follows that the desired electric field distribution is

$$E = E_o/(1 - \frac{x}{\ell_d}) \tag{5}$$

where $\ell_d \equiv 2K_+/bE_o$.

The charge distribution follows from Eq. 2

$$\rho = -\frac{\epsilon E_o}{\ell_d}\Big/\left(1 - \frac{x}{\ell_d}\right)^2 \tag{6}$$

The Einstein relation shows that $\ell_d = 2(kT/q)/E_o \simeq 2(25\times10^{-3})/10^4 = 5\mu m$

Prob. 10.8.1 (a) The appropriate solution to Eq. 8 is simply

$$\Phi = -S\,\frac{\cosh(x - \frac{\Delta}{2})}{\cosh(\Delta/2)} \tag{1}$$

Evaluated at the midplane, this gives

$$\Phi_c = -S/\cosh(\Delta/2) \tag{2}$$

(b) Symmetry demands that the slope of the potential vanish at the midplane. If the potential there is called Φ_c, evaluation of the term in brackets from Eq. 9 at the midplane gives $-\cosh\Phi_c$, and it follows that

$$\frac{1}{2}\left(\frac{d\Phi}{dx}\right)^2 - \cosh\Phi = -\cosh\Phi_c \tag{3}$$

so that instead of Eq. 10, the expression for the potential is that given in the problem.

(c) Evaluation of the integral expression at the midplane gives

$$\frac{\Delta}{2} = \int_{-S}^{\Phi_c} \frac{d\Phi}{\sqrt{2(\cosh\Phi - \cosh\Phi_c)}} \tag{4}$$

In principal, an iterative evaluation of this integral can be used to determine Φ_c and hence the potential distribution. However, the integrand is singular at the end point of the integration, so the integration in the vicinity of this end point is carried out analytically. In the neighborhood of Φ_c, $\cosh\Phi \simeq \cosh\Phi_c + \sinh\Phi_c(\Phi - \Phi_c)$ and the integrand of Eq. 4 is approximated by

$$\frac{1}{\sqrt{2}}(\cosh\Phi - \cosh\Phi_c)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}\left[\sinh\Phi_c(\Phi - \Phi_c)\right]^{-\frac{1}{2}}$$

With the numerical integration ending at $\Phi_c + \Delta\Phi$, short of Φ_c , the remainder of the integral is taken analytically.

Prob. 10.8.1 (cont.)

$$\frac{1}{\sqrt{2}}\int_{\Phi_c}^{\Phi_c+\Delta\Phi}\left[\sinh\Phi_c(\Phi-\Phi_c)\right]^{-\frac{1}{2}}d\Phi = \frac{2}{\sqrt{2}}\left(\frac{\Phi-\Phi_c}{\sinh\Phi_c}\right)^{1/2}\Bigg|_{\Phi_c}^{\Phi_c+\Delta\Phi} = \sqrt{2}\left(\frac{\Delta\Phi}{\sinh\Phi_c}\right) \quad (6)$$

Thus, the expression to be evaluated numerically is

$$\frac{\Delta}{2} = \int_{-\zeta}^{\Phi_c+\Delta\Phi}\frac{d\Phi}{\sqrt{2(\cosh\Phi-\cosh\Phi_c)}} = \sqrt{2}\left(\frac{\Delta\Phi}{\sinh\Phi_c}\right)^{1/2} \quad (7)$$

where Φ_c and $\Delta\Phi$ are negative quantities and ζ is a positive number. At least to obtain a rough approximation, Eq. 7 can be repeatedly evaluated with Φ_c altered to make Δ the prescribed value. For $\Delta/2 = 1$, $\zeta = -3$ the distribution is shown in Fig. P10.8.1 and $\Phi_c \simeq 1$.

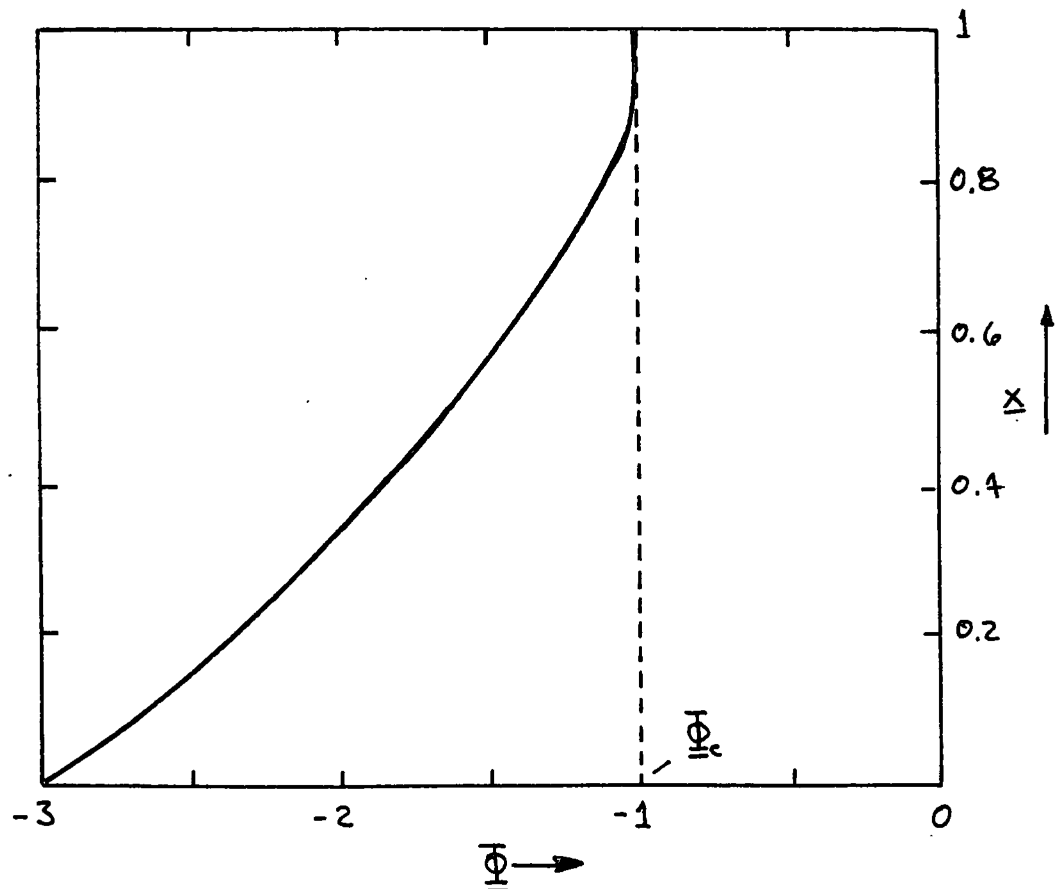


Fig. P10.8.1. Potential distribution over half of distance between parallel boundaries having zeta potentials ζ=-3.

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PROBLEM SET 11

3 (10.8.1)

$x=\Delta$ ———— $\underline{\Phi}(\underline{x}=\underline{\Delta}) = -\rho$

$x=0$ ———— $\underline{\Phi}(\underline{x}=0) = -\rho$

$x = \underline{x}\,\delta_D \qquad \Phi = \underline{\Phi}\,\frac{kT}{q}$

$\Delta = \underline{\Delta}\,\delta_D \qquad \delta_D = \sqrt{\frac{\varepsilon kT}{2 q n_0 q}}$

a. IN NORMALIZED TERMS, THE POTENTIAL DISTRIBUTION ACROSS THE ELECTROLYTE IS GIVEN BY

$$\frac{d^2\underline{\Phi}}{d\underline{x}^2} = \sinh(\underline{\Phi})$$

FOR $\underline{\Phi} \ll 1$, $\sinh \underline{\Phi} \simeq \underline{\Phi}$

$$\Rightarrow \quad \frac{d^2\underline{\Phi}}{d\underline{x}^2} - \underline{\Phi} = 0$$

THIS DIFFERENTIAL EQUATION HAS SOLUTIONS OF THE FORM: $\Phi \sim e^{\pm \underline{x}}, \sinh(\underline{x}), \cosh(\underline{x})$

IMPOSING THE POTENTIALS AT THE BOUNDARIES GIVES

$$\underline{\Phi} = \frac{-\rho}{\sinh(\underline{\Delta})}\left[\sinh(\underline{x}) - \sinh(\underline{x}-\underline{\Delta})\right] \left(= -\rho\,\frac{\cosh\left(\underline{x}-\frac{\underline{\Delta}}{2}\right)}{\cosh\left(\frac{\underline{\Delta}}{2}\right)}\right)$$

AT THE MIDPLANE, $\underline{x} = \frac{\underline{\Delta}}{2}$, $\underline{\Phi} = \underline{\Phi}_c \Rightarrow \underline{\Phi}_c = \frac{-\rho}{\sinh(\underline{\Delta})}\left[\sinh\left(\frac{\underline{\Delta}}{2}\right) - \sinh\left(\frac{-\underline{\Delta}}{2}\right)\right] = -\rho\,\frac{2\sinh\left(\frac{\underline{\Delta}}{2}\right)}{2\sinh\left(\frac{\underline{\Delta}}{2}\right)\cosh\left(\frac{\underline{\Delta}}{2}\right)}$

$$\Rightarrow \underline{\Phi}_c = \frac{-\rho}{\cosh\left(\frac{\underline{\Delta}}{2}\right)}$$

b. IN GENERAL $\quad \frac{d^2\underline{\Phi}}{d\underline{x}^2} = \sinh(\underline{\Phi}) \quad$ OR $\quad \frac{d^2\underline{\Phi}}{d\underline{x}^2} - \sinh(\underline{\Phi}) = 0$

MULTIPLICATION BY $\frac{d\underline{\Phi}}{d\underline{x}}$ GIVES $\quad \frac{d^2\underline{\Phi}}{d\underline{x}^2}\frac{d\underline{\Phi}}{d\underline{x}} - \sinh(\underline{\Phi})\frac{d\underline{\Phi}}{d\underline{x}} = 0$

NOW, NOTICE THAT $\quad \frac{d}{d\underline{x}}\left(\frac{1}{2}\left(\frac{d\underline{\Phi}}{d\underline{x}}\right)^2\right) = \frac{d\underline{\Phi}}{d\underline{x}}\frac{d^2\underline{\Phi}}{d\underline{x}^2}$

AND $\quad \frac{d}{d\underline{x}}\left[\cosh(\underline{\Phi})\right] = \sinh(\underline{\Phi})\frac{d\underline{\Phi}}{d\underline{x}}$

$$\Rightarrow \frac{d}{d\underline{x}}\left[\frac{1}{2}\left(\frac{d\underline{\Phi}}{d\underline{x}}\right)^2 - \cosh(\underline{\Phi})\right] = 0$$

OR

$$\frac{1}{2}\left(\frac{d\underline{\Phi}}{d\underline{x}}\right)^2 - \cosh(\underline{\Phi}) = C_1$$

DUE TO THE SYMMETRY OF THE PROBLEM, $\frac{d\underline{\Phi}}{d\underline{x}} = 0$ AT THE MIDPLANE, WHERE $\underline{\Phi} = \underline{\Phi}_c \Rightarrow C_1 = -\cosh(\underline{\Phi}_c)$

(OVER)

THIS YIELDS $\frac{1}{2}\left(\frac{d\Phi}{dx}\right)^2 = \cosh(\Phi) - \cosh(\Phi_c)$

OR $\frac{d\Phi}{dx} = \pm\sqrt{2[\cosh(\Phi) - \cosh(\Phi_c)]}$

INTEGRATION GIVES: $\int_0^x dx' = \pm\int_{-\rho}^{\Phi} \frac{d\Phi'}{\sqrt{2[\cosh(\Phi') - \cosh(\Phi_c)]}}$

$$\therefore \quad x = \pm\int_{-\rho}^{\Phi} \frac{d\Phi'}{\sqrt{2[\cosh(\Phi') - \cosh(\Phi_c)]}}$$

WITH THE + SIGN USED FOR $0 \le x < \frac{\Delta}{2}$ AND THE − SIGN USED FOR $\frac{\Delta}{2} < x \le \Delta$. THIS SEPARATION IS NECESSARY TO MAINTAIN THE "SYMMETRY", AND BECAUSE THE FUNCTIONAL TERM IN THE INTEGRAL GOES TO INFINITY AT $\Phi = \Phi_c$ OR $x = \frac{\Delta}{2}$

c. GIVEN $\Delta \equiv \hat{\delta}_D$, IT WOULD SEEM REASONABLE TO USE THE EQUATION IN PART b TO FIND Φ_c, BY FIRST GUESSING Φ_c, THEN NUMERICALLY SOLVING THE INTEGRAL TO $x = \frac{\Delta}{2}$. THE RESULT WOULD THEN BE USED TO MODIFY THE Φ_c TO WITHIN A GIVEN ERROR BY REPEATING THE PROCESS. UNFORTUNATELY, AT $x = \frac{\Delta}{2}$, $\Phi = \Phi_c \Rightarrow$ THE FUNCTION INSIDE THE INTEGRAL BLOWS UP (GOES TO INFINITY), SO A SIMPLE TRAPEZOIDAL INTEGRATION COULD LEAD TO NUMERICAL ERRORS. TO SIDESTEP THIS DIFFICULTY, THE DERIVATIVE OF THE POTENTIAL WILL BE USED IN A FINITE DIFFERENCE TECHNIQUE. WHILE NUMERICAL DIFFERENTIATION IS NOT A RECOMMENDED PROCEDURE IN GENERAL, THE FUNCTIONS ARE SMOOTH ENOUGH IN THIS CASE TO ALLOW THIS SOLUTION.

USING FINITE DIFFERENCES: $\frac{d\Phi}{dx} = \pm\sqrt{2[\cosh(\Phi) - \cosh(\Phi_c)]} \equiv \pm f(\Phi, \Phi_c)$

BUT $\frac{d\Phi}{dx} \simeq \frac{\Phi(x+\Delta x) - \Phi(x)}{\Delta x} \Rightarrow \Phi(x+\Delta x) \simeq \Phi(x) \pm f(\Phi(x), \Phi_c)\,\Delta x$ (A

NOW, AN INITIAL $\Phi_c = 0$ (WHICH IS THE MAXIMUM Φ) IS GUESSED, THEN EQ. (A) IS ITERATED UPON (WITH $\Phi(x=0) = -\rho$ AND Δx KNOWN) UNTIL $x = \frac{\Delta}{2}$, SO THAT $\Phi_c' = \Phi(x = \frac{\Delta}{2})$. THIS Φ_c' IS COMPARED TO Φ_c TO SEE IF THE DIFFERENCE IS SMALL. IF IT IS NOT, THEN THE PROCESS CAN BE REPEATED, WITH Φ_c REPLACED BY Φ_c'. ONCE Φ_c IS KNOWN, (A) CAN BE USED TO FIND $\Phi(x)$

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PROBLEM SET 11

3. (10.8.1) CONTINUED.

THIS ALGORITHM IS IMPLEMENTED BY THE PROGRAM LISTED ON THE FOLLOWING PAGES (AND IN PART d.)

AS A CHECK FOR THE PROGRAM'S COMPUTATION OF Φ_c, THE RESULTS OF PART a WERE USED

e.g. $\Phi_c \simeq -\dfrac{\rho}{\cosh(\frac{\Delta}{2})}$ FOR Φ SMALL.

AS A TEST, $\rho = 0.1$ AND $\Delta = 1$.

FROM PART a, $\Phi_c = -0.0887$.

FROM THE PROGRAM:

# OF STEPS	Φ_c	% DIFFERENCE
21	-0.0839	5.4%
63	-0.0859	3.2%
189	-0.0865	2.5%

AS ANOTHER TEST, $\rho = 0.025$ AND $\Delta = 1$

FROM PART a, $\Phi_c = -0.0222$

FROM MY PROGRAM, WITH 101 STEPS, $\Phi_c = -0.0216$

$\Rightarrow$ 2.7% DIFFERENCE.

IN BOTH CASES, THE FRACTIONAL NUMERICAL ERROR IN Φ_c IS 0.001 (0.1%), A SPECIFIED BY MY PROGRAM.

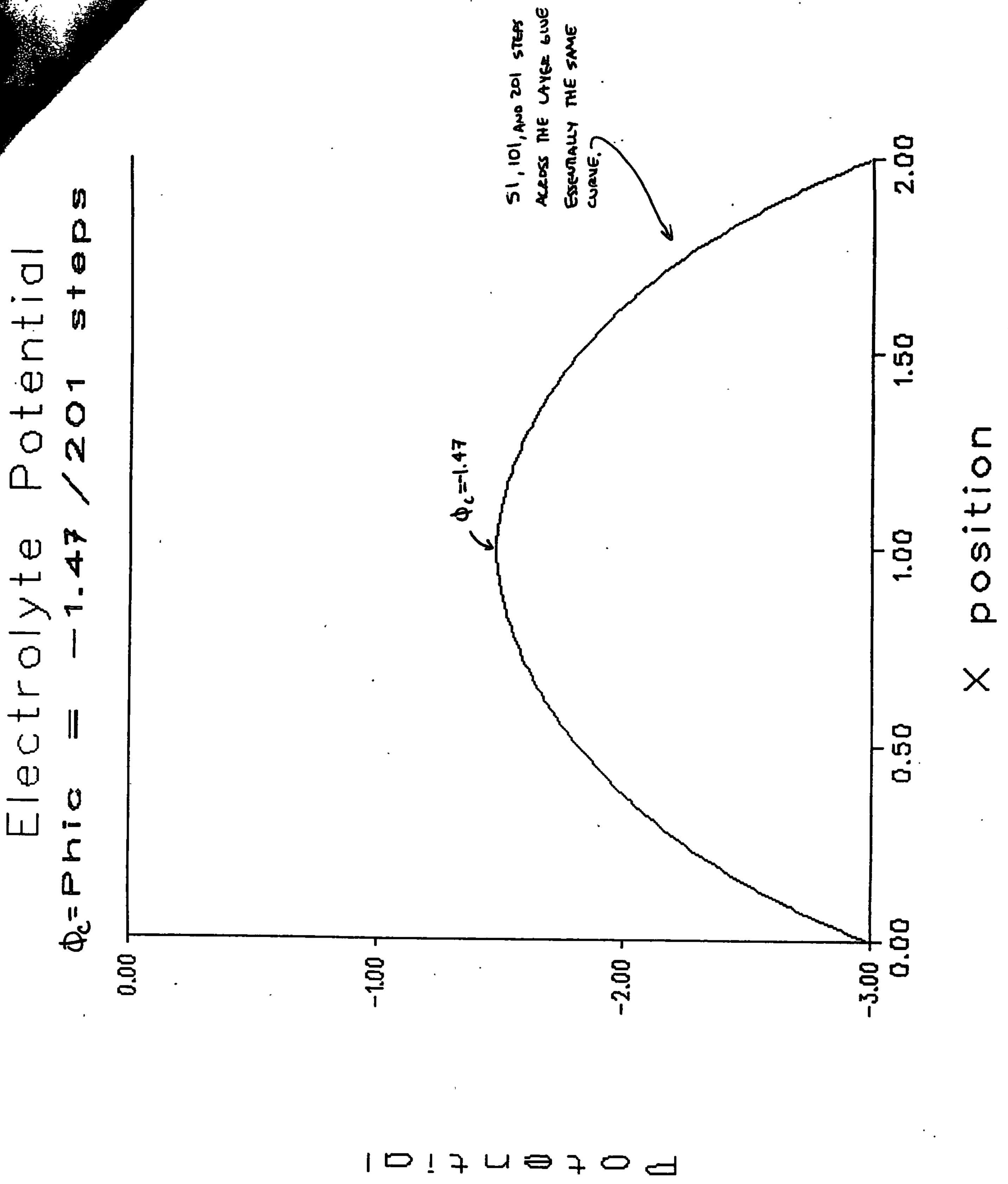
THESE TESTS LEAD ME TO BELIEVE THAT THE ALGORITHM DOES WORK SATISFACTORILY, EVEN WITH A SMALL NUMBER OF POINTS

d. WITH $\rho = 3$ AND $\Delta = 2$, THE PROGRAM WAS RUN AGAIN. IN THIS CASE, THE FOLLOWING VALUES OF Φ_c WERE FOUND:

# OF STEPS	Φ_c
51	-1.40
101	-1.45
201	-1.47

THIS INDICATES A CONVERGENCE OF $\Phi_c \simeq \underline{-1.5}$.

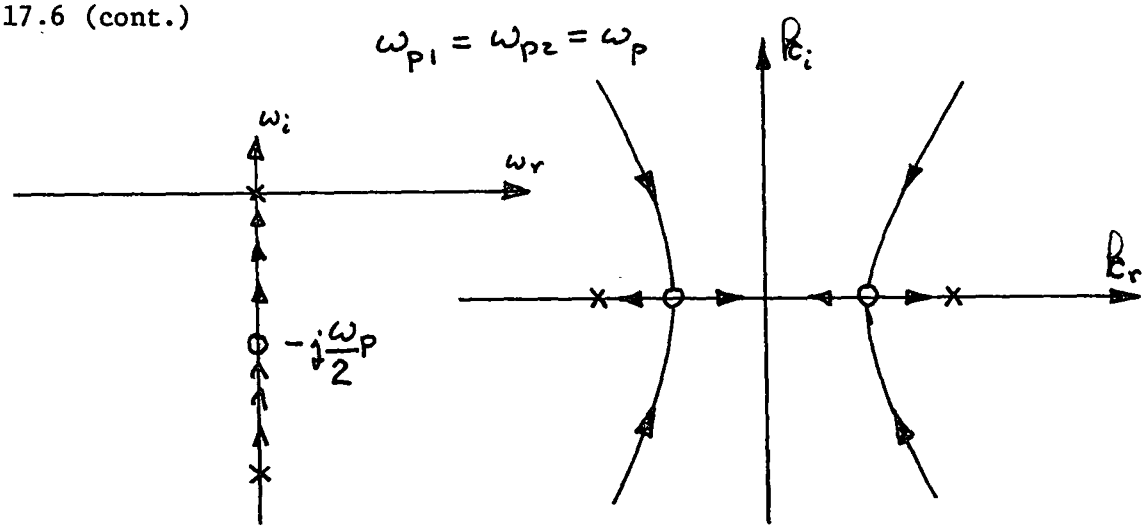
A PLOT OF THE POTENTIAL DISTRIBUTION IS ON THE NEXT PAGE.


Electrolyte Potential
ϕ_c=Phic = -1.47 / 201 steps
Potential
X position
0.00
-1.00
-2.00
-3.00
0.00
0.50
1.00
1.50
2.00
ϕ_c=-1.47
51, 101, AND 201 STEPS ACROSS THE LAYER GIVE ESSENTIALLY THE SAME CURVE.

```
      program Zeta_Potentials

      integer istep,imid
      real*4 delta,delx,phi(9999),phic,phierr,zeta,perror
      common istep,delta,delx,phi,phic,phierr,zeta

      call input
    2 delx = delta/real(istep-1)
      imid = 1 + istep/2
      phic = 0.0
    3 continue
C
C     CALCULATE THE VALUE OF PHIc

      do 4 i=1,imid-1
          phi(i+1) = phi(i) + delx * sqrt(2*(cosh(phi(i))-cosh(phic)))
    4 continue
C
C     DETERMINE IF THE UNCERTAINTY IN PHIc IS LESS THAN THE ERROR
      perror = (phi(imid)-phic)/(phic + 1.0e-06)
      if(abs(perror).gt.abs(phierr)) then
          phic = phi(imid)
          goto 3
      endif
C
C     PREPARE AND SEND THE DATA TO THE OUTPUT FILE
      do 5 i=1,imid-1
          phi(istep-i+1)=phi(i)
    5 continue
      call output
      STOP 'GOOD BYE'
      END


      SUBROUTINE INPUT
      integer istep
      real*4 delta,delx,phi(9999),phic,phierr,zeta
      common istep,delta,delx,phi,phic,phierr,zeta
C
C     INPUT THE NECESSARY PARAMETERS FOR THE PLOT
    8 write(*,*) 'Enter the zeta potential:'
       read(*,*,err=8) zeta
    9 write(*,*) 'Enter the normalized distance:'
       read(*,*,err=9) delta
   10 write(*,*) 'Enter the (odd) number of steps across the layer:'
       read(*,*,err=10) istep
   11 write(*,*) 'Enter the error fraction for the midplane phi:'
       read(*,*,err=11) phierr
      phi(1) = - zeta
      RETURN
      END
```

```
      SUBROUTINE OUTPUT
      integer istep
      real*4 delta,delx,phi(9999),phic,phierr,zeta,x
      common istep,delta,delx,phi,phic,phierr,zeta

      WRITE THE DESIRED DATA TO AN OUTPUT FILE, READY FOR ENABLE TO PLOT
      open(unit=6,file='e:zeta.out',status='new')
      write(6,*) 'The potential parameters are'
      write(6,9500) istep,delta,phic,zeta,phierr
9500  format(' Steps=  ', ',',I5,/,' Delta=  ', ',',F10.4,/,
     &       ' Phi_c=  ', ',',F10.4,/,' Zeta=  ', ',',F10.4,/,
     &       ' Error=  ', ',', F10.4)
      write(6,*) ' X position    Phi(x)  '
      do 100 i=1,istep
          x = real(i-1) * delx
          write(6,9510) x,phi(i)
 100  continue
9510  format(' ',F10.5,',',F10.5)
      close(unit=6)
      RETURN
      END
```

Prob. 11.17.6 (cont.)



See, Briggs, R.J., <u>Electron-Stream Interaction With Plasmas</u>, M.I.T. Press (1964) pp 32-34 and 42-44.

Prob. 10.9.1 (a) In using Eq. (a) of Table 9.3.1, the double layer is assumed to be inside the boundaries. (This is by contrast with the use made of this expression in the text, where the electrokinetics was represented by a slip boundary condition at the walls, and there was no interaction in the bulk of the fluid.) Thus, $v^{\alpha}=0, v^{\beta}=0$ and $T_{yx} = \epsilon E_y d\Phi/dx$. Because the potential has the same value on each of the walls, the last integral is zero.

$$\int_0^{\Delta} T_{yx}dx = \int_0^{\Delta} \epsilon E_y \frac{d\Phi}{dx}dx = \epsilon E_y[\Phi(\Delta) - \Phi(0)] = 0 \tag{1}$$

and the next to last integral becomes

$$\int_0^{x} T_{yx}dx = \epsilon E_y[\Phi(x) - \Phi(0)] = \epsilon E_y[\Phi(x) + \zeta] \tag{2}$$

Thus, the velocity profile is a superposition of the parabolic pressure driven flow and the potential distribution biased by the zeta potential so that it makes no contribution at either of the boundaries.

(b) If the Debye length is short compared to the channel width, then Φ =0 over most of the channel. Thus, Eqs. 1 and 2 inserted into Eq. (a) of Table 9.3.1 give the profile, Eq. 10.9.5.

(c) Division of Eq. (a) of Table 9.3.1 evaluated using Eqs. 1 and 2 by $\epsilon E_y kT/q\eta$ gives the desired normalized form. For example, if ζ= -3 and Δ =2, the electrokinetic contribution to the velocity profile is as shown in Fig. P10.8.1.

Prob. 10.9.2 (a) To find S_{yx}, note that from Eq. (a) of Table 9.3.1 with the wall velocities taken as $\epsilon\zeta E_y/\eta$

$$v_x = \frac{\epsilon\zeta E_y}{\eta} + \frac{\Delta^2}{2\eta}\frac{\partial p'}{\partial y}\left[\left(\frac{x}{\Delta}\right)^2 - \frac{x}{\Delta}\right] \tag{1}$$

Thus, the stress is

$$S_{yx} = \eta\frac{\partial v_y}{\partial x} = \frac{\Delta}{2}\frac{\partial p'}{\partial y}\left(\frac{2x}{\Delta} - 1\right) \tag{2}$$

This expression, evaluated at x=0, combines with Eqs. 10.9.11 and 10.9.12 to give the required expression.

(b) Under open circuit conditions, where the wall currents

Prob. 10.9.2 (cont.)

due to the external stress are returned in the bulk of the fluid and where the generated voltage also gives rise to a negative slip velocity that tends to decrease E_y, the generated potential is gotten by setting i in the given equation equal to zero and solving for E_y and hence v.

$$v = \frac{(\zeta \Delta\epsilon/\eta)\,\Delta p}{\left[\Delta\sigma + \dfrac{2\rho_o \zeta^2 \epsilon \delta_o}{\eta (RT/\zeta)}\right]} \tag{3}$$

Prob. 10.10.1 In Eq. 10.9.12, what is $(\zeta\epsilon\delta_D/2)E_y$ compared to $\delta_D^2 S_{yx}$? To approximate the latter, note that $S_{yx} \sim \eta U/R$, where from Eq. 10.10.10, U is at most $(\epsilon\zeta/\eta)E_o$. Thus, the stress term is of the order of $\delta_D^2 \epsilon\zeta/R$ and this is small compared to the surface current driven by the electric field if $R \gg \delta_D$.

Prob. 10.10.2 With the particle constrained and the fluid motionless at infinity, U=0 in Eq. 10.10.9. Hence, with the use of Eq. 10.10.7, that expression gives the force.

$$f_z = \frac{6\pi R\epsilon\zeta E_o}{1 + \dfrac{\sigma_s}{\sigma R}} \tag{1}$$

The particle is pulled in the same direction as the liquid in the diffuse part of the double layer. For a positive charge, the fluid flows from south to north over the surface of the particle and is returned from north to south at a distance on the order of R from the particle.

Prob. 10.10.3 Conservation of charge now requires that

$$-\sigma\frac{\partial\Phi}{\partial r} + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left[(\sigma_s E_\theta + \beta S_{\theta r})\sin\theta\right] = 0 \tag{1}$$

with K_θ again taking the form of Eq. 10.10.4. Using the stress functions with θ dependence defined in Table 7.20.1, Eq. 1 requires that

$$-\sigma\left(-E_o - \frac{2A}{R^3}\right) + \frac{2}{R}\left[\sigma_s\left(-E_o + \frac{A}{R^3}\right) + \beta\,\tilde{S}^{\beta}_{\theta r}\right] = 0 \tag{2}$$

Prob. 10.10.3 (cont.)

The viscous shear stress can be substituted into this expression using Eq. 10.10.8b with $\tilde{v}_\theta$ given by Eq. 7 and E_θ in turn written using Eq. 10.10.4. Hence,

$$\sigma\left(E_o + \frac{2A}{R^3}\right) + \frac{2}{R}\sigma_s\left(-E_o + \frac{A}{R^3}\right) - \frac{2\beta\eta}{R^2}\left[\frac{3}{2}U + \frac{3\epsilon\zeta}{\eta}\left(-E_o + \frac{A}{R^3}\right)\right] = 0 \quad (3)$$

This expression can be solved for A/R^3

$$\frac{A}{R^3} = \frac{E_o\left(-\sigma + \frac{2\sigma_s}{R} - \frac{6\beta\epsilon\zeta}{R^2}\right) + \frac{3\beta\eta}{R^2}U}{2\sigma + \frac{2\sigma_s}{R} - \frac{6\beta\epsilon\zeta}{R^2}} \quad (4)$$

Substituted into Eq. 10.10.4, this expression determines the potential distribution. With no flow at infinity, the field consists of the uniform imposed field plus a dipole field with moment proportional to A. Note that the terms in β resulting from the shear stress contributions are negligible compared to those in σ_s, provided that $\delta_D \ll R$. With no applied field, the shear stress creates a streaming current around the particle that influences the surrounding potential much as if there were a dipole current source at the origin. The force can be evaluated using Eq. 10.10.9.

$$f_z = -\pi R\eta\left\{\frac{U\left(12\sigma + \frac{12\sigma_s}{R} - \frac{24\beta\epsilon\zeta}{R^2}\right) - \frac{12\epsilon\zeta\sigma}{\eta}E_o}{2\sigma + \frac{2\sigma_s}{R} - \frac{6\beta\epsilon\zeta}{R^2}}\right\} \quad (5)$$

Again, note that, because $\delta_D \ll R$, all terms involving β are negligible. Thus, Eq. 5 reduces to

$$f_z = -6\pi\eta RU + \frac{6\epsilon\zeta\sigma E_o}{\eta\left(\sigma + \frac{\sigma_s}{R}\right)} \quad (6)$$

which makes it clear that Stoke's drag prevails in the absence of an applied electric field.

Prob. 10.11.1 From Eq. 10.11.6, the total charge of a clean surface is

$$q_d = A\sigma_d \tag{1}$$

For the Helmholtz layer,

$$\sigma_d = \frac{\epsilon v_d}{\Delta} \tag{2}$$

Thus, Eq. 10.11.9 gives the coenergy function as

$$W_s' = -\int_{A_o}^{A} \gamma_o \delta A + \epsilon A \int_{\Phi_d}^{v_d} \frac{v_d}{\Delta}\delta v_d = -\gamma_o(A - A_o) + \frac{\epsilon A}{2\Delta}(v_d^2 - \Phi_d^2) \tag{3}$$

In turn, Eq. 10.11.10 gives the surface tension function as

$$\gamma_e = \gamma_o - \int_{\Phi_d}^{v_d} \frac{\epsilon v_d}{\Delta}\delta v_d = \gamma_o - \frac{\epsilon}{2\Delta}(v_d^2 - \Phi_d^2) \tag{4}$$

and Eq. 10.11.11 provides the incremental capacitance.

$$C_d = \frac{\partial \sigma_d}{\partial v_d} = \frac{\epsilon}{\Delta} \tag{5}$$

The curve shown in Fig. 10.11.2b is essentially of the form of Eq. 4. The surface charge density shows some departure from being the predicted linear function of v_d, while the incremental capacitance is quite different from the constant predicted by the Helmholtz model.

Prob. 10.11.2 (a) From the diagram, vertical force equilibrium for the control volume requires that

$$\pi R^2(p^\alpha - p^\beta) + 2\pi R(\gamma_o - \tfrac{1}{2}\epsilon E_\nu^2 \Delta) = 0 \tag{1}$$

so that

$$p^\alpha - p^\beta = \frac{-2}{R}(\gamma_o - \tfrac{1}{2}\epsilon E_\nu^2 \Delta) \tag{2}$$

and because $E_\nu = v_d/\Delta$,

$$p^\alpha - p^\beta = -\frac{2}{R}\left(\gamma_o - \frac{1}{2}\epsilon \frac{v_d^2}{\Delta}\right) \tag{3}$$

Compare this to the prediction from Eq. 10.11.1 (with a clean interface so that $\nabla_\Sigma \to 0$ and with $R_1 = R_2 = R$)

$$p^\alpha - p^\beta = T_r = -\frac{2\gamma_e}{R} \tag{4}$$

With the use of Eq. 4 from Prob. 10.11.1 with $\Phi_d = 0$, this becomes

Prob. 10.11.2 (cont.)

$$p^{\alpha} - p^{\beta} = -\frac{2}{R^2}\left(\gamma_o - \frac{\epsilon}{2\Delta}v_d\right) \tag{5}$$

in agreement with Eq. 3. Note that the shift from the origin in the potential for maximum γ_e is not represented by the simple picture of the double layer as a capacitor.

(b) From Eq. 5 with $R \to R + \delta\xi$

$$P_o^{\alpha} - P_o^{\beta} + \delta p = -\frac{2}{R + \delta\xi}\left(\gamma_o - \frac{\epsilon}{2\Delta}v_d^2\right) \simeq -\frac{2}{R}\left(\gamma_o - \frac{\epsilon}{2\Delta}v_d^2\right) + \frac{2}{R^2}\left(\gamma_o - \frac{\epsilon v_d^2}{2\Delta}\right)\delta\xi \tag{6}$$

and it follows from the perturbation part of this expression that

$$\delta p = \frac{2}{R^2}\left(\gamma_o - \frac{\epsilon}{2\Delta}v_d^2\right)\delta\xi \tag{7}$$

If the volume "within" the double-layer is preserved, then the thickness of the layer must vary as the radius of the interface is changed in accordance with

$$(\Delta + \delta\Delta)4\pi(R + \delta\xi)^2 = \Delta 4\pi R^2 \Rightarrow \delta\Delta = -\frac{2\Delta\delta\xi}{R} \tag{8}$$

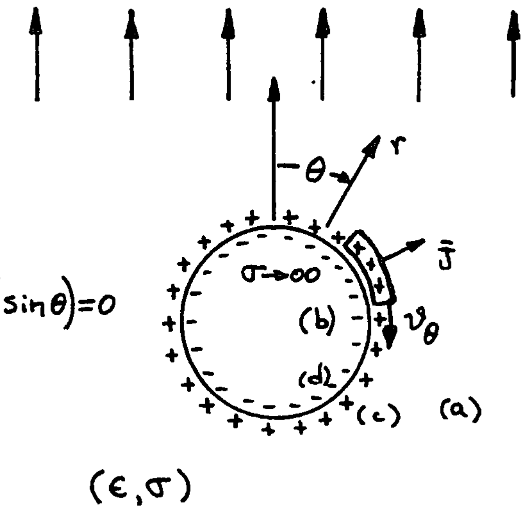
It follows from the evaluation of Eq. 3 with the voltage across the layer held fixed, that

$$\begin{aligned} p^{\alpha} - p^{\beta} + \delta p &= -\frac{2}{R + \delta\xi}\left(\gamma_o - \frac{1}{2}\frac{\epsilon v_d^2}{\Delta + \delta\Delta}\right) \\ &\simeq -\frac{2}{R}\left(\gamma_o - \frac{1}{2}\frac{\epsilon v_d^2}{\Delta}\right) + 2\left(\gamma_o - \frac{1}{2}\frac{\epsilon v_d^2}{\Delta}\right)\frac{\delta\xi}{R^2} - \frac{2}{R}\frac{1}{2}\frac{\epsilon v_d^2}{\Delta^2}\delta\Delta \end{aligned} \tag{9}$$

In view of Eq. 8,

$$\delta p = \frac{2}{R^2}\left(\gamma_o - \frac{1}{2}\frac{\epsilon v_d^2}{\Delta}\right) + \frac{2}{R^2}\frac{\epsilon v_d^2}{\Delta}\delta\xi = \frac{2}{R^2}\left(\gamma_o + \frac{1}{2}\frac{\epsilon v_d^2}{\Delta}\right)\delta\xi \tag{10}$$

What has been shown is that if the volume were actually preserved, then the effect of the potential would be just the opposite of that portrayed by Eq. 7. Thus, Eq. 10 does not represent the observed electrocapillary effect. By contrast with the "volume-conserving" interface, a "clean" interface is one made by simply exposing to each other the materials on each side of the interface.



Prob. 10.12.1 Conservation of charge for the double layer is represented using the volume element shown in the figure.

$$\sigma E_r^c + \nabla_\Sigma \cdot \sigma_d \bar{v} = 0 \Rightarrow -\sigma\left(\frac{\partial \Phi}{\partial r}\right)^c + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left(\sigma_d v_\theta^c \sin\theta\right) = 0$$

It is assumed that the drop remains spherical and is biased away from the maximum in the electrocapillary curve at $\sigma_d = \sigma_o$. Thus, with the electric potential around the drop represented by

$$\Phi = -E_o r\cos\theta + \frac{A}{r^2}\cos\theta \tag{2}$$

Eq. 1 becomes

$$-\sigma\left(-E_o\cos\theta - \frac{2A}{R^3}\cos\theta\right) + \sigma_o \frac{2\sin\theta\cos\theta}{R\sin\theta}\tilde{v}_\theta^c = 0$$

and it follows that the θ dependence cancels out so that

$$\frac{2\sigma_o}{R}\tilde{v}_\theta^c + \frac{2\sigma}{R^3}A = -\sigma E_o \tag{3}$$

Normal stress equilibrium requires that

$$S_{rr}^a - S_{rr}^b - \frac{2\gamma_e}{R} = 0 \tag{4}$$

With the equilibrium part of this expression subtracted out, it follows that

$$\tilde{S}_{rr}^a 2\cos\theta - \tilde{S}_{rr}^b 2\cos\theta + \frac{2\sigma_o}{R}\tilde{\Phi}^c = 0 \tag{5}$$

In view of the stress-velocity relations for creep flow, Eqs. 7.21.23 and 7.21.24, this boundary condition becomes

$$-\frac{(6\eta_b + 3\eta_a)}{R}\tilde{v}_\theta^c + \frac{2\sigma_o}{R^3}A + \frac{3}{2R}\eta_a U = 2\sigma_o E_o \tag{6}$$

where additional boundary conditions that have been used are $v_\theta^d = v_\theta^c$ and $v_r^d = v_r^c = 0$. The shear stress balance requires that

$$\tilde{S}_{\theta r}^c \sin\theta - \tilde{S}_{\theta r}^d \sin\theta + \sigma_o E_\theta^c = 0 \tag{7}$$

In view of Eq. 2 and these same stress-velocity relations, it follows that

$$\frac{3}{R}(\eta_a + \eta_b)\tilde{v}_\theta^c - \frac{\sigma_o}{R^3}A + \frac{3\eta_a}{2R}U = -\sigma_o E_o \tag{8}$$

Prob. 10.12.1 (cont.)

Simultaneous solution of Eqs. 3, 6 and 8 for U gives the required relationship between the velocity at infinity, U, and the applied electric field, E_o.

$$U = \frac{\sigma_o R E_o}{2\eta_a + 3\eta_b + \frac{\sigma_o^2}{\sigma}} \tag{9}$$

To make the velocity at infinity equal to zero, the drop must move in the z-direction with this velocity. Thus, the drop moves in a direction that would be consistent with thinking of the drop as having a net charge having the same sign as the charge on the "drop-side" of the double layer.

Prob. 10.12.2 In the sections that have both walls solid, Eq. (a) of Table 9.3.1 applies with $v^\alpha=0$ and $v^\beta=0$.

$$v(x) = \frac{a^2}{2\eta_a}\left(\frac{\partial p}{\partial y}\right)_a^I \left[\left(\frac{x}{a}\right)^2 - \frac{x}{a}\right] \tag{1}$$

Integration relates the pressure gradient in the electrolyte (region a) and in this mercury free section (region I) to the volume rate of flow.

$$Q_a^I = w\int_0^a v\,dx = -\frac{a^3 w}{12\eta_a}\left(\frac{\partial p}{\partial y}\right)_a^I \tag{2}$$

Similarly, in the upper and lower sections where there is mercury and electrolyte, these same relations apply with the understanding that for the upper region, x=0 is the mercury interface, while for the mercury, x=b is the interface.

$$v_a(x) = U\left(1-\frac{x}{a}\right) + \frac{a^2}{2\eta_a}\left(\frac{\partial p}{\partial y}\right)_a^{II}\left[\left(\frac{x}{a}\right)^2 - \frac{x}{a}\right] \tag{3}$$

$$v_b(x) = U\frac{x}{b} + \frac{b^2}{2\eta_b}\left(\frac{\partial p}{\partial y}\right)_b^{II}\left[\left(\frac{x}{b}\right)^2 - \frac{x}{b}\right] \tag{4}$$

The volume rates of flow in the upper and lower parts of Section II are then

$$Q_a^{II} = \frac{Uaw}{2} - \frac{a^3 w}{12\eta_a}\left(\frac{\partial p}{\partial y}\right)_a^{II} \tag{5}$$

$$Q_b^{II} = \frac{Ubw}{2} - \frac{b^3 w}{12\eta_b}\left(\frac{\partial p}{\partial y}\right)_b^{II} \tag{6}$$

Because gravity tends to hold the interface level, these pressure gradients

Prob. 10.12.2 (cont.)

need not match. However, the volume rate of flow in the mercury must be zero.

$$Q_b^{II} = 0 \Rightarrow \left(\frac{\partial p}{\partial y}\right)_b^{II} = \frac{6\eta_b U}{b^2} \tag{7}$$

and the volume rates of flow in the electrolyte must be the same

$$Q_a^{I} = Q_a^{II} \Rightarrow \left(\frac{\partial p}{\partial y}\right)_a^{II} = \frac{3\eta_a U}{a^2} \tag{8}$$

Hence, it has been determined that given the interfacial velocity U, the velocity profile in Section II is

$$v_a(x) = U\left\{\left(1 - \frac{x}{a}\right) + \frac{3}{2}\left[\left(\frac{x}{a}\right)^2 - \frac{x}{a}\right]\right\} \tag{9}$$

$$v_b(x) = U\left\{\frac{x}{b} + 3\left[\left(\frac{x}{b}\right)^2 - \frac{x}{b}\right]\right\} \tag{10}$$

Stress equilibrium at the mercury-electrolyte interface determines U. First, observe that the tangential electric field at this interface is approximately

$$E_y = \frac{I}{2\sigma a w} \tag{11}$$

Thus, stress equilibrium requires that

$$\frac{\sigma_o I}{2\sigma a w} + \eta_a \frac{\partial v_a}{\partial x}\bigg|_{x=0} - \eta_b \frac{\partial v_b}{\partial x}\bigg|_{x=b} = 0 \tag{12}$$

where the first term is the double layer surface force density acting in shear on the flat interface. Evaluated using Eqs. 9 and 10, Eq. 12 shows that the interfacial velocity is

$$U = \frac{\sigma_o I}{2\sigma w\left(\frac{5}{2}\eta_a + 4\frac{a}{b}\eta_b\right)} \tag{13}$$

Finally, the volume rate of flow follows from Eqs. 5 and 8 as

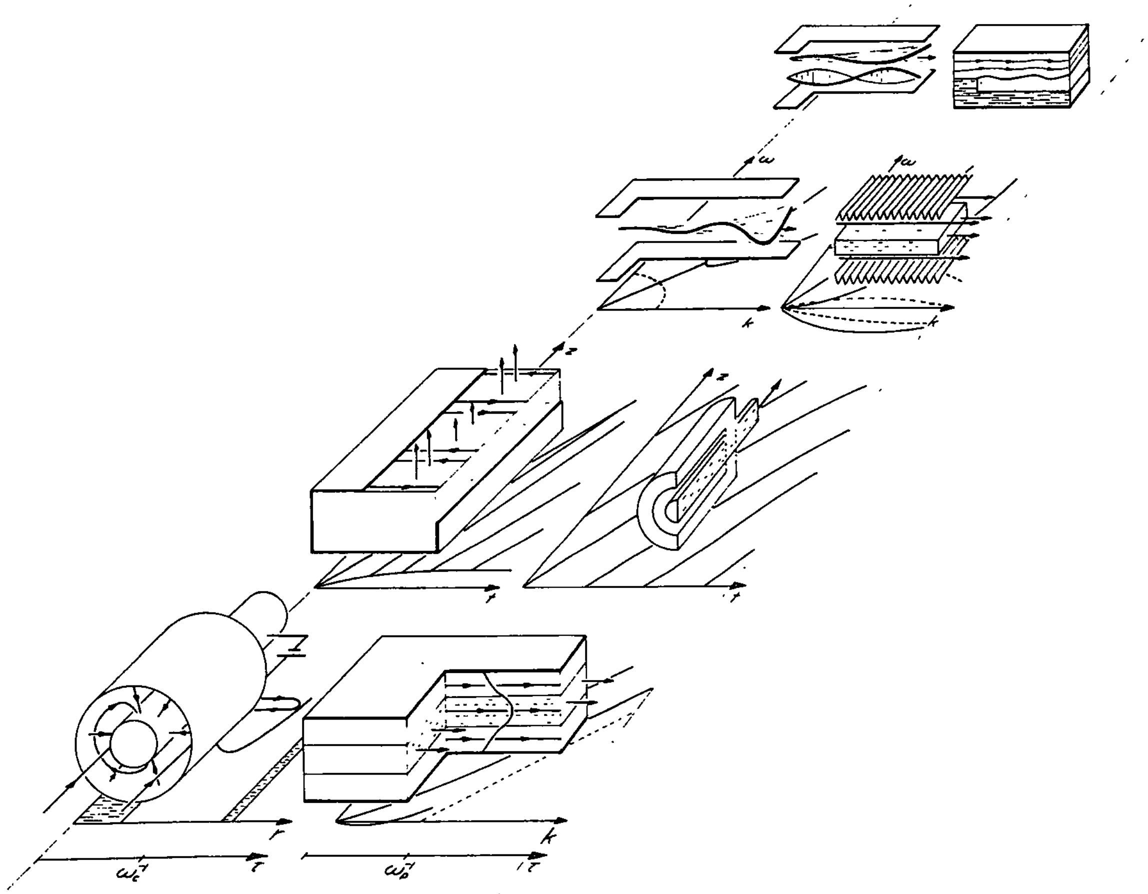
$$Q_a = \frac{U a w}{4} \tag{14}$$

Thus, Eqs. 13 and 14 combine to give the required dependence of the electrolyte volume rate of flow as a function of the driving current I.

$$Q_a = \frac{a\left(\frac{\sigma_o}{\sigma}\right) I}{4\left(5\eta_a + 8\eta_b\frac{a}{b}\right)} \tag{15}$$

11

Streaming Interactions



Prob. 11.2.1 With the understanding that the time derivative on the left is the rate of change of $\bar{v}$ for a given particle (for an observer moving with the particle velocity $\bar{v}$) the equation of motion is

$$m\frac{\partial \bar{v}}{\partial t} = q\left(\bar{E} + \bar{v}\times\mu_o\bar{H}\right) \tag{1}$$

Substitution of $\bar{E} = -\nabla\Phi$ and dot multiplication of this expression with $\bar{v}$ gives

$$\bar{v}\cdot\left[m\frac{\partial \bar{v}}{\partial t} = -q\nabla\Phi + q\bar{v}\times\mu_o\bar{H}\right] \tag{2}$$

Because $\bar{v}\times\mu_o\bar{H}$ is perpendicular to $\bar{v}$,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}m\bar{v}\cdot\bar{v}\right) = -q\bar{v}\cdot\nabla\Phi \tag{3}$$

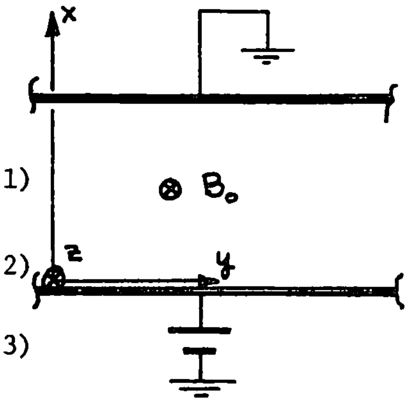
By definition, the rate of change of Φ with respect to time is

$$\frac{D\Phi}{Dt} = \frac{\partial\Phi}{\partial t} + \bar{v}\cdot\nabla\Phi = \bar{v}\cdot\nabla\Phi \tag{4}$$

where here it is understood that $\partial\Phi/\partial t$ means the partial is taken holding the Eulerian coordinates (x,y,z) fixed. Thus, this partial derivative is zero. It follows that because the del operator used in expressing Eq. 3 is also written in Eulerian coordinates, that the right-hand side of Eq. 4 can be taken as the rate of change of a spatially varying Φ with respect to time as observed by a particle. So, now with the understanding that the partial is taken holding the identity of a particle fixed (for example, using the initial coordinates of the particle as the independent spatial variables) Eq. 3 becomes the desired energy conservation statement.

$$\frac{\partial}{\partial t}\left[\frac{1}{2}m\bar{v}\cdot\bar{v} + q\Phi\right] = 0 \tag{5}$$

Prob. 11.3.1 (a) Using (x,y,z) to denote the cartesian coordinates of a given electron between the electrodes shown to the right, the particle equations of motion (Eq. 11.2.2) are simply



$$m\frac{d^2x}{dt^2} = -\frac{eV}{a} - B_o e\frac{dy}{dt} \qquad (1)$$

$$m\frac{d^2y}{dt^2} = B_o e\frac{dx}{dt} \qquad (2)$$

$$m\frac{d^2z}{dt^2} = 0 \qquad (3)$$

There is no initial velocity in the z direction, so it follows from Eq. 3 that the motion in the z direction can be taken as zero.

(b) To obtain the required expression for x(t), take the time derivative of Eq. (1) and replace the second derivative of y using Eq. (2). Thus,

$$m\frac{d^3x}{dt^3} = -\frac{(B_o e)^2}{m}\frac{dx}{dt} \Rightarrow \frac{d}{dt}\left(\frac{d^2x}{dt^2} + \omega_c^2 x\right) = 0\,;\ \omega_c^2 \equiv \left(\frac{B_o e}{m}\right)^2 \qquad (4)$$

When the electron is at x = 0,

$$\frac{dx}{dt} = 0\ ;\ \frac{dy}{dt} = 0 \Rightarrow (\text{Eq } 2 \text{ at } x=0)\ \ m\frac{d^2x}{dt^2} = -\frac{eV}{a} \qquad (5)$$

So that Eq. 4 becomes

$$\frac{d^2x}{dt^2} + \omega_c^2 x = \frac{-eV}{am} \qquad (6)$$

Note that for operation with electrons, $V < 0$.

(c) This expression is most easily solved by adding to the particular solution, $qV/am\omega_c^2$, the combination of $\sin \omega_c x$ and $\cos \omega_c x$ (the homogeneous solutions) required to satisfy the initial conditions. However, to proceed in a manner analogous to that required in the text, Eq. 6 is multiplied by dx/dt and the resulting expression written in the form

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \omega_c^2\frac{x^2}{2} + \frac{eV}{am}x\right] = 0 \qquad (7)$$

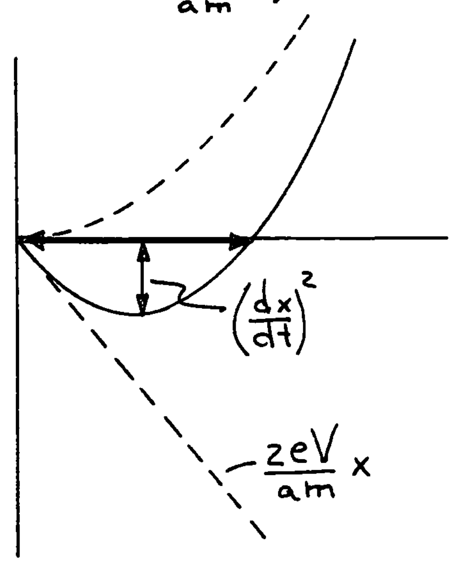
so that it is evident that the quantity in brackets is conserved. To satisfy the condition of Eq. 5, the constant of integration is zero

Prob. 11.3.1 (cont.)

(the initial total energy is zero) so it follows from Eq. 7 that

$$\frac{dx}{dt} = \pm\sqrt{\frac{-2eV}{am}x - \omega_c^2 x^2} \qquad \pm\sqrt{0 - \left(\omega_c^2 x^2 + \frac{2eV}{am}x\right)} \tag{8}$$

where $eV < 0$.



The potential well picture given by this expression is shown at the right. Rearrangement of Eq. 8 puts it in a form that can be integrated. First, it is written as

$$\pm\int_0^x \frac{dx}{\sqrt{-\frac{2eV}{am}x - \omega_c^2 x^2}} = \int_0^t dt \tag{9}$$

Then, integration gives

$$\cos^{-1}\left[\frac{\frac{-eV}{am\omega_c^2} - x}{\frac{-eV}{am\omega_c^2}}\right] = \omega_c t \Rightarrow x = \frac{eV}{am\omega_c^2}(\cos\omega_c t - 1) \tag{10}$$

Of course, this is just the combination of particular and homogeneous solutions to Eq. 6 required to satisfy the initial condition.

The associated motion in the y direction follows by using Eq. 10 to evaluate the right-hand side of Eq. 2. Then, integration gives the velocity

$$\frac{dy}{dt} = \frac{B_o e^2 V}{am^2\omega_c^2}(\cos\omega_c t - 1) \tag{11}$$

where the integration constant is evaluated to satisfy Eq. 5. A second integration, this time with the constant of integration evaluated to make y=0 when t=0, gives (note that $\omega_c = -B_o e/m$).

$$y = \frac{Ve}{\omega_c^2 am}(\sin\omega_c t - \omega_c t) \tag{12}$$

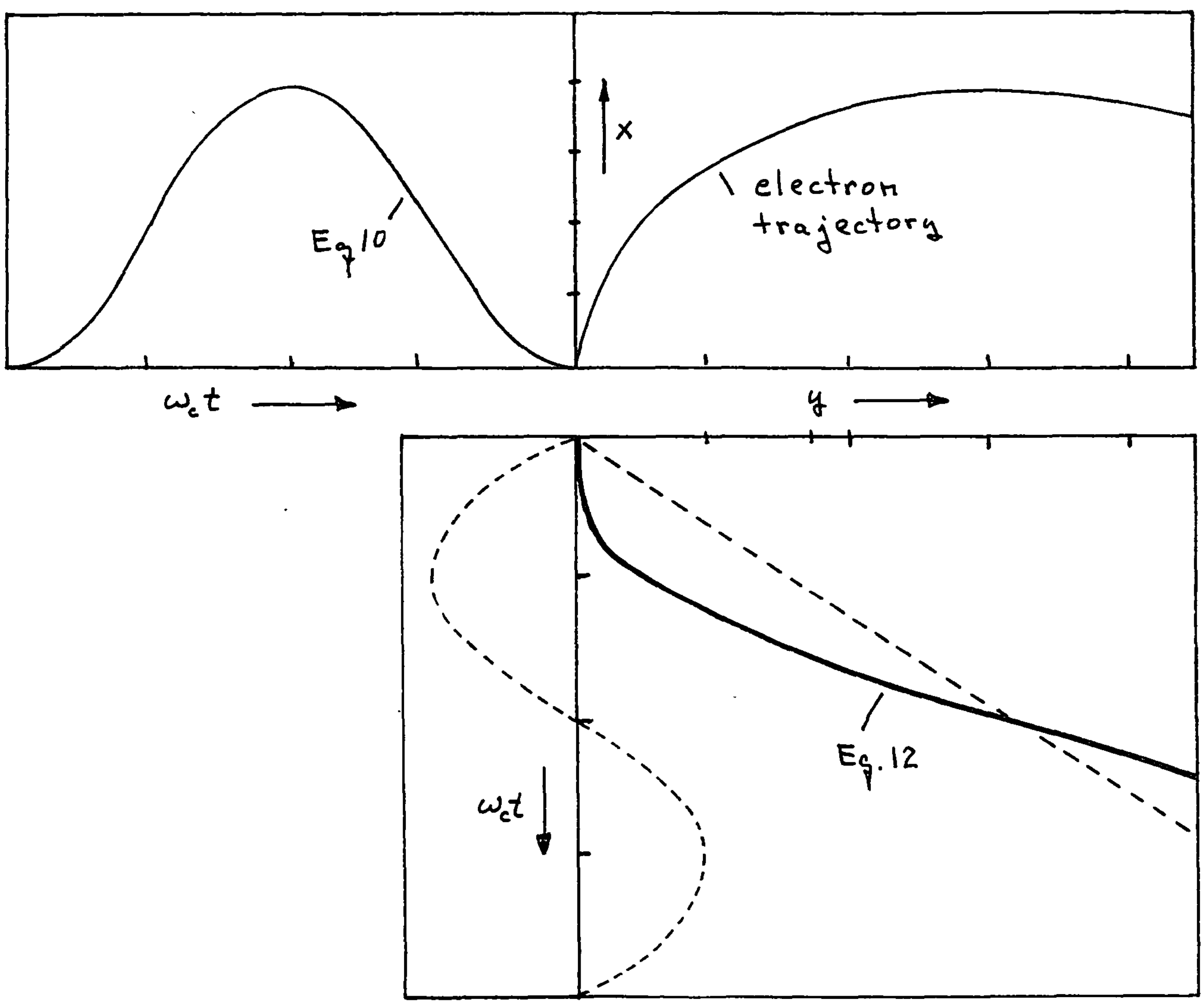
Thus, with t as a parameter, Eqs. 10 and 12 give the trajectory of a particle starting out from the origin when t=0. Electrons coming from the cathode at other times or other locations along the y axis have similar trajectories.

Prob. 11.3.1 (cont.)

(d) The construction shown in the figure is useful in picturing particle motions that are the planar analogous of those found in cylindrical geometry in the text.

(e) The trajectory just grazes the anode if the peak amplitude given by Eq. 10 is just equal to the spacing, a. The potential resulting from this equality is then the critical one.

$$V_c = -a^2 m \omega_c^2 / 2e \tag{13}$$



P11.3 (cont.)

c) To find Φ_c,

$$\frac{\Delta}{2} = \int_{-\zeta}^{\Phi_c} \frac{d\Phi}{\sqrt{2\cosh\Phi - 2\cosh\Phi_c}} \qquad (8)$$

must be numerically evaluated. The procedure would be given values for Δ and ζ and would start with a "best guess" value of Φ_c (perhaps $\varnothing$). It would then determine equal interval spacings Φ so that a specified number of points would be used to do the numerical evaluation. A simple numerical integration, such as trapezoidal areas, would then be used to evaluate the integrand of (8). The resulting integration would then be compared to $\Delta/2$ for a re-evaluation of Φ_c. The process would iterate until an appropriate answer of the evaluated integral falls within specified error tolerances of $\Delta/2$.

A potentially sticky situation appears as $\Phi \to \Phi_c$. The integrand is singular at that value of Φ. One way around this is to use small enough interval spacing of Φ so that the $\Phi = \Phi_c$ value can be neglected. Another way is to expand the denominator of the integrand into a Taylor expansion around Φ_c,

$$2\cosh\Phi - 2\cosh\Phi_c \cong 2\left(\cancel{\cosh\Phi_c} + \sinh\Phi_c\cdot(\Phi-\Phi_c) - \cancel{\cosh\Phi_c}\right)\Big|_{\Phi\to\Phi_c}$$

$$\cong 2\sinh\Phi_c\cdot(\Phi-\Phi_c)\Big|_{\Phi\to\Phi_c} \qquad (9)$$

As the numerical Φ's approach Φ_c, (9) would be plugged into the integral of (8). The integration would still need to stop before $\Phi = \Phi_c$ is reached.

Once Φ_c is determined, $\Phi(x)$ is easily evaluated by numerical integration.

d) given $\Delta = 2$ and $\zeta = 3$, I did the integration using Lotus 123. The worksheet is shown on pgs 5 and 6 while the graph is on page 7. To understand the worksheet,

Col. A = % of way through numerical integration x 100
Col. B = Potential (where end of Col. B is Φ_c)
Col. C = Cosh of potential (I had to make a Cosh func. from exponetials)
Col. D = Value of integrand with given Potential in B
Col. E = Trapezoidal area integration, e.g. $E_2 = (D_1 + D_2)*(B_2 - B_1)/2$
Col. F = Sum of Col. E, i.e. $F1 = \Delta/2$
Col. G = Φ_c
Col. H = X as a function of Φ.

The result: $\Phi_c = -1.38$.

The Plot is on pg. 7.

123 Worksheet used to do calculations for 6.672 P10.8.2

	Potential		Integrand		midplane	Phi of mid.	
	ζ				Δ/2	Φ_c	
1	-3	10.06766	0.250715	0	1.009043	-1.38	0
2	-2.9676	9.748310	0.255904	0.008207			0.008207
3	-2.9514	9.592493	0.258556	0.004167			0.012374
4	-2.9352	9.439193	0.261247	0.004210			0.016584
5	-2.919	9.288371	0.263979	0.004254			0.020839
6	-2.9028	9.139986	0.266752	0.004298			0.025138
7	-2.8866	8.994000	0.269567	0.004344			0.029482
8	-2.8704	8.850375	0.272425	0.004390			0.033872
9	-2.8542	8.709072	0.275327	0.004436			0.038309
10	-2.838	8.570055	0.278276	0.004484			0.042793
11	-2.8218	8.433287	0.281270	0.004532			0.047325
12	-2.8056	8.298732	0.284313	0.004581			0.051906
13	-2.7894	8.166356	0.287405	0.004630			0.056537
14	-2.7732	8.036122	0.290548	0.004681			0.061219
15	-2.757	7.907998	0.293742	0.004732			0.065952
16	-2.7408	7.781949	0.296990	0.004784			0.070737
17	-2.7246	7.657942	0.300293	0.004838			0.075575
18	-2.7084	7.535945	0.303652	0.004891			0.080466
19	-2.6922	7.415926	0.307069	0.004946			0.085413
20	-2.676	7.297853	0.310546	0.005002			0.090416
21	-2.6598	7.181696	0.314084	0.005059			0.095476
22	-2.6436	7.067423	0.317686	0.005117			0.100593
23	-2.6274	6.955005	0.321353	0.005176			0.105769
24	-2.6112	6.844412	0.325087	0.005236			0.111005
25	-2.595	6.735616	0.328891	0.005297			0.116303
26	-2.5788	6.628588	0.332766	0.005359			0.121662
27	-2.5626	6.523299	0.336715	0.005422			0.127085
28	-2.5464	6.419722	0.340740	0.005487			0.132572
29	-2.5302	6.317830	0.344844	0.005553			0.138125
30	-2.514	6.217596	0.349029	0.005620			0.143746
31	-2.4978	6.118993	0.353299	0.005688			0.149435
32	-2.4816	6.021997	0.357656	0.005758			0.155193
33	-2.4654	5.926581	0.362103	0.005830			0.161023
34	-2.4492	5.832721	0.366643	0.005902			0.166926
35	-2.433	5.740391	0.371280	0.005977			0.172903
36	-2.4168	5.649568	0.376018	0.006053			0.178957
37	-2.4006	5.560227	0.380859	0.006130			0.185087
38	-2.3844	5.472346	0.385809	0.006210			0.191297
39	-2.3682	5.385901	0.390871	0.006291			0.197588
40	-2.352	5.300870	0.396050	0.006374			0.203962
41	-2.3358	5.217229	0.401351	0.006458			0.210421
42	-2.3196	5.134958	0.406778	0.006545			0.216967
43	-2.3034	5.054035	0.412337	0.006634			0.223602
44	-2.2872	4.974438	0.418033	0.006726			0.230328
45	-2.271	4.896146	0.423872	0.006819			0.237148
46	-2.2548	4.819140	0.429862	0.006915			0.244063
47	-2.2386	4.743398	0.436007	0.007013			0.251076
48	-2.2224	4.668901	0.442316	0.007114			0.258191
49	-2.2062	4.595630	0.448797	0.007218			0.265409
50	-2.19	4.523564	0.455456	0.007324			0.272733
51	-2.1738	4.452686	0.462304	0.007433			0.280167

52	-2.1576	4.382977	0.469350	0.007546	0.287714
53	-2.1414	4.314417	0.476603	0.007662	0.295376
54	-2.1252	4.246990	0.484075	0.007781	0.303157
55	-2.109	4.180678	0.491777	0.007904	0.311062
56	-2.0928	4.115462	0.499722	0.008031	0.319093
57	-2.0766	4.051327	0.507923	0.008161	0.327255
58	-2.0604	3.988255	0.516395	0.008296	0.335552
59	-2.0442	3.926230	0.525154	0.008436	0.343988
60	-2.028	3.865235	0.534217	0.008580	0.352569
61	-2.0118	3.805255	0.543604	0.008730	0.361300
62	-1.9956	3.746273	0.553334	0.008885	0.370185
63	-1.9794	3.688275	0.563429	0.009045	0.379231
64	-1.9632	3.631244	0.573916	0.009212	0.388443
65	-1.947	3.575167	0.584819	0.009385	0.397829
66	-1.9308	3.520028	0.596170	0.009566	0.407395
67	-1.9146	3.465812	0.608001	0.009753	0.417149
68	-1.8984	3.412506	0.620348	0.009949	0.427098
69	-1.8822	3.360096	0.633252	0.010154	0.437252
70	-1.866	3.308568	0.646757	0.010368	0.447621
71	-1.8498	3.257908	0.660914	0.010592	0.458213
72	-1.8336	3.208103	0.675779	0.010827	0.469040
73	-1.8174	3.159140	0.691416	0.011074	0.480114
74	-1.8012	3.111006	0.707897	0.011334	0.491449
75	-1.785	3.063688	0.725305	0.011608	0.503058
76	-1.7688	3.017175	0.743731	0.011899	0.514957
77	-1.7526	2.971453	0.763286	0.012206	0.527164
78	-1.7364	2.926511	0.784092	0.012533	0.539697
79	-1.7202	2.882337	0.806295	0.012882	0.552580
80	-1.704	2.838920	0.830065	0.013254	0.565834
81	-1.6878	2.796248	0.855602	0.013653	0.579488
82	-1.6716	2.754310	0.883145	0.014083	0.593572
83	-1.6554	2.713094	0.912981	0.014548	0.608120
84	-1.6392	2.672591	0.945459	0.015053	0.623174
85	-1.623	2.632789	0.981006	0.015604	0.638778
86	-1.6068	2.593678	1.020155	0.016209	0.654988
87	-1.5906	2.555247	1.063579	0.016878	0.671866
88	-1.5744	2.517487	1.112144	0.017623	0.689489
89	-1.5582	2.480388	1.166981	0.018460	0.707950
90	-1.542	2.443940	1.229610	0.019412	0.727363
91	-1.5258	2.408134	1.302123	0.020507	0.747870
92	-1.5096	2.372959	1.387499	0.021785	0.769656
93	-1.4934	2.338407	1.490157	0.023309	0.792965
94	-1.4772	2.304469	1.616992	0.025167	0.818132
95	-1.461	2.271135	1.779507	0.027511	0.845644
96	-1.4448	2.238398	1.998733	0.030603	0.876248
97	-1.4286	2.206248	2.318587	0.034970	0.911218
98	-1.4124	2.174677	2.852775	0.041888	0.953106
99	-1.3962	2.143677	4.053033	0.055937	1.009043
100	-1.38	2.113240	ERR	0.034481	1.043524
	2.113240				

Divide by Zero error

Could be fixed as explained in part (c) of problem.

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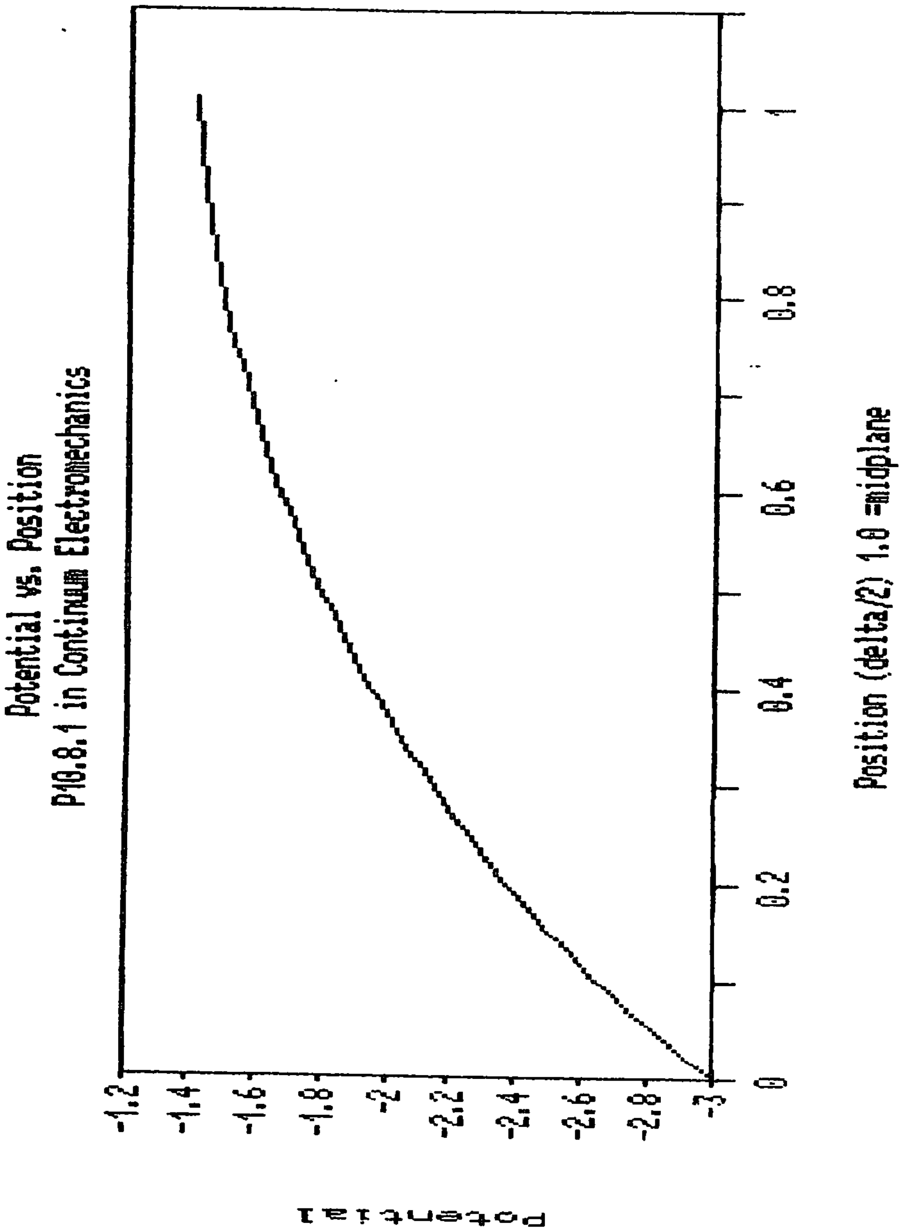


Courtesy of James Brennan. Used with permission.

Prob. 11.4.1 The point in this problem is to appreciate the quasi-one-dimensional model represented by the paraxial ray equation. First, observe that it is not simply a one-dimensional version of the general equations of motion. The exact equations are satisfied identically in a region where E_r, E_z and H_r are zero by the solution r = constant, θ = constant and a uniform motion in the z direction, z=Ut. That the magnetic field, B_z, has a z variation (and hence that there are radial components of $\bar{B}$) is implied by the use of Busch's Theorem (Eq. 11.4.2). The angular velocity implicit in writing the radial force equation reflects the arrival of the electron at the point in question from a region where there is no magnetic flux density. It is the centrifugal force caused by the angular velocity created in the transition from the field free region to the one where B_z is uniform that appears in Eq. 11.4.9, for example.

Prob. 11.4.2 The theorem is a consequence of the property of solutions to Eq. 11.4.9.

$$-\frac{d^2r}{dz^2} = \chi^2 r \tag{1}$$

In this expression, $\chi = \chi(z)$, reflecting the possibility that the B_z varies in an arbitrary way in the z-direction. Integration of Eq. 1 gives

$$-\int_0^z \frac{d}{dz}\left(\frac{dr}{dz}\right)dz = \int_0^z \chi^2 r\,dz \Rightarrow \left.\frac{dr}{dz}\right|_0 - \left.\frac{dr}{dz}\right|_z = \int_0^z \chi^2 r\,dz > 0 \tag{2}$$

Because the quantity on the right is positive definite, it follows that the derivative at some downstream location is less than that at the entrance.

$$\left.\frac{dr}{dz}\right|_0 > \left.\frac{dr}{dz}\right|_z \tag{3}$$

Prob. 11.4.3 For the magnetic lens, Eq. 11.4.8 reduces to

$$\frac{d^2r}{dz^2} + \frac{e}{8\Phi m}B_z^2 r = 0 \tag{1}$$

Integration through the length of the lens gives

$$\int_{z_-}^{z_+} \frac{d}{dz}\left(\frac{dr}{dz}\right)dz + \int_{z_-}^{z_+} \frac{1}{8\Phi}\frac{e}{m}B_z^2 r\,dz = 0 \tag{2}$$

Prob. 11.4.3 (cont.)

and this expression becomes

$$\left.\frac{dr}{dz}\right|_{z_+} - \left.\frac{dr}{dz}\right|_{z_-} = -\int_{z_-}^{z_+} \frac{e}{8\Phi m} B_z^2 r\,dz = -\frac{er}{8\Phi m}\int_{z_-}^{z_+} B_z^2\,dz \tag{3}$$

On the right it has been assumed that the variation through the "weak" lens of the radial position is negligible. The definition of f that follows from Fig. 11.4.2 is

$$\frac{dr}{dz} = -\frac{r}{f} \tag{4}$$

so that for electrons entering the lens as parallel rays, it follows from Eq. 3 that

$$\frac{r}{f} = \frac{er}{8\Phi m}\int_{z_-}^{z_+} B_z^2\,dz \tag{5}$$

which can be solved for f to obtain the expression given. As a check, observe for the example given in the text where $B_z = B_o$ over the length of the lens,

$$\int_{z_-}^{z_+} B_z^2\,dz = B_o^2\,\ell \tag{6}$$

and it follows from Eq. 5 that

$$f = \frac{8\Phi m}{e\ell B_o^2} \tag{7}$$

This same expression is found from Eq. 11.4.12 in the limit $\ell\kappa \ll 1$.

Prob. 11.4.4 For the given potential distribution

$$\Phi = V_o J_o(\gamma r)\,e^{-\gamma z} \tag{1}$$

the coefficients in Eq. 11.4.8 are

$$A = -\frac{\gamma}{2}\;;\; C = \frac{\gamma^2}{4} \tag{2}$$

and the differential equation reduces to one having constant coefficients.

$$\frac{d^2r}{dz^2} - \frac{\gamma}{2}\frac{dr}{dz} + \frac{\gamma^2}{4} r = 0 \tag{3}$$

At $z = z_+$, just to the downstream side of the plane z=0, boundary conditions are

$$r = r_o\;;\; \frac{dr}{dz} = 0 \tag{4}$$

Prob. 11.4.4 (cont.)

Solutions to Eq. 3 are of the form

$$r = D e^{P_1 z} + F e^{P_2 z}; \quad P_{\substack{1\\2}} \equiv \frac{\gamma}{4}(1 \pm j\sqrt{3}) \tag{5}$$

and evaluation of the coefficients by using the conditions of Eq. 4 results in the desired electron trajectory.

$$r = r_o e^{\frac{\gamma z}{4}} \left(\cos\frac{\sqrt{3}\gamma}{4} z - \frac{1}{\sqrt{3}} \sin\frac{\sqrt{3}\gamma}{4} z \right) \tag{6}$$

Prob. 11.5.1 In Cartesian coordinates, the transverse force equations are

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial z}\right) v_x = \frac{e}{m}\frac{\partial \Phi}{\partial x} - \frac{e}{m} B_o v_y \tag{1}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial z}\right) v_y = \frac{e}{m}\frac{\partial \Phi}{\partial y} + \frac{e}{m} B_o v_x \tag{2}$$

With the same substitution as used in the zero order equations, these relations become

$$\begin{bmatrix} j(\omega - kU) & \frac{e}{m} B_o \\ -\frac{e}{m} B_o & j(\omega - kU) \end{bmatrix} \begin{bmatrix} \hat{v}_x \\ \hat{v}_y \end{bmatrix} = \begin{bmatrix} \frac{e}{m}\frac{d\hat{\Phi}}{dx} \\ -j\frac{e}{m} k_y \hat{\Phi} \end{bmatrix} \tag{3}$$

where the potential distributions on the right are predetermined from the zero order fields. For example, solution of Eqs. 3 gives

$$\hat{v}_x = \frac{j(\omega - kU)\frac{e}{m}\frac{d\hat{\Phi}}{dx} + j\frac{e}{m}\left(\frac{B_o e}{m}\right) k_y \hat{\Phi}}{\left(\frac{e}{m}B_o\right)^2 - (\omega - kU)^2} \tag{4}$$

If the Doppler shifted frequency is much less than the electron cyclotron frequency, $\omega_c = eB_o/m$,

$$\left(\frac{e}{m}B_o\right)^2 >> (\omega - kU)^2$$

Typically, $|d\hat{\Phi}/dx| \sim |k_z \hat{\Phi}|$ and $k_y \sim k_z$ so that Eqs. 4 and 11.5.5 show

Prob. 11.5.1 (cont.)

that

$$\frac{|\hat{v}_x|}{|\hat{v}_z|} = \frac{(\omega - kU)^2}{\omega_c^2} + \frac{(\omega - kU)}{\omega_c} \tag{6}$$

so, if $|\omega - kU| < \omega_c$, then the transverse motions are negligible compared to the longitudinal ones. Most likely $\omega - kU \sim \omega_p$ so the requirement is essentially that the plasma frequency be low compared to the electron cyclotron frequency.

Prob. 11.5.2 (a) Equations 11.5.5 and 11.5.6 remain valid in cylindrical geometry. However, Eq. 11.5.7 is replaced by the circular version of Eq. 11.5.4 combined with Eq. 11.5.6

$$\frac{d^2\hat{\Phi}}{dr^2} + \frac{1}{r}\frac{d\hat{\Phi}}{dr} - \left(\frac{m^2}{r^2} + \gamma^2\right)\hat{\Phi} = 0 \tag{1}$$

Thus, it has the form of Bessel's equation, Eq. 2.16.19, with $k \rightarrow \gamma$. The derivation of the transfer relations in Table 2.16.2 remains valid because the displacement vector is found from the potential by taking the radial derivative and that involves γ and not k. (If the derivation involved a derivative with respect to z, there would be two ways in which k entered in the original derivation, and γ could not be unambiguously identified with k everywhere.)

(b) Using (c), (d) and (e) to designate the radii r=a and r=+b and -b respectively, the solid circular beam is described by

$$\hat{D}_r^e = \epsilon_o f_m(0,b,\gamma)\hat{\Phi}^e \tag{2}$$

while the free space annulus has

$$\begin{bmatrix} \hat{D}_r^c \\ \hat{D}_r^d \end{bmatrix} = \epsilon_o \begin{bmatrix} f_m(b,a,k) & g_m(a,b,k) \\ g_m(b,a,k) & f_m(a,b,k) \end{bmatrix} \begin{bmatrix} \hat{\Phi}^c \\ \hat{\Phi}^d \end{bmatrix} \tag{3}$$

Thus, in view of the conditions that $\hat{D}_x^d = \hat{D}_x^e$ and $\hat{\Phi}^d = \hat{\Phi}^e$, Eqs. 2 and 3b show that

$$\hat{\Phi}^e = \frac{g_m(b,a,k)\hat{\Phi}^c}{f_m(0,b,\gamma) - f_m(a,b,k)} \tag{4}$$

Prob. 11.5.2 (cont.)

This expression is then substituted into Eq. 3a to show that

$$\hat{D}_r^c = \frac{\epsilon_o\left[f_m(0,b,\gamma)f_m(b,a,k) - f_m(b,a,k)f_m(a,b,k) + g_m(a,b,k)g_m(b,a,k)\right]}{f_m(0,b,\gamma) - f_m(a,b,k)}\hat{\Phi}^c \tag{5}$$

which is the desired driven response.

(c) The dispersion equation follows from Eq. 5, and takes the same form as Eq. 11.5.12

$$f_m(0,b,\gamma) = f_m(a,b,k) \tag{6}$$

For the temporal modes, what is on the right (a function of geometry and the wavenumber) is real. From the properties of the f_m determined in Sec. 2.17, $f_m(a,b,k) > 0$ for $a > b$ and $f_m(0,b,\gamma) < 0$, so it is clear that for γ real, Eq. 6 cannot be satisfied. However, for $\gamma = -j\alpha$ where α is defined as real, Eq. 6 becomes

$$-\alpha\frac{J_m'(\alpha b)}{J_m(\alpha b)} = f_m(a,b,k) \tag{7}$$

This expression can be solved graphically to find an infinite number of solutions, α_n. Given these values, the eigenfrequencies follow from the definition of γ given with Eq. 11.5.7.

$$\omega_n = kU \pm \frac{\omega_p}{\sqrt{1 + \left(\frac{\alpha_n}{k}\right)^2}} \tag{8}$$

Prob. 11.6.1 The system of m first order differential equations takes the form

$$\sum_{j=1}^{m}\left(F_{ij}\frac{\partial x_j}{\partial t} + G_{ij}\frac{\partial x_j}{\partial z}\right) = 0 \tag{1}$$

where i = 1 .... m generates the m equations.

(a) Following the method of "undetermined multipliers, multiply the ith equation by λ_i and add all m equations

$$\begin{aligned}
\lambda_1 \sum_{j=1}^{m}\left(F_{1j}\frac{\partial x_j}{\partial t} + G_{1j}\frac{\partial x_j}{\partial z}\right) &= 0\\
&\vdots\\
\lambda_i \sum_{j=1}^{m}\left(F_{ij}\frac{\partial x_j}{\partial t} + G_{ij}\frac{\partial x_j}{\partial z}\right) &= 0\\
&\vdots\\
\lambda_m \sum_{j=1}^{m}\left(F_{mj}\frac{\partial x_j}{\partial t} + G_{mj}\frac{\partial x_j}{\partial z}\right) &= 0
\end{aligned} \tag{2}$$

$$\sum_{j=1}^{m}\sum_{i=1}^{m}\left(\lambda_i F_{ij}\frac{\partial x_j}{\partial t} + \lambda_i G_{ij}\frac{\partial x_j}{\partial z}\right) = 0 \tag{3}$$

Now, for directional derivatives of each x_j to be the same

$$\frac{dz}{dt} = \frac{\sum_{i=1}^{m}\lambda_i G_{ij}}{\sum_{i=1}^{m}\lambda_i F_{ij}} \tag{4}$$

These expressions, j = 1 .... m can be written as m equations in the λ_i's.

Prob. 11.6.1 (cont.)

$$\sum_{i=1}^{m}\left(F_{ij}\frac{dz}{dt}-G_{ij}\right)\lambda_i=0$$

The first characteristic equations are given by the condition that the determinant of the coefficients of the λ_i's vanish.

$$\mathrm{Det}\left[\sum_{i=1}^{m}\left(F_{ij}\frac{dz}{dt}-G_{ij}\right)\right]=0$$

(b) Now, to form the coefficient matrix, write Eq. 1 as the first m of the 2 m expressions

$$\begin{bmatrix} F_{11} & G_{11} & F_{12} & G_{12} & \cdots & F_{1m} & G_{1m} \\ F_{21} & G_{21} & F_{22} & G_{21} & \cdots & F_{2m} & G_{2m} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ F_{m1} & G_{m1} & F_{m2} & G_{m2} & \cdots & F_{mn} & G_{mn} \\ dt & dz & 0 & 0 & & 0 & 0 \\ \vdots & \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & dt & dz \end{bmatrix} \begin{bmatrix} x_{1,t} \\ x_{1,z} \\ x_{2,t} \\ \vdots \\ \vdots \\ x_{m,t} \\ x_{m,z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ dx_1 \\ \vdots \\ dx_m \end{bmatrix}$$

The second m of these expressions are

$$dx_i=\frac{\partial x_i}{\partial t}dt+\frac{\partial x_i}{\partial z}dz\ ;\quad i=1\cdots m$$

To show that determinant of these coefficients is the same as Eq. 6, operate on Eq. 7 in ways motivated by the special case of obtaining Eq. 11.6.19 from Eq. 11.6.17. Multiply the (m+1)'st equation through 2m'th equation (the last m equations) by dt^{-1}. Then, these last m

Prob. 11.6.1 (cont.)

rows (m+1....2m) are first respectively multiplied by $F_{11}, F_{12} \ldots F_{1m}$ and subtracted from the first equation. The process is then repeated using of $F_{21}, F_{22} \ldots F_{2m}$ and the result subtracted from the second equation, and so on to the mth equation. Thus, Eq. 7 becomes

$$\begin{bmatrix} 0 & G_{11}-F_{11}\frac{dz}{dt} & 0 & G_{12}-F_{12}\frac{dz}{dt} & \cdots & 0 & G_{1m}-F_{1m}\frac{dz}{dt} \\ \vdots & & & & & & \\ 0 & G_{m1}-F_{m1}\frac{dz}{dt} & 0 & G_{m2}-F_{m2}\frac{dz}{dt} & \cdots & 0 & G_{mm}-F_{mm}\frac{dz}{dt} \\ 1 & \frac{dz}{dt} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & \frac{dz}{dt} \end{bmatrix} = 0 \tag{9}$$

Now, this expression is expanded by "minors" about the 1's that appear as the only entries in the odd columns to obtain

$$\begin{bmatrix} G_{11}-F_{11}\frac{dz}{dt} & G_{12}-F_{12}\frac{dz}{dt} & \cdots & G_{1m}-F_{1m}\frac{dz}{dt} \\ \vdots & & & \vdots \\ G_{m1}-F_{m1}\frac{dz}{dt} & G_{m2}-F_{m2}\frac{dz}{dt} & \cdots & G_{mm}-F_{mm}\frac{dz}{dt} \end{bmatrix} \tag{10}$$

Multiplied by (-1) this is the same as Eq. 6.

Prob. 11.7.1 Eqs. 9.13.11 and 9.13.12, with $V=0$ and $b=0$ are

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial z}+g\frac{\partial \xi}{\partial t}=0 \tag{1}$$

$$\frac{\partial \xi}{\partial t}+\frac{\partial}{\partial z}(v\xi)=0 \tag{2}$$

In a uniform channel, the compressible equations of motion are Eqs. 11.6.3 and 11.6.4

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial z}+\frac{a^2}{\rho}\frac{\partial \rho}{\partial z}=0 \tag{3}$$

$$\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial z}+\rho\frac{\partial v}{\partial z}=0 \tag{4}$$

These last expressions are identical to the first two if the identification is made $v\rightarrow v$, $\rho\rightarrow\xi$ and $a^2/\rho\rightarrow g$. Because $a=a(\rho)$ (Eq. 11.6.2) the analogy is not complete unless a^2/ρ is independent of ρ . This requires that (from Eq. 11.6.2)

$$\frac{a^2}{\rho}=\gamma\frac{P_o}{\rho_o}\left(\frac{\rho}{\rho_o}\right)^{\gamma-1}\Big/\rho \tag{5}$$

be independent of ρ , which it is if $\rho^{\gamma-1}/\rho=\rho^{\gamma-2}=1$, or if $\gamma=2$.

Prob. 11.7.2 Eqs. 9.13.4 and 9.13.9 with A and f defined by $f = -\frac{1}{2}(\epsilon-\epsilon_o)\frac{V^2}{\pi\xi^2}+\frac{\gamma}{\xi}$ and $A = \pi\xi^2/2$ are

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} + \frac{\partial}{\partial z}\left[\frac{1}{2}\frac{(\epsilon-\epsilon_o)}{\rho}\frac{V^2}{\pi^2\xi^2} - \frac{\gamma}{\rho\xi}\right] = 0 \tag{1}$$

$$\frac{\partial}{\partial t}\xi^2 + \frac{\partial}{\partial z}(\xi^2 v) = 0 \tag{2}$$

These form the first two of the following 4 equations.

$$\begin{bmatrix} 1 & v & 0 & \frac{(\epsilon-\epsilon_o)V^2}{\pi^2\rho}\frac{1}{\xi^3} - \frac{\gamma}{\rho\xi^2} \\ 0 & \xi^2 & 2\xi & 2v\xi \\ dt & dz & 0 & 0 \\ 0 & 0 & dt & dz \end{bmatrix}\begin{bmatrix} v_t \\ v_z \\ \xi_t \\ \xi_z \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ dv \\ d\xi \end{bmatrix} \tag{3}$$

The last two state that dv and $d\xi$ are computed along the characteristic lines.

The Ist characteristic equations follow from requiring that the determinant of the coefficients vanish.

To reduce this determinant divide the third and fourth columns by dt and $dt/2\xi$ respectively, and subtract from the first and second respectively. Then expand by minors to obtain the new determinant

$$\begin{bmatrix} v - \frac{dz}{dt} & \frac{(\epsilon-\epsilon_o)V^2}{\pi^2\rho\xi^3} - \frac{\gamma}{\rho\xi^2} \\ \xi^2 & 2\xi\left[v - \frac{dz}{dt}\right] \end{bmatrix} = 0 \tag{4}$$

Prob. 11.7.2 (cont.)

Thus, the Ist characteristic equations are

$$\left(\frac{dz}{dt}-v\right)^2=\frac{1}{2}\xi\left[\frac{(\epsilon-\epsilon_0)V^2}{\pi^2\rho\xi^3}-\frac{\gamma}{\rho\xi^2}\right] \tag{5}$$

or

$$\frac{dz}{dt}=v\pm a(\xi)\;;\;a(\xi)\equiv\left[\frac{(\epsilon-\epsilon_0)V^2}{2\pi^2\rho\xi^2}-\frac{\gamma}{2\rho\xi}\right]^{1/2}\text{ on } C^{\pm} \tag{6}$$

The IInd characteristics are found from the determinant obtained by substituting the column matrix on the right for the column on the left.

$$\begin{bmatrix} 0 & v & 0 & \frac{(\epsilon-\epsilon_0)V^2}{\pi^2\rho}\frac{1}{\xi^3}-\frac{\gamma}{\rho\xi^2} \\ 0 & \xi^2 & 2\xi & 2v\xi \\ dv & dz & 0 & 0 \\ d\xi & 0 & dt & dz \end{bmatrix}=0 \tag{7}$$

Solution, expanding in minors about dv and $d\xi$, gives

$$dv\left\{v\left(2\xi\frac{dz}{dt}-2v\xi\right)+\xi^2\left(\frac{2a^2}{\xi}\right)\right\}$$
$$+\,d\xi\left\{2\xi\frac{dz}{dt}\left(\frac{2a^2}{\xi}\right)\right\}=0 \tag{8}$$

With the understanding the $\pm$ signs mean that the relations pertain to $C^{\pm}$, Eq. 6 reduces this expression to the IInd characteristic equations.

$$\frac{2a}{\xi}d\xi\pm dv=0\quad\text{on}\quad C^{\pm} \tag{9}$$

Prob. 11.7.3 (a) The equations of motion are 9.13.11 and 9.13.12 with $V = 0$ and $b = 0$.

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} + g\frac{\partial \xi}{\partial z} = 0 \tag{1}$$

$$\frac{\partial \xi}{\partial t} + v\frac{\partial \xi}{\partial z} + \xi\frac{\partial v}{\partial z} = 0 \tag{2}$$

These are the first two of the following relations

$$\begin{bmatrix} 1 & v & 0 & g \\ 0 & \xi & 1 & v \\ dt & dz & 0 & 0 \\ 0 & 0 & dt & dz \end{bmatrix} \begin{bmatrix} v_{,t} \\ v_{,z} \\ \xi_{,t} \\ \xi_{,z} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ dv \\ d\xi \end{bmatrix} \tag{3}$$

The last two define dv and $d\xi$ as the differentials computed in the characteristic directions.

The determinant of the coefficients gives the Ist characteristics. Using the same reduction as in going from Eq. 11.6.18 to 11.6.19 gives

$$\begin{bmatrix} v - \frac{dz}{dt} & g \\ \xi & v - \frac{dz}{dt} \end{bmatrix} = \left(v - \frac{dz}{dt}\right)^2 - g\xi = 0 \tag{4}$$

or

$$\frac{dz}{dt} = v \pm \sqrt{g\xi} = v \pm \tfrac{1}{2}R(\xi)\,;\; R(\xi) \equiv 2\sqrt{g\xi} \tag{5}$$

Prob. 11.7.3 (cont.)

The second characteristics are this same determinant with the column matrix on the right substituted for the first column on the left.

$$\begin{bmatrix} 0 & v & 0 & g \\ 0 & \xi & 1 & v \\ dv & dz & 0 & 0 \\ d\xi & 0 & dt & dz \end{bmatrix} = dv\left[v(dz - v\,dt) + \xi(g\,dt)\right] + d\xi(g\,dz) \tag{6}$$

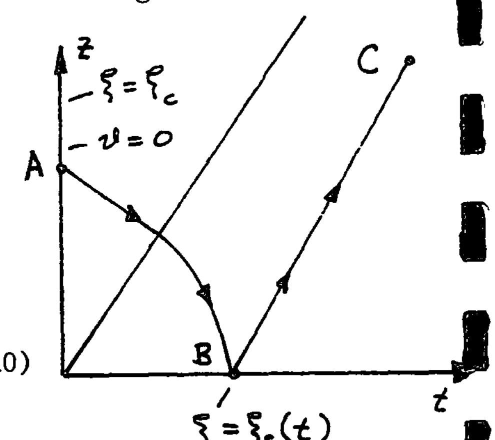
In view of Eq. 5, this expression becomes

$$dv \pm \sqrt{\frac{g}{\xi}}\, d\xi = 0 \quad ; \; C^{\pm} \tag{7}$$

Integration gives

$$v \pm R(\xi) = c_{\pm} \quad ; \; C^{\pm} \tag{8}$$

(b) The initial and boundary conditions are as shown to the right. C^{+} characteristics are straight lines.



On C^{-} from $A \rightarrow B$ the invariant is

$$-R(\xi_c) = c_- \tag{9}$$

At B, it follows that

$$v_B = c_- + R(\xi_s) = R(\xi_s) - R(\xi_c) \tag{10}$$

and hence from $B \rightarrow C$

$$c_+ = v_B + R(\xi_B) = R(\xi_s) - R(\xi_c) + R(\xi_s) = 2R(\xi_s) - R(\xi_c) \tag{11}$$

Also, from $B \rightarrow C$

$$c_- = -R(\xi_c) \tag{12}$$

Prob. 11.7.3 (cont.)

Eq. 8 shows that at a point where C^+ and C^- characteristics cross

$$\vartheta = \frac{c_+ + c_-}{2} \tag{13}$$

$$R(\xi) = \frac{c_+ - c_-}{2} \tag{14}$$

So, at any point on $B \to C$, these equations are evaluated using Eqs. 11 and 12 to give

$$\vartheta = R(\xi_s) - R(\xi_c) \tag{15}$$

$$R(\xi) = R(\xi_s) \tag{16}$$

Further, the slope of the line is the constant, from Eq. 5,

$$\begin{aligned} \frac{dz}{dt} &= 2R(\xi_s) + \frac{1}{2}\left[R(\xi_s) - R(\xi_c)\right] \\ &= \frac{3}{2}R(\xi_s) - R(\xi_c) \end{aligned} \tag{17}$$

Thus, the response on all C^+ characteristics originating on the t axis is determined. For those originating on the z axis, the solution is $\vartheta = 0$ and $\xi = \xi_c$.

(c) Initial conditions set the invariants $c_\pm$

$$c_\pm = \vartheta \pm 2\sqrt{g\xi} = 1 \pm 2\sqrt{\xi} \tag{18}$$

The numerical values are shown on the respective characteristics in Fig. 11.7.3a to the left of the z axis.

(d) At the intersections of the characteristics, ϑ and ξ follow from Eqs. 13 and 14

Prob. 11.7.3 (cont.)

$$\vartheta = \frac{1}{2}(c_+ + c_-) \tag{19}$$

$$\xi = \left(\frac{c_+ - c_-}{4}\right)^2 \tag{20}$$

The numerical values are displayed above the intersections in the figure as (ϑ, ξ). Note that the characteristic lines in this figure are only schematic.

(e) The slopes of the characteristics at each intersection now follow from Eq. 5.

$$\left(\frac{dz}{dt}\right)_{\pm} = \vartheta \pm \sqrt{\xi} \tag{21}$$

The numerical values are displayed under the characteristic intersections as $\left[\left(\frac{dz}{dt}\right)_+, \left(\frac{dz}{dt}\right)_-\right]$. Based on these slopes, the characteristics are drawn in Fig. P11.7.3b.

(f) Note (ϑ, ξ) are constant along characteristics $C^{\pm}$ leaving the "cone". All other points outside the "cone" have characteristics originating where $\vartheta = 1$ and $\xi = 1$ (constant state) and hence at these points the solution is $\vartheta = 1$ and $\xi = 1$. The velocity is shown as a function of z when $t = 0$, and 4 in Fig. P11.7.3c. As can be seen from either these plots or the characteristics, the wavefronts steepen into shocks.

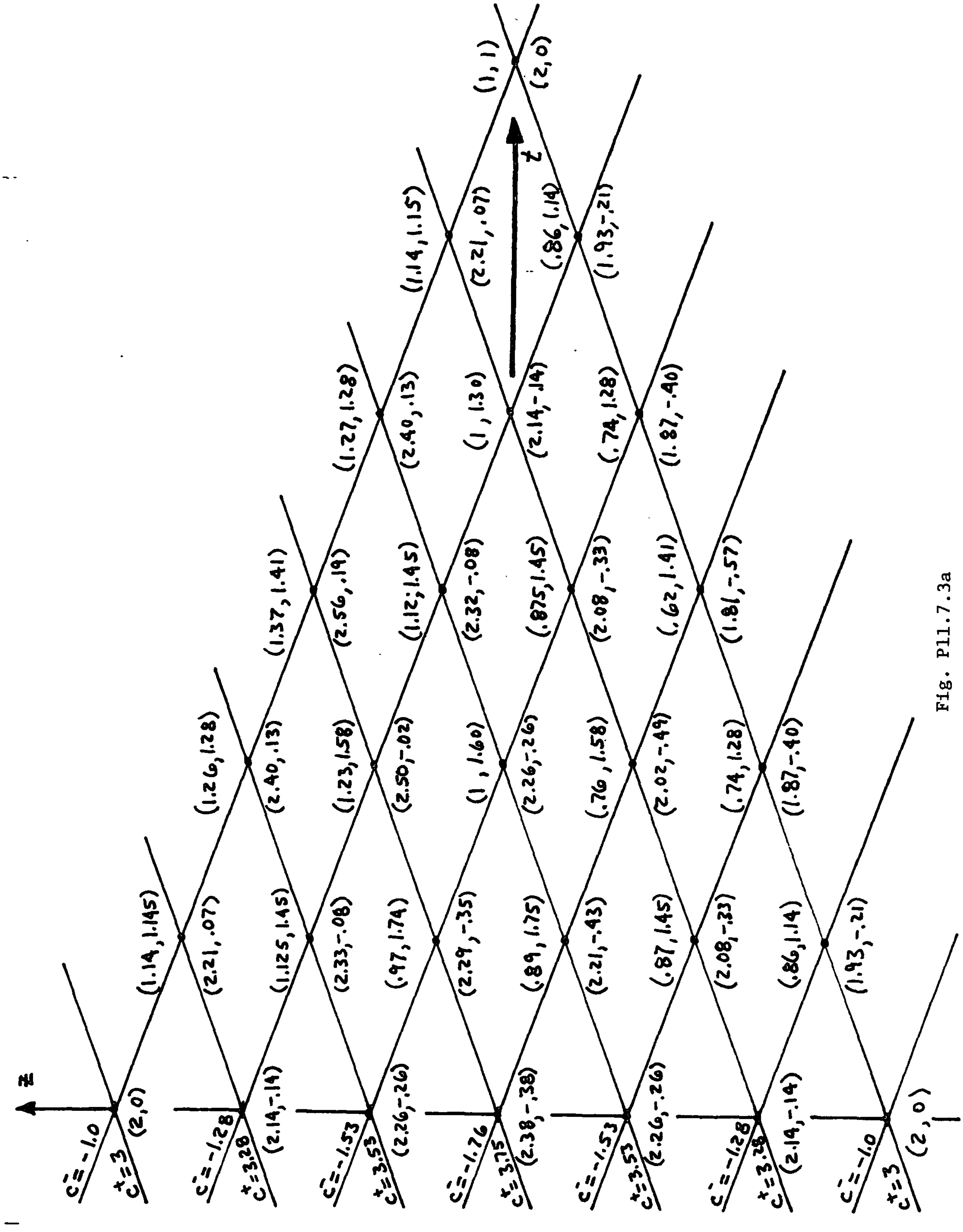

(1, 1)
(2, 0)
t
(1.14, 1.15)
(2.21, .07)
(.86, 1.14)
(1.93, -.21)
(1.27, 1.28)
(2.40, .13)
(1, 1.30)
(2.14, -.14)
(.74, 1.28)
(1.87, -.40)
(1.37, 1.41)
(2.56, .19)
(1.12, 1.45)
(2.32, -.08)
(.875, 1.45)
(2.08, -.33)
(.62, 1.41)
(1.81, -.57)
(1.26, 1.28)
(2.40, .13)
(1.23, 1.58)
(2.50, -.02)
(1, 1.60)
(2.26, -.26)
(.76, 1.58)
(2.02, -.49)
(.74, 1.28)
(1.87, -.40)
(1.14, 1.145)
(2.21, .07)
(1.125, 1.45)
(2.33, -.08)
(.97, 1.74)
(2.29, -.35)
(.89, 1.75)
(2.21, -.43)
(.87, 1.45)
(2.08, -.33)
(.86, 1.14)
(1.93, -.21)
z
(2, 0)
c⁻ = -1.0
c⁺ = 3
(2.14, -.14)
c⁻ = -1.28
c⁺ = 3.28
(2.26, -.26)
c⁻ = -1.53
c⁺ = 3.53
(2.38, -.38)
c⁻ = -1.76
c⁺ = 3.75
(2.26, -.26)
c⁻ = -1.53
c⁺ = 3.53
(2.14, -.14)
c⁻ = -1.28
c⁺ = 3.28
(2, 0)
c⁻ = -1.0
c⁺ = 3

Fig. P11.7.3a

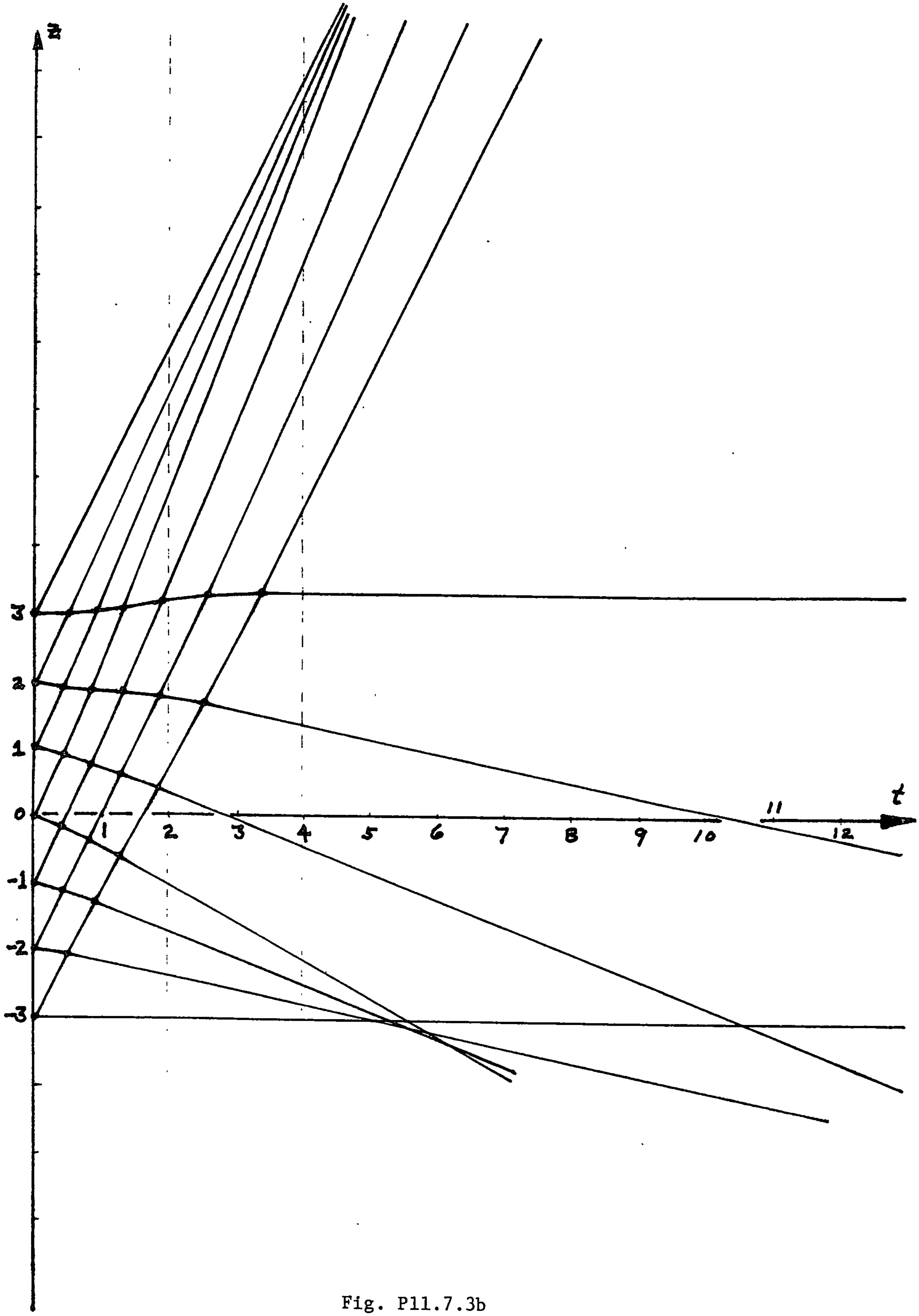


Fig. P11.7.3b

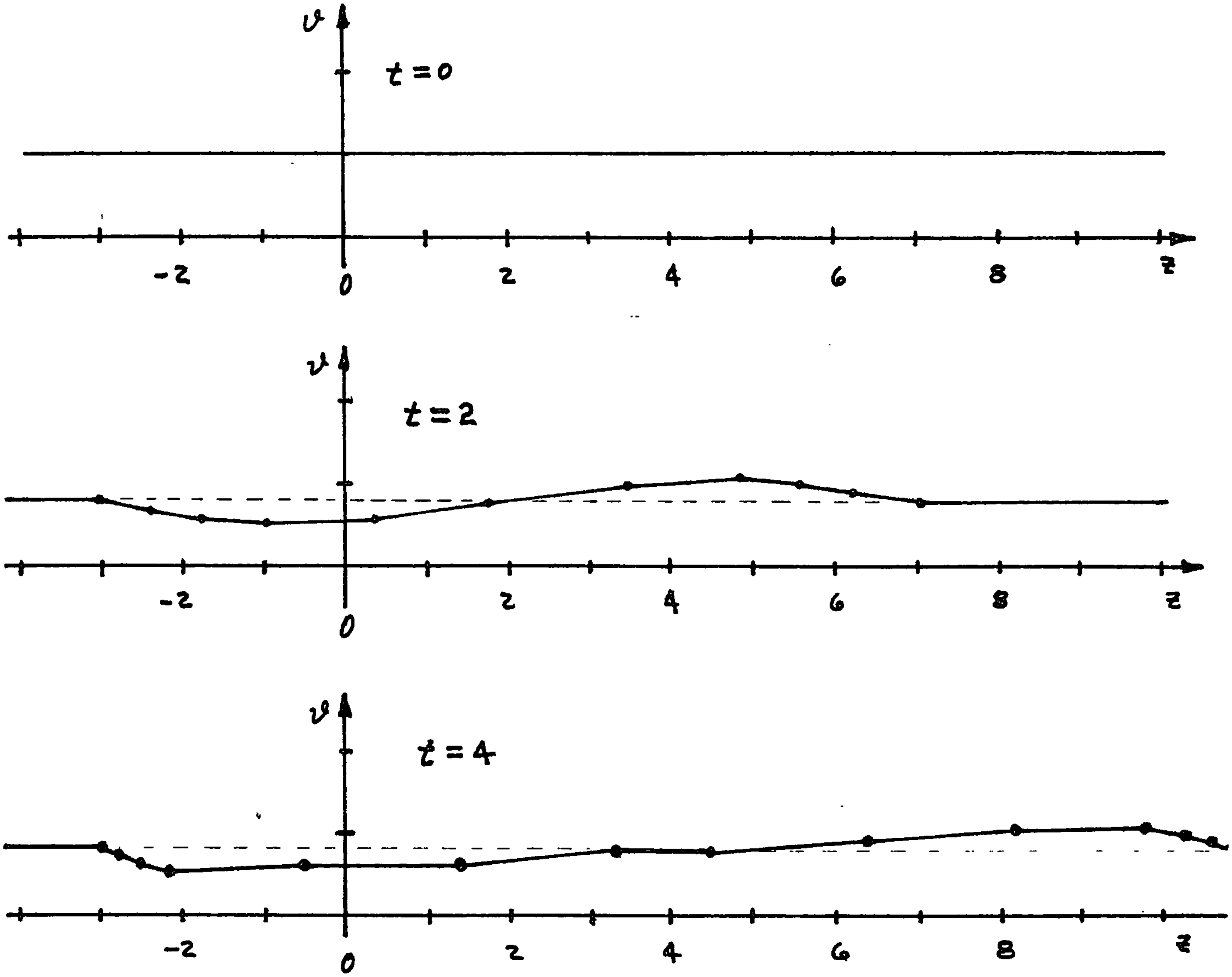


Fig. P11.7.3c

Prob. 11.7.4 (a) Faraday's and Ampere's laws for fields of the given forms reduce to

$$\begin{bmatrix} \bar{i}_x & \bar{i}_y & \bar{i}_z \\ 0 & 0 & \frac{\partial}{\partial z} \\ E & 0 & 0 \end{bmatrix} = \bar{i}_y \frac{\partial E}{\partial z} = -\bar{i}_y \mu_o \frac{\partial H}{\partial t} \tag{1}$$

$$\begin{bmatrix} \bar{i}_x & \bar{i}_y & \bar{i}_z \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & H & 0 \end{bmatrix} = -\bar{i}_x \frac{\partial H}{\partial z} = \bar{i}_x \left[\epsilon + 3\delta E^2 \right] \frac{\partial E}{\partial t} \tag{2}$$

The fields are transverse and hence solenoidal, as required by the remaining two equations with $\rho_f = 0$.

(b) The characteristic equations follow from

$$\begin{bmatrix} 0 & 1 & \mu_o & 0 \\ \epsilon + 3\delta E^2 & 0 & 0 & 1 \\ dt & dz & 0 & 0 \\ 0 & 0 & dt & dz \end{bmatrix} \begin{bmatrix} E_t \\ E_z \\ H_t \\ H_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ dE \\ dH \end{bmatrix} \tag{3}$$

The I'st characteristic equations follow by setting the determinant of the coefficients equal to zero. Expanding by minors about the two terms in the first row gives

$$-(dt)^2 + \mu_o (dz)^2 (\epsilon + 3\delta E^2) = 0 \Rightarrow \frac{dz}{dt} = \frac{\pm 1}{\sqrt{\mu_o(\epsilon + 3\delta E^2)}} \text{ on } C^{\pm} \tag{4}$$

Prob. 11.7.4 (cont.)

The IInd characteristic equations follow from the determinant formed by substituting the column matrix on the right in Eq. 3 for the first column on the left.

$$\begin{bmatrix} 0 & 1 & \mu_0 & 0 \\ 0 & 0 & 0 & 1 \\ dE & dz & 0 & 0 \\ dH & 0 & dt & dz \end{bmatrix} = 0 \tag{5}$$

Expansion about the two terms in the first column gives

$$-dE\,dt - dH(dz\,\mu_0) = 0 \Rightarrow dE + \mu_0 dH \frac{dz}{dt} = 0 \tag{6}$$

With dz/dt given by Eq. 4, this becomes

$$dE \pm \sqrt{\frac{\mu_0}{\epsilon + 3\delta E^2}}\, dH = 0 \Rightarrow dH \pm \sqrt{\frac{\epsilon + 3\delta E^2}{\mu_0}}\, dE = 0 \tag{7}$$

This expression is integrated to obtain

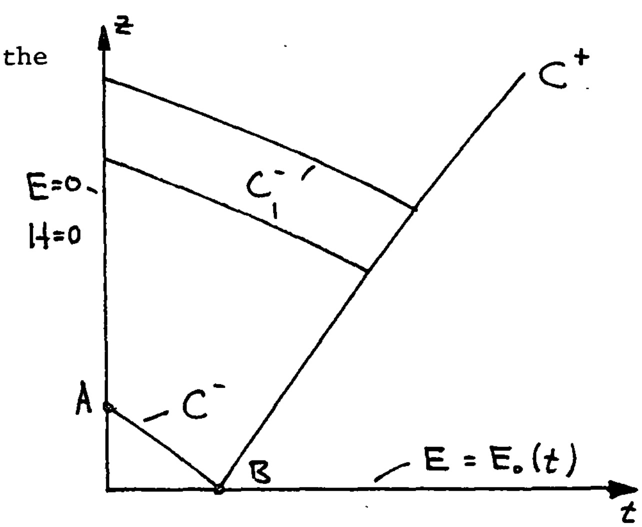
$$H \pm \mathcal{R}(E) = c_\pm \tag{8}$$

where

$$\mathcal{R}(E) \equiv \left\{ E\sqrt{E^2 + \frac{\epsilon}{3\delta}} + \frac{\epsilon}{3\delta} \ln\left(E + \sqrt{E^2 + \frac{\epsilon}{3\delta}}\right) \right\} \sqrt{\frac{3\delta}{4\mu_0}}$$

(c) At point A on the t=0 axis the invariant follows from Eq. 8 as

$$c_- = -\mathcal{R}(0) = \frac{-\epsilon}{3\delta} \ln\sqrt{\frac{\epsilon}{3\delta}}\, \sqrt{\frac{3\delta}{4\mu_0}} \tag{9}$$



Prob. 11.7.4 (cont.)

Evaluation of the same equation at B when $E = E_o(t)$ then gives

$$H_B - \mathcal{R}(E_o) = -\mathcal{R}(0) \Rightarrow H_B = -\mathcal{R}(0) + \mathcal{R}(E_o) \tag{10}$$

Thus, it is clear that if H were also given $(H_o(t))$ at z=0, the problem would be overspecified.

On the C^+ characteristic, Eqs. 8 and 11 and the fact that $E=E_o$ at B serve to evaluate

$$c_+ = H_B + \mathcal{R}(E_o) = -\mathcal{R}(0) + 2\mathcal{R}(E_o) \tag{11}$$

Because _ is the same for all C^- characteristics coming from the z axis, it follows from Eqs. 8, 9 and 12 that

$$H + \mathcal{R}(E) = -\mathcal{R}(0) + 2\mathcal{R}(E_o) \tag{12}$$

$$H - \mathcal{R}(E) = -\mathcal{R}(0) \tag{13}$$

So, on the C^+ characteristics originating on the t axis,

$$H = \mathcal{R}(E_o) - \mathcal{R}(0) \tag{14}$$

$$\mathcal{R}(E) = \mathcal{R}(E_o) \tag{15}$$

Because the slope of this line is given by Eq. 4

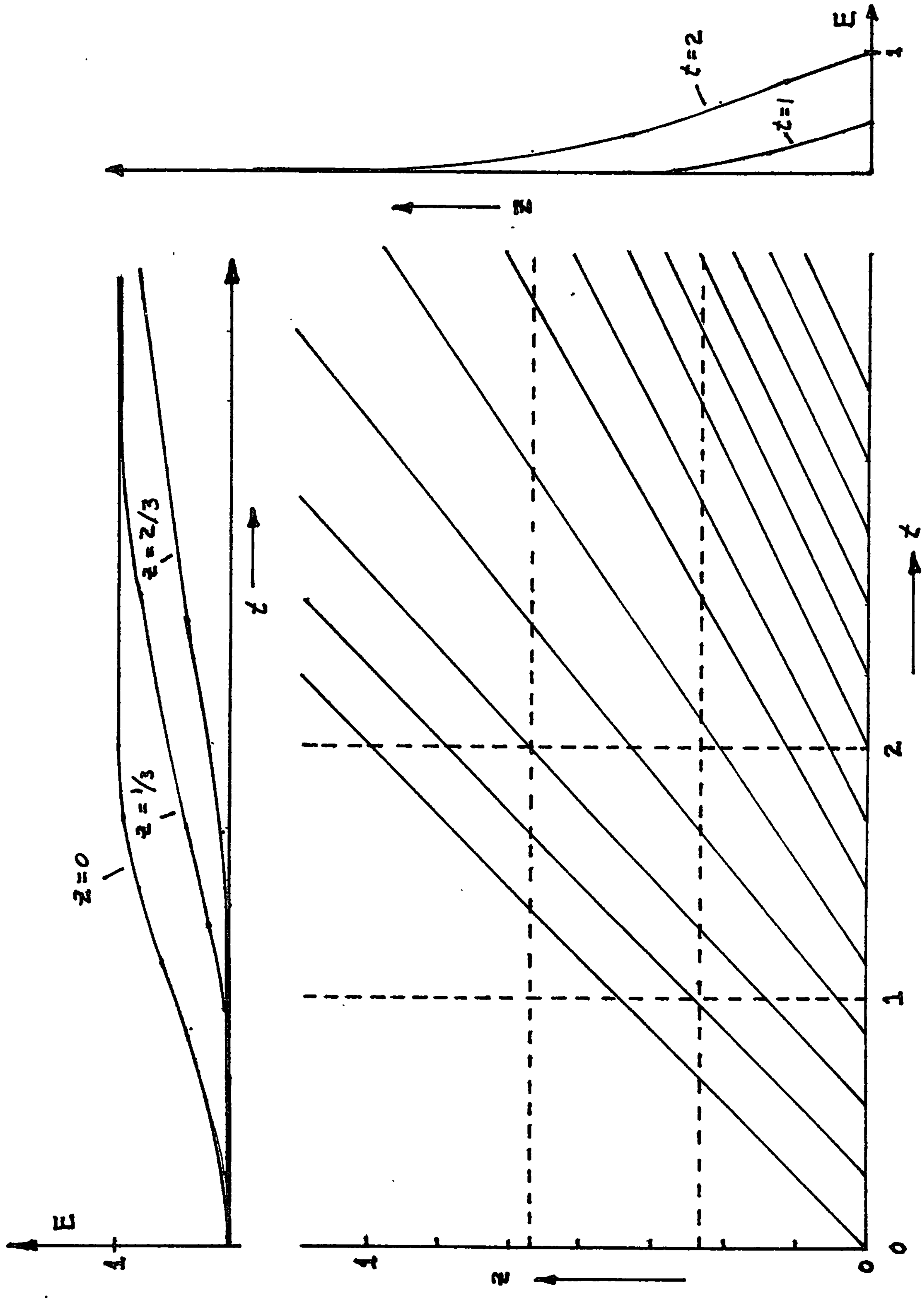
$$\frac{dz}{dt} = \frac{1}{\sqrt{\mu_o(\epsilon + 3\delta E^2)}} \tag{16}$$

evaluated using E inferred from Eq. 16, it follows that the slope is the same at each point on the line.

For $\mu_o = \epsilon = \delta$, the C^+ characteristics have the slopes

$$\frac{dz}{dt} = \frac{1}{\sqrt{1 + 3E_o^2}}$$

and hence values shown in the table. These lines are drawn in the figure. Remember that E is constant along these lines. Thus, it is possible to



E
t=2
t=1
z
t
z=0
z=1/3
z=2/3
4
0
1
2

Prob. 11.7.4 (cont.)

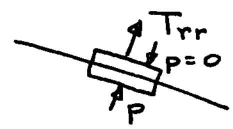
plot either the z or t dependence of E, as shown. Note that the wave front tends to smooth out.

t	E_o	dz/dt
0	0	1
0.285	0.0493	0.996
0.571	0.188	0.951
0.857	0.389	0.829
1.14	0.609	0.688
1.43	0.813	0.579
1.71	0.950	0.519
2.0	1.0	0.50

<u>Prob. 11.7.5</u> (a) Conservation of total flux requires that

$$B_o\pi(a^2-\xi_o^2) = B_z\pi(a^2-\xi^2) \Rightarrow B_z = B_o\frac{(a^2-\xi_o^2)}{(a^2-\xi^2)} \tag{1}$$

Thus, for long wave deformations, radial stress equilibrium at the interface requires that



$$p = -T_{rr} = \frac{1}{2}\mu_o B_z^2 = \frac{1}{2}\mu_o\frac{(a^2-\xi_o^2)}{(a^2-\xi^2)} \tag{2}$$

By replacing $\pi\xi^2 = A(z)$, the function on the right in Eq. (2) takes the form of Eq. 9.13.5. Thus, the desired equations of motion are Eq. 9.13.9

$$\frac{\partial A}{\partial t} + v\frac{\partial A}{\partial z} + A\frac{\partial v}{\partial z} = 0 \tag{3}$$

and Eq. 9.13.4

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} + \frac{c^2}{A}\frac{\partial A}{\partial z} = 0 \tag{4}$$

where $$c^2 \equiv \frac{B_o^2}{\mu_o\rho}\frac{(\pi a^2 - A_o)^2}{(\pi a^2 - A)^3}A$$

Prob. 11.7.5 (cont.)

Then, the characteristic equations are formed from

$$\begin{bmatrix} 1 & v & 0 & A \\ 0 & \frac{c^2}{A} & 1 & v \\ dt & dz & 0 & 0 \\ 0 & 0 & dt & dz \end{bmatrix} \begin{bmatrix} A_t \\ A_z \\ v_t \\ v_z \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ dA \\ dv \end{bmatrix} \tag{5}$$

The determinant of the coefficients gives the I'st characteristics

$$\frac{dz}{dt} = v \pm c \tag{6}$$

while the second follows from

$$\begin{bmatrix} 0 & v & 0 & A \\ 0 & \frac{c^2}{A} & 1 & v \\ dA & dz & 0 & 0 \\ dv & 0 & dt & dz \end{bmatrix} = 0 \tag{7}$$

which is

$$dA\left[v\left(\frac{dz}{dt} - v\right) + \alpha^2\right] + dv\left(\frac{dz}{dt} A_o\right) = 0 \tag{8}$$

With the use of Eq. 6, this becomes

$$dv \pm c\,\frac{dA}{A_o} = 0 \tag{9}$$

The integral of this expression is

$$v \pm \mathcal{R}(A) = C_\pm \tag{10}$$

where

$$\mathcal{R}(A) = \int \frac{c}{A}\, dA = \sqrt{\frac{B_o^2(\pi a^2 - A_o)^2}{\mu_o \rho A_o}}\ \frac{2}{\pi a^2} \sqrt{\frac{A}{\pi a^2 - A}} \tag{11}$$

Prob. 11.7.5 (cont.)

Now, given initial conditions

$$\xi = \xi_o(z) \Rightarrow A = A_o(z) \; ; \; v = 0 \tag{12}$$

where the maximum $A_o(z)$ is A_{max} , invariants follow from Eq. 10 as

$$c_+ = \mathcal{R}(A_B) \; ; \; c_- = -\mathcal{R}(A_C) \tag{13}$$

so solution at D is

$$\mathcal{R}(A_D) = \frac{c_+ - c_-}{2} = \frac{\mathcal{R}(A_B) + \mathcal{R}(A_C)}{2}$$

Thus, the solution R at D is the mean of that at B and C. The largest possible value for A at D is therefore obtained if either B or C is at the maximum in A. Because this implies that the other characteristic comes from a lesser value of A, it follows that A at D is smaller than A_{max}.

Prob. 11.8.1 For "plane-wave" motions of arbitrary orientation, $\bar{v} = \bar{v}(x,t)$ and $\bar{H} = \bar{H}(x,t)$, the general laws are:

Mass Conservation

$$\frac{\partial \rho}{\partial t} + v_x \frac{\partial \rho}{\partial x} + \rho \frac{\partial v_x}{\partial x} = 0 \tag{1}$$

Momentum Conservation (three components)

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x}\right) + \frac{\partial p}{\partial x} = \frac{\partial T_{xx}}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{2}\mu_o\left(H_x^2 - H_y^2 - H_z^2\right)\right] \tag{2}$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x}\right) = \frac{\partial T_{yx}}{\partial x} = \frac{\partial}{\partial x}\left(\mu_o H_x H_y\right) \tag{3}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x}\right) = \frac{\partial T_{zx}}{\partial x} = \frac{\partial}{\partial x}\left(\mu_o H_x H_z\right) \tag{4}$$

Energy Conservation (which reduces to the insentropic equation of state)

$$\left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x}\right)\left(p \rho^{-\gamma}\right) = 0 \tag{5}$$

The laws of Faraday, Ampere and Ohm (for perfect conductor), Eq. 6.2.3

$$\frac{\partial H_x}{\partial t} = 0 \tag{6}$$

$$\frac{\partial H_y}{\partial t} = \frac{\partial}{\partial x}\left(-v_x H_y + v_y H_x\right) \tag{7}$$

$$\frac{\partial H_z}{\partial t} = \frac{\partial}{\partial x}\left(v_z H_x - v_x H_z\right) \tag{8}$$

These eight equations represent the evolution of the dependent variables

$$(\rho, p, v_x, v_y, v_z, H_x, H_y, H_z)$$

From Eq. 6, (as well as the requirement that $\bar{H}$ is solenoidal) it follows that H_x is independent of both t and x. Hence, H_x can be eliminated from Eq. 2 and considered a constant in Eqs. 3, 4, 7 and 8. Equations 1-5, 7 and 8 are now written as the first 7 of the following 14 equations.

$$
\begin{bmatrix}
1 & v_x & 0 & 0 & 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \rho & \rho v_x & 0 & 0 & 0 & 0 & 0 & \mu_0 H_y & 0 & \mu_0 H_z \\
0 & 0 & 0 & 0 & 0 & 0 & \rho & \rho v_x & 0 & 0 & 0 & -\mu_0 H_x & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho & \rho v_x & 0 & 0 & 0 & -\mu_0 H_x \\
-\frac{\gamma p}{\rho} & -\frac{\gamma v_x p}{\rho} & 1 & v_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -H_y & 0 & H_x & 0 & 0 & -1 & -v_x & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -H_z & 0 & 0 & 0 & H_x & 0 & 0 & -1 & -v_x \\
dt & dx & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & dt & dx & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & dt & dx & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & dt & dx & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & dt & dx & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & dt & dx & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & dt & dx & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\rho_{,t} \\ \rho_{,x} \\ p_{,t} \\ p_{,x} \\ v_{x,t} \\ v_{x,x} \\ v_{y,t} \\ v_{y,x} \\ v_{z,t} \\ v_{z,x} \\ H_{y,t} \\ H_{y,x} \\ H_{z,t} \\ H_{z,x}
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ d\rho \\ dp \\ dv_x \\ dv_y \\ dv_z \\ dH_y \\ dH_z
\end{bmatrix}
$$

Prob. 11.8.1 (cont.)

Following steps illustrated by Eq. 11.15.19, the determinant of the coefficients is reduced to

$$\begin{vmatrix} v_x - \frac{dx}{dt} & 0 & \rho & 0 & 0 & 0 & 0 \\ 0 & 1 & \rho(v_x - \frac{dx}{dt}) & 0 & 0 & \mu_o H_y & \mu_o H_z \\ 0 & 0 & 0 & \rho(v_x - \frac{dx}{dt}) & 0 & -\mu_o H_x & 0 \\ 0 & 0 & 0 & 0 & \rho(v_x - \frac{dx}{dt}) & 0 & -\mu_o H_x \\ -\frac{\gamma p}{\rho}(v_x - \frac{dx}{dt}) & (v_x - \frac{dx}{dt}) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -H_y & H_x & 0 & -(v_x - \frac{dx}{dt}) & 0 \\ 0 & 0 & -H_z & 0 & H_x & 0 & -(v_x - \frac{dx}{dt}) \end{vmatrix} = 0 \qquad (10)$$

The quantity $v_x - \frac{dx}{dt}$ can be factored out of the fifth row. That row is then subtracted from the second so that there are all zeros in the second column except for the A_{52} term. Expansion by minors about this term then gives

$$(v_x - \frac{dx}{dt})\begin{vmatrix} \rho(v_x - \frac{dx}{dt}) & \rho & 0 & 0 & 0 & 0 \\ \gamma p & \rho(v_x - \frac{dx}{dt}) & 0 & 0 & \mu_o H_y & \mu_o H_z \\ 0 & 0 & \rho(v_x - \frac{dx}{dt}) & 0 & -\mu_o H_x & 0 \\ 0 & 0 & 0 & \rho(v_x - \frac{dx}{dt}) & 0 & -\mu_o H_x \\ 0 & -H_y & H_x & 0 & -(v_x - \frac{dx}{dt}) & 0 \\ 0 & -H_z & 0 & H_x & 0 & -(v_x - \frac{dx}{dt}) \end{vmatrix} = 0 \qquad (11)$$

Multiplication of the second row by $(v_x - \frac{dx}{dt})\rho/\gamma p$ and subtraction from the

Prob. 11.8.1 (cont.)

first generates all zeros in the first row except for the A_{12} term. Expansion about that term then gives

$$\left(v_x-\frac{dx}{dt}\right)\begin{bmatrix} \rho-\frac{\rho^2}{\gamma p}\left(v_x-\frac{dx}{dt}\right)^2 & 0 & 0 & -\frac{\mu_o H_y \rho}{\gamma p}\left(v_x-\frac{dx}{dt}\right) & -\frac{\mu_o H_z \rho}{\gamma p}\left(v_x-\frac{dx}{dt}\right) \\ 0 & \rho\left(v_x-\frac{dx}{dt}\right) & 0 & -\mu_o H_x & 0 \\ 0 & 0 & \rho\left(v_x-\frac{dx}{dt}\right) & 0 & -\mu_o H_x \\ -H_y & H_x & 0 & -\left(v_x-\frac{dx}{dt}\right) & 0 \\ -H_z & 0 & H_x & 0 & -\left(v_x-\frac{dx}{dt}\right) \end{bmatrix}=0 \qquad (12)$$

Multiplication of the second column by $\mu_o H_x/\rho\left(v_x-\frac{dx}{dt}\right)$ and addition to the fourth column generates all zeros in the second row except for the A_{22} term, while multiplication of the third column by $\mu_o H_x/\rho\left(v_x-\frac{dx}{dt}\right)$ and addition to the last column gives all zeros in the third row except for the A_{33} term. Thus, expansion by minors about the A_{22} and A_{33} terms gives

$$\left(v_x-\frac{dx}{dt}\right)\begin{bmatrix} \rho-\frac{\rho^2}{\gamma p}\left(v_x-\frac{dx}{dt}\right)^2 & -\frac{\mu_o H_y \rho}{\gamma p}\left(v_x-\frac{dx}{dt}\right) & -\frac{\mu_o H_z \rho}{\gamma p}\left(v_x-\frac{dx}{dt}\right) \\ -H_y & -\left(v_x-\frac{dx}{dt}\right)+\frac{\mu_o H_x^2}{\rho\left(v_x-\frac{dx}{dt}\right)} & 0 \\ -H_z & 0 & -\left(v_x-\frac{dx}{dt}\right)+\frac{\mu_o H_x^2}{\rho\left(v_x-\frac{dx}{dt}\right)} \end{bmatrix}=0 \qquad (13)$$

Prob. 11.8.1 (cont.)

This third order determinant is then expanded by minors to give

$$\frac{\rho^4}{\gamma p}\left(v_x-\frac{dx}{dt}\right)\left[-\left(v_x-\frac{dx}{dt}\right)^2+\frac{\mu_o H_x^2}{\rho}\right]\cdot \tag{14}$$

$$\left\{\left[\left(v_x-\frac{dx}{dt}\right)^2\right]^2-\left(v_x-\frac{dx}{dt}\right)^2\left[\frac{\gamma p}{\rho}+\frac{\mu_o}{\rho}\left(H_x^2+H_y^2+H_z^2\right)\right]+\frac{\gamma p}{\rho}\frac{\mu_o H_x^2}{\rho}\right\}=0$$

This expression has been factored to make evident the 7 characteristic lines. First, there is the particle line, evident from the outset (Eq. 5) as the line along which the isentropic invariant propagates.

$$\frac{dx}{dt}=v_x \tag{15}$$

The second represents the two Alfvén waves

$$\frac{dx}{dt}=v_x\pm a_a \quad ; \quad a_a\equiv\sqrt{\frac{\mu_o H_x^2}{\rho}} \tag{16}$$

and the last represents four magnetoacoustic waves

$$\frac{dx}{dt}=v_x\pm\begin{Bmatrix}a_{b+}\\ a_{b-}\end{Bmatrix} \tag{17}$$

where

$$a_{b\pm}^2\equiv\frac{1}{2}\left(a^2+a_a^2+a_b^2\right)\pm\frac{1}{2}\sqrt{\left(a^2+a_a^2+a_b^2\right)^2-4a^2a_a^2}$$

$$a\equiv\sqrt{\frac{\gamma p}{\rho}}=\sqrt{\gamma RT}$$

$$a_b\equiv\sqrt{\frac{\mu_o}{\rho}\left(H_y^2+H_z^2\right)}$$

Prob. 11.9.1 Linearized, Eq. 11.9.17 becomes

$$\frac{d\overset{\circ}{e}}{dn} = \frac{-n}{\overset{\circ}{e}} \tag{1}$$

Thus,

$$\overset{\circ}{e}\, d\overset{\circ}{e} = -n\, dn \tag{2}$$

and integration gives

$$\overset{\circ}{e}{}^2 + \overset{\circ}{n}{}^2 = \text{constant} = \overset{\circ}{e}{}_i^2 \tag{3}$$

where the constant of integration is evaluated at the upstream grid where $n=0$ and $\overset{\circ}{e}=\overset{\circ}{e}_i$.

Prob. 11.9.2 Linearized, Eqs. 11.9.9 and 11.9.10 reduce to

$$\frac{dn}{dt} = -\overset{\circ}{e} \tag{1}$$

$$\frac{d\overset{\circ}{e}}{dt} = n \tag{2}$$

Elimination of $\underline{e}$ between these gives

$$\frac{d^2 n}{dt^2} + n = 0 \tag{3}$$

The solution to this equation giving $n=0$ when $t=t_o$ is

$$n = A(t_o)\sin(t-t_o) = A\left(t-\frac{z}{U}\right)\sin\left(\frac{z}{U}\right) \tag{4}$$

and it follows from Eq. 1 that

$$\overset{\circ}{e} = -A(t_o)\cos(t-t_o) = -A\left(t-\frac{z}{U}\right)\cos\left(\frac{z}{U}\right) \tag{5}$$

To establish $A(t_o)$ it is necessary to use Eq. 11.9.15, which requires that

$$-A(t) = -\frac{V(t)}{U} + \frac{1}{U}\int_0^1 \int_0^z A\left(t-\frac{z'}{U}\right)\sin\left(\frac{z'}{U}\right)dz'\,dz \tag{6}$$

For the specific excitation

$$V = \mathrm{Re}\,\hat{V}\exp j\omega t \tag{7}$$

it is reasonable to search for a solution to Eq. 6 in which the phase and amplitude of the response at z=0 are unknown, but the frequency is the same as that of the driving voltage.

Prob. 11.9.2 (cont.)

$$A = \mathcal{R}e\,\hat{A}\exp j\omega t \tag{8}$$

Observe that

$$A\left(t-\frac{z'}{U}\right) = \mathcal{R}e\left(\hat{A}\,e^{j\omega t}e^{-j\frac{\omega z'}{U}}\right) = \frac{1}{2}\hat{A}\,e^{j\left(\omega t-\frac{\omega z'}{U}\right)} + \frac{1}{2}\hat{A}^{*}e^{-j\left(\omega t-\frac{\omega z'}{U}\right)} \tag{9}$$

and

$$\sin\frac{z'}{U} = \frac{1}{2j}\left(e^{j\frac{z'}{U}} - e^{-j\frac{z'}{U}}\right) \tag{10}$$

Thus,

$$\int_0^1\int_0^z A\left(t-\frac{z}{U}\right)\sin\frac{z'}{U}\,dz' =$$

$$\mathcal{R}e\,\frac{U\hat{A}}{2j}\,e^{j\omega t}\left\{-\frac{\left(e^{j\left(\frac{-\omega+1}{U}\right)}-1\right)}{\frac{1}{U}(-\omega+1)^2} + \frac{\left(e^{-j\left(\frac{\omega+1}{U}\right)}-1\right)}{\frac{1}{U}(\omega+1)^2} - \frac{1}{j(-\omega+1)} - \frac{1}{j(\omega+1)}\right\} \tag{11}$$

Substitution of Eqs. 7, 8 and 11 into Eq. 6 then gives an expression that can be solved for $\hat{A}$.

$$\hat{A} = \frac{\hat{V}}{U}\left\{1-\frac{U}{4j}\left[\frac{\left(e^{j\left(\frac{-\omega+1}{U}\right)}-1\right)U}{(1-\omega)^2} - \frac{\left(e^{-j\left(\frac{\omega+1}{U}\right)}-1\right)U}{(1+\omega)^2} + \frac{2}{j(1-\omega^2)}\right]\right\} \tag{12}$$

Thus, the solution taking the form of Eq. 4 is

$$n(z,t) = \mathcal{R}e\,\hat{A}\,e^{j\omega\left(t-\frac{z}{U}\right)}\sin\left(\frac{z}{U}\right) \tag{13}$$

where $\underline{A}$ is given by Eq. 12.

Prob. 11.10.1 With $p = 0$, Eqs. 11.10.7 and 11.10.8 are

$$dv + de(M \mp 1) = 0 \tag{1}$$

$$\frac{dz}{dt} = M \pm 1 \;;\; C^{\pm} \tag{2}$$

In this limit, Eq. 1 can be integrated.

$$v + (M \mp 1)e = c_{\pm} \tag{3}$$

Initial conditions are

$$\xi = \xi_o(z,0) \Rightarrow e = \frac{\partial \xi_o}{\partial z} = e_o(z,0) \tag{4}$$

$$v = v_o(z,0) \tag{5}$$

These serve to evaluate $C_{\pm}$ in Eq. 3

$$c_{\pm} = v_o + (M \mp 1)e_o \tag{6}$$

At a point C where the characteristics cross Eq. 3 can be solved simultaneously to give

$$\begin{bmatrix} 1 & M-1 \\ 1 & M+1 \end{bmatrix} \begin{bmatrix} v \\ e \end{bmatrix} = \begin{bmatrix} C_+ \\ C_- \end{bmatrix} \Rightarrow \begin{array}{l} v = \frac{1}{2}\left[(M+1)c_+ - (M-1)c_-\right] \\ e = \frac{1}{2}\left[c_- - c_+\right] \end{array} \tag{7}$$

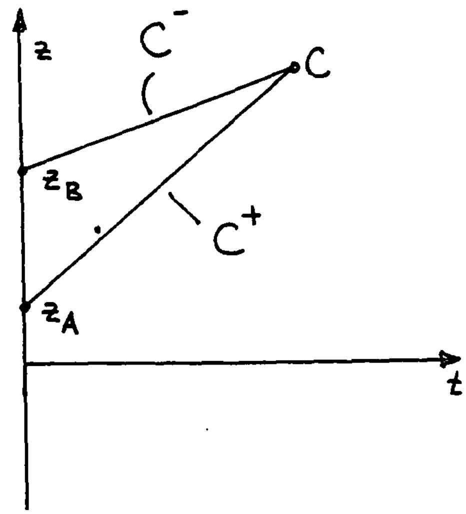
Integration of Eqs. 2 to give the characteristic lines shown gives

$$z = (M \pm 1)t + z_{\substack{A \\ B}} \tag{8}$$

Prob. 11.10.1 (cont.)

For these lines, the invariants of Eqs. 6 are

$$C_{\pm} = v_o\left(z_{\substack{A\\B}}\right) + (M \mp 1)e_o\left(z_{\substack{A\\B}}\right) \tag{9}$$



With z_A and z_B evaluated using Eq. 8, these invarients are written in terms of the (z,t) at point C.

$$C_{\pm} = v_o[z-(M\pm 1)t] + (M\mp 1)e_o[z-(M\pm 1)t] \tag{10}$$

and, finally, the solutions at C, Eq. 7, are written in terms of the (z,t) at C.

$$\begin{aligned} v = \frac{1}{2}\Big\{&(M+1)v_o[z-(M+1)t] + (M-1)(M+1)e_o[z-(M+1)t] \\ &- (M-1)v_o[z-(M-1)t] - (M+1)(M-1)e_o[z-(M-1)t] \end{aligned} \tag{11}$$

$$\begin{aligned} e = \frac{1}{2}\Big\{&v_o[z-(M-1)t] + (M+1)e_o[z-(M-1)t] \\ &- v_o[z-(M+1)t] - (M-1)e_o[z-(M+1)t]\Big\} \end{aligned} \tag{12}$$

Prob. 11.10.2 (a) With $\gamma = 0$, Eqs. 11.10.1 and 11.10.2 combine to give

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial z}\right)^2 \xi = \frac{\epsilon_o}{2\Delta\rho}\left[\frac{(aE_o^2)}{(a-\xi)^2} - \frac{(aE_o^2)}{(a+\xi)^2}\right] \tag{1}$$

Normalization of this expression is such that

$$\underline{\xi} = \xi/a\,,\quad \underline{t} = t/\tau\,,\quad \underline{z} = z/\tau U \tag{2}$$

gives

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right)^2 \xi = \frac{P}{4}\left[\frac{1}{(1-\xi)^2} - \frac{1}{(1+\xi)^2}\right] \tag{3}$$

where

$$P \equiv 2\epsilon_o E_o^2 \tau^2/\Delta\rho a$$

(b) With the introduction of v as a variable, Eq. 3 becomes

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right) v = -\frac{\partial \mathcal{E}}{\partial \xi} \tag{4}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right) \xi = v \tag{5}$$

where

$$\mathcal{E} = -\frac{P}{4}\left(\frac{1}{1-\xi} + \frac{1}{1+\xi}\right)$$

The characteristics could be found by one of the approaches outlined, but here they are obvious. On the I'st characteristics

$$\frac{dz}{dt} = 1 \tag{6}$$

the II'nd characteristic equations both apply and are

Prob. 11.10.2 (cont.)

$$\frac{d\upsilon}{dt} = -\frac{\partial E}{\partial \xi} \tag{7}$$

$$\frac{d\xi}{dt} = \upsilon \tag{8}$$

Multiply the left-hand side of Eq. 7 by the right-hand side of Eq. 8 and similarly, the right-hand side of Eq. 7 by the left-hand side of Eq. 8.

$$\upsilon\frac{d\upsilon}{dt} = -\frac{\partial E}{\partial \xi}\frac{d\xi}{dt} \Rightarrow \frac{d}{dt}\left[\frac{1}{2}\upsilon^2 + E(\xi)\right] = 0 \tag{9}$$

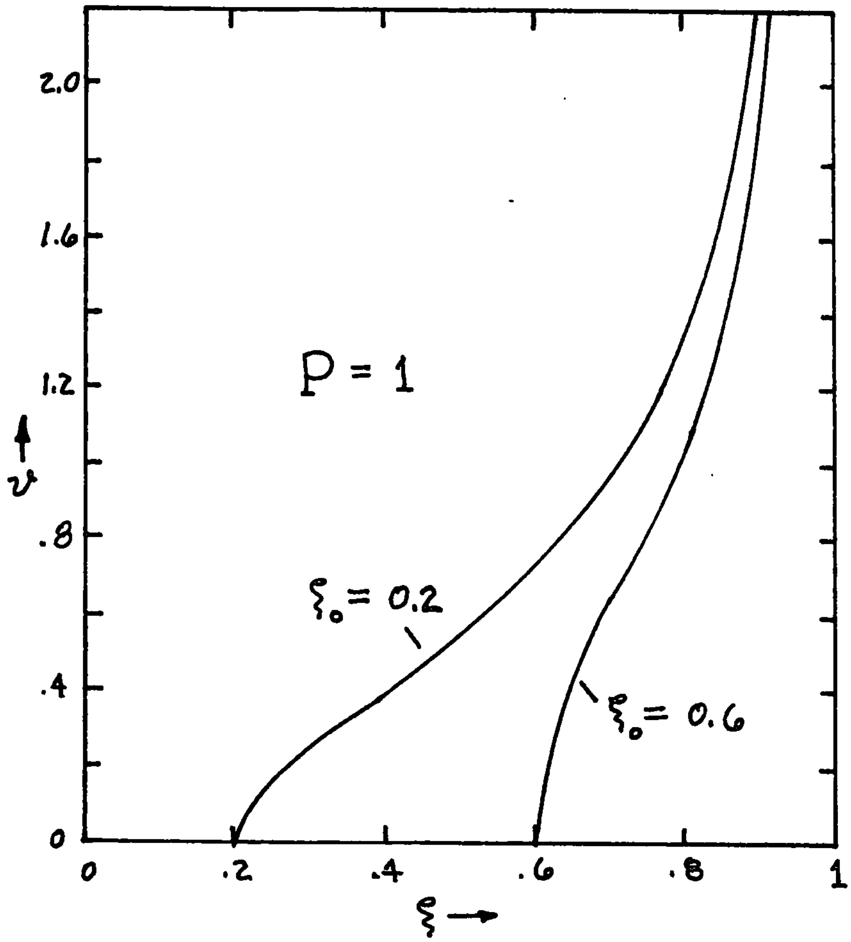
(c) It follows from Eq. 9 that

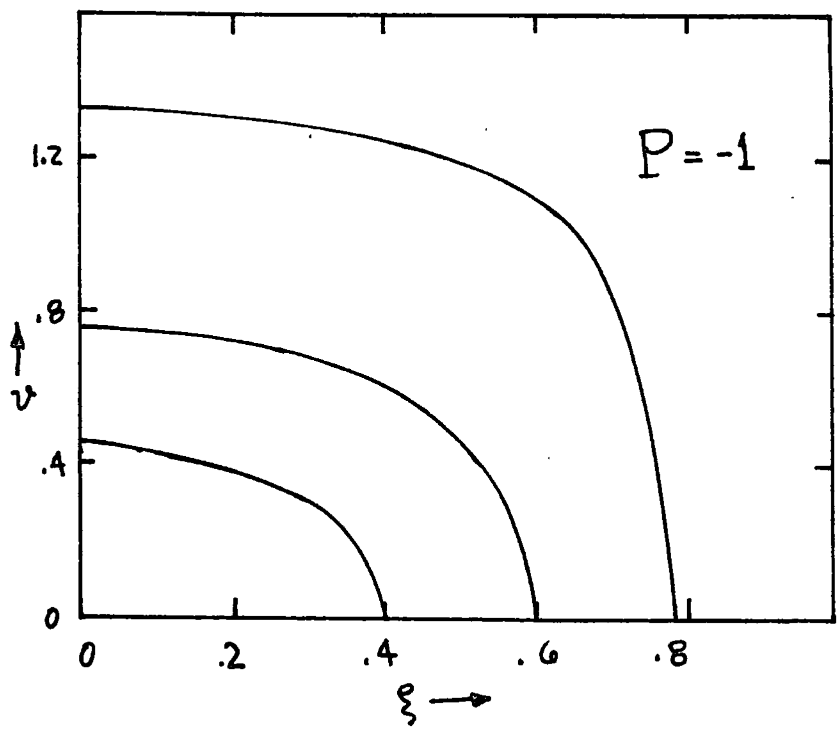
$$\frac{1}{2}\upsilon^2 + E(\xi) = \frac{1}{2}\upsilon_o^2 + E(\xi_o) \tag{10}$$

or specifically

$$\frac{1}{2}\upsilon^2 - \frac{P}{4}\left[\frac{1}{1-\xi} + \frac{1}{1+\xi}\right] = \frac{1}{2}\upsilon_o^2 - \frac{P}{4}\left[\frac{1}{1-\xi_o} + \frac{1}{1+\xi_o}\right]$$

Phase-plane plots are shown in the first quandrant. Reflecting the unstable nature of the dynamics, the trajectories are open for $P>1$, showing a deflection that has $\upsilon \to \infty$ as $\xi \to 1$ (the sheet approaches one or the other of the electrodes). The oscillatory nature of the response with $P = -1$ is apparent from the closed trajectories.


2.0
1.6
1.2
.8
.4
0
v
P = 1
$\xi_0 = 0.2$
$\xi_0 = 0.6$
0
.2
.4
.6
.8
1
ξ


1.2
.8
.4
0
v
P = -1
0
.2
.4
.6
.8
ξ

Prob. 11.10.3 The characteristic equations follow from Eqs. 11.10.19-11.10.22 written as

$$\begin{bmatrix} 1 & M_1 & M_1 & M_1^2-1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ dt & dz & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & dt & dz & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & M_2 & M_2 & M_2^2-1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & dt & dz & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & dt & dz \end{bmatrix} \begin{bmatrix} v_{1,t} \\ v_{1,z} \\ e_{1,t} \\ e_{1,z} \\ v_{2,t} \\ v_{2,z} \\ e_{2,t} \\ e_{2,z} \end{bmatrix} = \begin{bmatrix} Pf_1 \\ 0 \\ dv_1 \\ de_1 \\ Pf_2 \\ 0 \\ dv_2 \\ de_2 \end{bmatrix} \tag{1}$$

Also included are the 4 equations representing the differentials $dv_1 \dots de_2$. These expressions have been written in such an order that the lack of coupling between streams is exploited. Thus, the determinant of the coefficients can be reduced by independently manipulating the first 4 rows and first 4 columns or the second 4 rows and second four columns. Thus, the determinant is reduced by dividing the third rows by dt and subtracting from the first and adding the third column to the second.

$$\begin{bmatrix} 0 & 2M_1-\frac{dz}{dt} & M_1 & M_1^2-1 \\ 0 & 0 & -1 & 0 \\ dt & dz & 0 & 0 \\ 0 & dt & dt & dz \end{bmatrix} \begin{bmatrix} 0 & 2M_2-\frac{dz}{dt} & M_2 & M_2^2-1 \\ 0 & 0 & -1 & 0 \\ dt & dz & 0 & 0 \\ 0 & dt & dt & dz \end{bmatrix} \tag{2}$$

$$= \left[\left(2M_1-\frac{dz}{dt}\right)dz-(M_1^2-1)dt\right]\left[\left(2M_2-\frac{dz}{dt}\right)dz-(M_2^2-1)dt\right]=0$$

Prob. 11.10.3 (cont.)

This expression reduces to

$$(dt)^2\left[\left(\frac{dz}{dt}-M_1\right)^2-1\right]\left[\left(\frac{dz}{dt}-M_2\right)^2-1\right]=0 \tag{3}$$

and it follows that the Ist characteristic equations are Eqs. 11.10.24 and 11.10.26.

The IInd characteristics follow from

$$\begin{bmatrix} Pf_1 & M_1 & M_1 & M_1^2-1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ d\vartheta_1 & dz & 0 & 0 & 0 & 0 & 0 & 0 \\ de_1 & 0 & dt & dz & 0 & 0 & 0 & 0 \\ Pf_2 & 0 & 0 & 0 & 1 & M_2 & M_2 & M_2^2-1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ d\vartheta_2 & 0 & 0 & 0 & dt & dz & 0 & 0 \\ de_2 & 0 & 0 & 0 & 0 & 0 & dt & dz \end{bmatrix}=0 \tag{4}$$

Expanded by minors about the left column, this determinant becomes

$$\begin{aligned} &Pf_1(-dz)(-1)dz\,D_2+d\vartheta_1(-1)\left[2M_1dz-dt\left(M_1^2-1\right)\right]D_2 \\ &-de_1(dz)(1)\left(M_1^2-1\right)D_2=0 \end{aligned} \tag{5}$$

Thus, so long as $D_2 \neq 0$ (not on the second characteristic equation) Eq. 5 reduces to

$$d\vartheta_1\left[2M_1\frac{dz}{dt}-\left(M_1^2-1\right)\right]+\left(M_1^2-1\right)\frac{dz}{dt}de_1 = Pf_1\frac{(dz)^2}{dt}$$

In view of Eq. 2, this becomes

$$d\vartheta_1\left(\frac{dz}{dt}\right)^2+\left(M_1^2-1\right)\frac{dz}{dt}de_1 = Pf_1\frac{(dz)^2}{dt} \tag{6}$$

Prob. 11.10.3 (cont.)

Now, using Eq. 5a,

$$d\vartheta_1(M_1 \pm 1)^2 + (M_1 - 1)(M_1 + 1)(M_1 \pm 1)de_1 = Pf_1(M_1 \pm 1)^2 dt \qquad (7)$$

and finally, Eq. 11.10.23 is obtained

$$d\vartheta_1 + (M_1 \mp 1)de_1 = Pf_1 dt \qquad (8)$$

These equations apply on $C_1^{\pm}$ respectively. To recover the IInd characteristics, which apply where $D_2 = 0$ and hence Eq. 4 degenerates, substitute the column on the right in Eq. 1 for the fifth column on the left. The situation is then analogous to the one just considered. The characteristic equations are written with $d\vartheta_1 \rightarrow \Delta\vartheta_{1A}$ on C^+ originating at A, etc. The subscripts A, B, C and D designate the change in the variable along the line originating at the subscript point. The superscripts designate the positive or negative characteristic lines. Thus, Eqs. 11.10.23 and 11.10.25 become the first, second, fifth and sixth of the following eight equations.

$$\begin{bmatrix} 1 & 0 & M_1-1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & M_1+1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & M_2-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & M_2+1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \Delta\vartheta_{1A}^+ \\ \Delta\vartheta_{1B}^- \\ \Delta e_{1A}^+ \\ \Delta e_{1B}^- \\ \Delta\vartheta_{2C}^+ \\ \Delta\vartheta_{2D}^- \\ \Delta e_{2C}^+ \\ \Delta e_{2D}^- \end{bmatrix} = \begin{bmatrix} Pf_1(\xi_{1A}, \xi_{2A})\Delta t \\ Pf_1(\xi_{1B}, \xi_{2B})\Delta t \\ -(\vartheta_{1A} - \vartheta_{1B}) \\ -(e_{1A} - e_{1B}) \\ Pf_2(\xi_{1C}, \xi_{2C})\Delta t \\ Pf_2(\xi_{1D}, \xi_{2D})\Delta t \\ -(\vartheta_{2C} - \vartheta_{2D}) \\ -(e_{2C} - e_{2D}) \end{bmatrix} \qquad (9)$$

Prob. 11.10.3 (cont.)

The third, fourth and last two equations require that

$$\begin{aligned} \vartheta_{1E} &= \vartheta_{1A} + \Delta\vartheta_{1A}^{+} = \vartheta_{1B} + \Delta\vartheta_{1B}^{-}\,;\ \Delta e_{1E} = e_{1A} + \Delta e_{1A}^{+} = e_{1B} + \Delta e_{1B}^{-} \\ \vartheta_{2E} &= \vartheta_{2C} + \Delta\vartheta_{2C}^{+} = \vartheta_{2D} + \Delta\vartheta_{2D}^{-}\,;\ \Delta e_{2E} = e_{2C} + \Delta e_{2C}^{+} = e_{2D} + \Delta e_{2D}^{-} \end{aligned} \tag{10}$$

Clearly, the first four equations are coupled to the second four only through the inhomogeneous terms. Thus, solution for $\Delta\vartheta_{1A}^{+}$ and Δe_{1A}^{+} involves the inversion of the first 4 expressions.

The determinant of the respective 4x4 coefficients are

$$D_1 = -2\ ;\ D_2 = -2 \tag{11}$$

and hence

$$\Delta\vartheta_{1A}^{+} = -\frac{1}{2}\begin{vmatrix} Pf_1(\xi_{1A},\xi_{2A})\Delta t & 0 & M_1-1 & 0 \\ Pf_1(\xi_{1B},\xi_{2B})\Delta t & 1 & 0 & M_1+1 \\ -(\vartheta_{1A}-\vartheta_{1B}) & -1 & 0 & 0 \\ -(e_{1A}-e_{1B}) & 0 & 1 & -1 \end{vmatrix}$$

$$= -\frac{1}{2}\begin{vmatrix} Pf_1(\xi_{1A},\xi_{2A})\Delta t & 0 & M_1-1 & 0 \\ Pf_1(\xi_{1B},\xi_{2B})\Delta t-(\vartheta_{1A}-\vartheta_{1B}) & 0 & 0 & M_1+1 \\ -(\vartheta_{1A}-\vartheta_{1B}) & -1 & 0 & 0 \\ -(e_{1A}-e_{1B}) & 0 & 1 & -1 \end{vmatrix} \tag{12}$$

$$= -\frac{1}{2}\Big[-Pf_1(\xi_{1A},\xi_{2A})\Delta t\,(M_1+1) + Pf_1(\xi_{1B},\xi_{2B})(M_1-1)\Delta t - (\vartheta_{1A}-\vartheta_{1B})(M_1-1) - (e_{1A}-e_{1B})(M_1-1)(M_1+1)\Big]$$

Prob. 11.10.3 (cont.)

which is Eq. 11.10.27. Similarly,

$$\Delta e_{1A}^{+} = -\frac{1}{2}\begin{vmatrix} 1 & 0 & Pf_{1A}\Delta t & 0 \\ 0 & 1 & Pf_{1B}\Delta t & M_1+1 \\ 1 & -1 & -(\vartheta_{1A}-\vartheta_{1B}) & 0 \\ 0 & 0 & -(e_{1A}-e_{1B}) & -1 \end{vmatrix}$$

$$= -\frac{1}{2}\begin{vmatrix} 0 & 1 & Pf_{1A}\Delta t+(\vartheta_{1A}-\vartheta_{1B}) & 0 \\ 0 & 1 & Pf_{1B}\Delta t & M_1+1 \\ 1 & -1 & -(\vartheta_{1A}-\vartheta_{1B}) & 0 \\ 0 & 0 & -(e_{1A}-e_{1B}) & -1 \end{vmatrix} \qquad (\underline{13})$$

$$= -\frac{1}{2}\begin{vmatrix} 0 & Pf_{1A}\Delta t+(\vartheta_{1A}-\vartheta_{1B})-Pf_{1B}\Delta t & -(M_1+1) \\ 1 & Pf_{1B}\Delta t & M_1+1 \\ 0 & -(e_{1A}-e_{1B}) & -1 \end{vmatrix}$$

$$= \frac{1}{2}\Big[-P(f_{1A}-f_{1B})\Delta t - (\vartheta_{1A}-\vartheta_{1B}) - (M_1+1)(e_{1A}-e_{1B})\Big]$$

which is the same as Eq. 11.10.28.

The expressions for $\Delta\vartheta_{2c}^{+}$ and Δe_{2c}^{+} are found in the same way from the second set of 4 equations rather than the first. The calculation is the same except that $A \to C$, $B \to D$, $1 \to 2$ and $2 \to 1$.

Prob. 11.11.1 In the long-wave limit, the magnetic field intensity above and below the sheet is given by the statement of flux conservation

$$\mu_o H_z(a \mp \xi) = -\mu_o H_o a \pm A_d(t) \tag{1}$$

Thus, the x-directed force per unit area on the sheet is

$$T = -\frac{1}{2}\mu_o [\![H_z^2]\!] = -\frac{1}{2}\mu_o\left[\frac{(-\mu_o H_o a + A_d)^2}{\mu_o^2(a-\xi)^2} - \frac{(-\mu_o H_o a - A_d)^2}{\mu_o^2(a+\xi)^2}\right] \tag{2}$$

This expression is linearized to obtain

$$\begin{aligned} T \cong -\frac{1}{2}\frac{1}{\mu_o}\Big\{&[(-\mu_o H_o a)^2 + 2(-\mu_o H_o a)A_d][\frac{1}{a^2} + \frac{\xi}{a^3}] \\ &- [(-\mu_o H_o a)^2 + 2(-\mu_o H_o a)(-A_d)][\frac{1}{a^2} - \frac{\xi}{a^3}]\Big\} \end{aligned} \tag{3}$$

$$\cong \frac{2 H_o A_d}{a} - 2\mu_o H_o^2 \frac{\xi}{a}$$

Thus, the equation of motion for the sheet is

$$\Delta\rho\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial z}\right)^2 \xi = 2\gamma\frac{\partial^2 \xi}{\partial z^2} - 2\mu_o H_o^2 \frac{\xi}{a} + \frac{2 H_o A_d}{a} \tag{4}$$

Normalization such that

$$t = \underline{t}\,\tau \;, \quad z = \underline{z}\,\tau V \;, \quad V \equiv \sqrt{2\gamma/\Delta\rho} \tag{5}$$

gives

$$\left(\frac{\partial}{\partial \underline{t}} + \frac{U}{V}\frac{\partial}{\partial \underline{z}}\right)^2 \underline{\xi} = \frac{2\gamma\tau^2}{\Delta\rho(\tau V)^2}\frac{\partial^2 \underline{\xi}}{\partial \underline{z}^2} - \frac{2\mu_o H_o^2 \tau^2 \underline{\xi}}{\Delta\rho a} + \frac{2\mu_o H_o^2 \tau^2 A_d}{\Delta\rho a \mu_o H_o a} \tag{6}$$

which becomes the desired result, Eq. 11.11.3

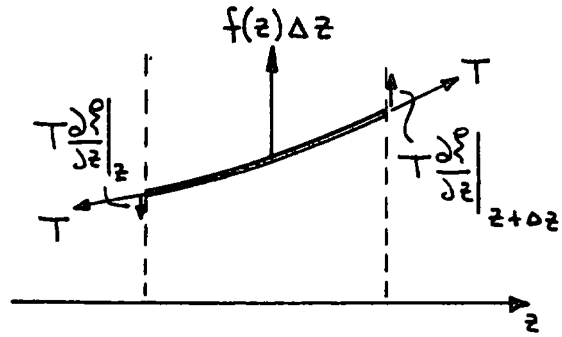
$$\left(\frac{\partial}{\partial t} + M\frac{\partial}{\partial z}\right)^2 \xi = \frac{\partial^2 \xi}{\partial z^2} + P\xi - Pf \tag{7}$$

where

$$\underline{P} = -\frac{2\mu_o H_o^2 \tau^2}{\Delta\rho a} \;; \quad M = \frac{U}{V} \;; \quad f = A_d/\mu_o H_o a$$

Prob. 11.11.2 The transverse force equation for the "wire" is written by considering the incremental length Δz shown in the figure

$$\Delta z\, m \frac{\partial^2 \xi}{\partial t^2} = T\left[\frac{\partial \xi}{\partial z}\bigg|_{z+\Delta z} - \frac{\partial \xi}{\partial z}\bigg|_{z}\right] + f(z)\Delta z \quad (1)$$



Divided by Δz and in the limit $\Delta z \to 0$, this expression becomes

$$m\frac{\partial^2 \xi}{\partial t^2} = T\frac{\partial^2 \xi}{\partial z^2} + f(z) \quad (2)$$

The force per unit length is

$$f = (\bar{I} \times \bar{B})_x = I\,\bar{i}_z \times \left[\frac{B_o}{d}(y\,\bar{i}_x + x\,\bar{i}_y)\right]_x = \frac{IB_o}{d}x \quad (3)$$

Evaluated at the location of the wire, $x = \xi$, this expression is inserted into Eq. 2 to give

$$m\frac{\partial^2 \xi}{\partial t^2} = T\frac{\partial^2 \xi}{\partial z^2} + \frac{IB_o}{d}\xi \quad (4)$$

This takes the form of Eq. 11.11.3 with $M = 0$ and $f = 0$ with $t = \underline{t}\,\tau$, $z = \underline{z}\,\tau V$, $V \equiv \sqrt{T/m}$ and

$$P \equiv \frac{IB_o\tau^2}{md} \quad (5)$$

Prob. 11.11.3 The solution is given by evaluating $\hat{A}$ and $\hat{B}$ in Eq. 11.11.9. With the deflection made zero at $z = \ell$, the first of the following two equations is obtained ($\bar{z} = \ell \Rightarrow \underline{z} = \underline{\ell}$ where $\underline{\ell} \equiv \ell/\tau V$)

$$\begin{bmatrix} e^{-jk_1\ell} & e^{-jk_2\ell} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \hat{A} \\ \hat{B} \end{bmatrix} = \begin{bmatrix} 0 \\ \hat{\xi}_d \end{bmatrix} \tag{1}$$

The second assures that $\xi(0,t) = Re\,\hat{\xi}_d\, e^{j\omega_o t}$. Solution for $\hat{A}$ and $\hat{B}$ gives

$$\hat{A} = \frac{-\hat{\xi}_d\, e^{-jk_2\ell}}{e^{-jk_1\ell} - e^{-jk_2\ell}} \quad ; \quad \hat{B} = \frac{\hat{\xi}_d\, e^{-jk_1\ell}}{e^{-jk_1\ell} - e^{-jk_2\ell}} \tag{2}$$

and Eq. 11.11.9 becomes

$$\xi = Re\,\hat{\xi}_d\, \frac{-e^{-jk_1z}\, e^{-jk_2\ell} + e^{-jk_2z}\, e^{-jk_1\ell}}{e^{-jk_1\ell} - e^{-jk_2\ell}}\, e^{j\omega_o t} \tag{3}$$

With the definitions

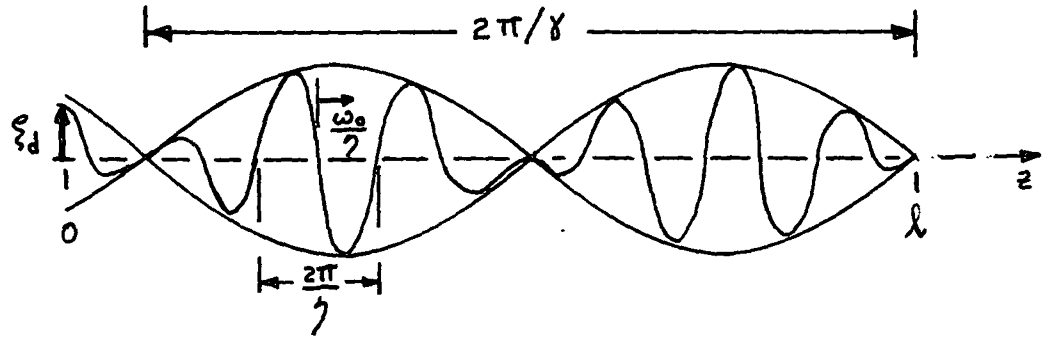
$$k_{1,2} = \eta \pm \gamma \;;\; \eta \equiv \frac{\omega_o M}{M^2 - 1} \;;\; \gamma = \frac{\sqrt{\omega_o^2 + P(1 - M^2)}}{M^2 - 1} \tag{4}$$

Eq. 3 is written as Eq. 11.11.13

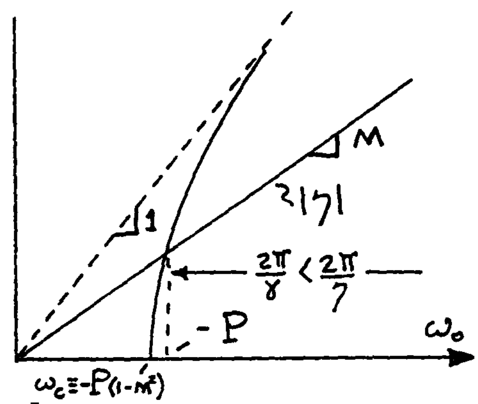
$$\xi = -Re\,\hat{\xi}_d \frac{\left[e^{-j(\gamma z - \gamma\ell)} - e^{j(\gamma z - \gamma\ell)}\right]}{e^{-j\gamma\ell} - e^{j\gamma\ell}}\, e^{j(\omega_o t - \eta z)} = -Re\,\hat{\xi}_d \frac{\sin\gamma(z-\ell)}{\sin\gamma\ell}\, e^{j(\omega_o t - \eta z)} \tag{5}$$

For $\omega_o^2 > P\,(M^2 - 1)$ (sub-magnetic, $P < 0$ and $M^2 < 1$), γ is real. The deflection is then as sketched

Prob. 11.11.3 (cont.)

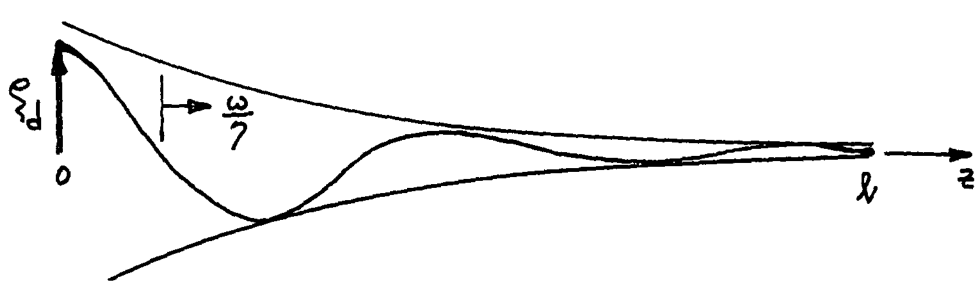


Note that for $M^2 < 1$, $\eta < 1$ and the phases propagate in the -z direction. The picture is for the wavelength of the envelope greater than that of the propagating wave ($2\pi/\gamma > 2\pi/\eta \Rightarrow |\gamma| < |\eta|$). The relationship of wavelengths depends on ω_o , as shown in the figure, and is as sketched in the frequency range $\omega_c < \omega_o < \sqrt{-P}$. For frequencies $\omega_o > \sqrt{-P}$, the deflections are more complex to picture because the wavelength of the envelope is shorter than that of the traveling wave. With the frequency below cut-off, γ becomes imaginary. Let $\gamma = j\alpha$ and Eq. 5 becomes



$$\xi = -Re\, \hat{\xi}_d \frac{\sinh \alpha(z-\ell)}{\sinh \alpha\ell} e^{j(\omega_o t - \eta z)} \tag{6}$$

Now, the picture is as shown below



Again, the phases propogate upstream. The decay of the envelope is likely to be so rapid that the traveling wave would be difficult to discern.

Prob. 11.11.4 Solutions have the general form of Eq. 11.11.9 where $\hat{f} = 0$.

$$\xi = Re\left(\hat{A}\, e^{-jk_1 z} + \hat{B}\, e^{-jk_2 z}\right) e^{j\omega_o t} \tag{1}$$

Thus

$$\frac{\partial \xi}{\partial z} = Re\left(-jk_1 \hat{A}\, e^{-jk_1 z} - jk_2 \hat{B}\, e^{-jk_2 z}\right) e^{j\omega_o t} \tag{2}$$

and the boundary conditions that $\xi(0,t) = Re\, \hat{\xi}_d\, e^{j\omega_o t}$ and $\partial\xi/\partial z$ evaluated at $z = 0$ be zero require that

$$\begin{bmatrix} 1 & 1 \\ -jk_1 & -jk_2 \end{bmatrix} \begin{bmatrix} \hat{A} \\ \hat{B} \end{bmatrix} = \begin{bmatrix} \hat{\xi}_d \\ 0 \end{bmatrix} \tag{3}$$

so that

$$\hat{A} = \frac{k_2 \hat{\xi}_d}{k_2 - k_1}\ ; \quad \hat{B} = \frac{-k_1 \hat{\xi}_d}{k_2 - k_1} \tag{4}$$

and Eq. 1 becomes

$$\xi = Re\, \hat{\xi}_d \frac{\left(k_2 e^{-jk_1 z} - k_1 e^{-jk_2 z}\right)}{k_2 - k_1} e^{j\omega_o t} \tag{5}$$

With the definitions

$$k_{1,2} = \eta \pm \gamma\ ; \quad \eta = \frac{\omega_o M}{M^2 - 1}\ ; \quad \gamma = \frac{\sqrt{\omega_o^2 + P(1 - M^2)}}{M^2 - 1} \tag{6}$$

Eq. 5 becomes

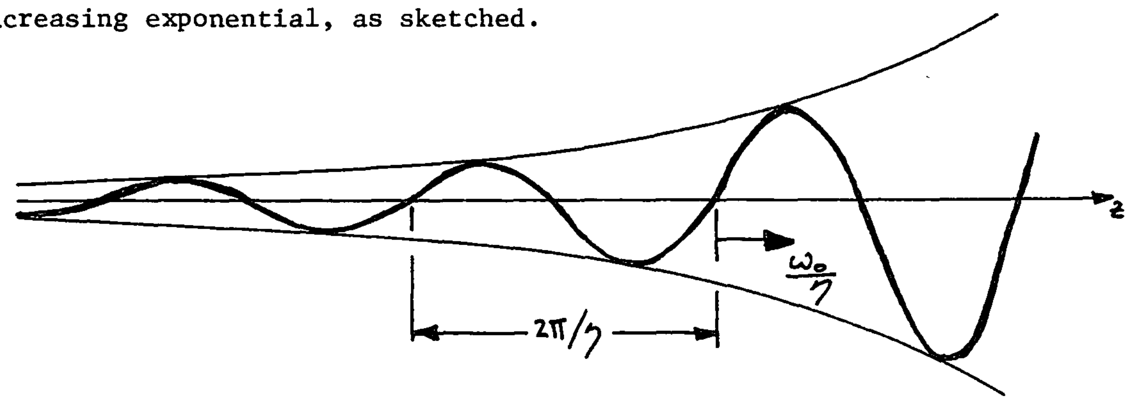
Prob. 11.11.4 (cont.)

$$\xi = \mathcal{R}e\, \hat{\xi}_d \frac{(\mathcal{B}_2 e^{-j\gamma z} - \mathcal{B}_1 e^{j\gamma z})}{-2\gamma} e^{j(\omega_o t - \eta z)} \tag{7}$$

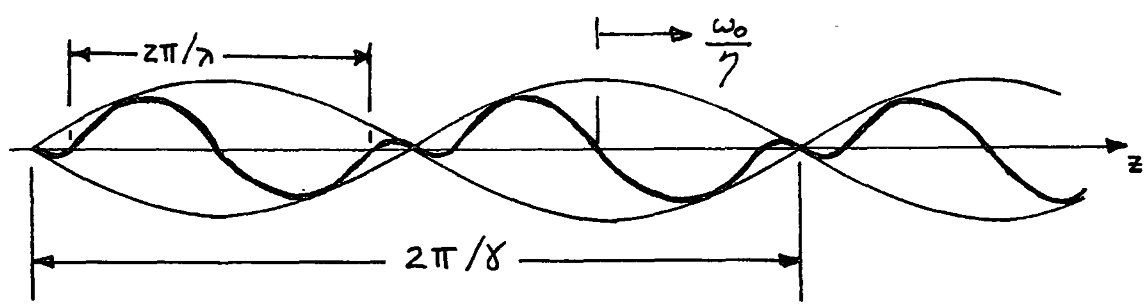
For $\omega_o^2 + P(1 - M^2) < 1$ (super electric below "cut-off") γ is imaginary, $\gamma = j\alpha$. Then, Eq. 7 becomes

$$\xi = \mathcal{R}e\, \hat{\xi}_d \frac{(\mathcal{B}_2 e^{\alpha z} - \mathcal{B}_1 e^{-\alpha z})}{-2j\alpha} e^{j(\omega_o t - \eta z)} \tag{8}$$

Note that the phases propagate downstream with an envelope that eventually is an increasing exponential, as sketched.



This is illustrated by the experiment of Fig. 11.11.5. If the frequency is so high that $\omega_o^2 + P(1 - M^2) > 0$, the envelope is a standing wave



Note that at cut-off, where $\omega_o^2 = P(M^2 - 1)$, the envelope has an infinite wavelength. As the frequency is raised, this wavelength shortens. This is illustrated with $P = 0$ by the experiment of Fig. 11.11.4.

Prob. 11.11.5 (a) The analysis is as described in Prob. 8.13.1 except that there is now a coaxial cylinder. Thus, instead of Eq. 10 from the solution to Prob. 8.13.1, the transfer relation is Eq. (a) of Table 2.16.2 with $\hat{\Phi}^{\alpha} = 0$ because the outer electrode is an equipotential.

$$\hat{e}_r^a = f_m(a, R)\hat{\Phi}^a \tag{1}$$

Then it follows that (m=1)

$$-(\omega - kU)^2 \rho F_1(0, R) = \frac{\epsilon_o E_o^2}{R} - \epsilon_o E_o^2 f_1(a, R) + \frac{\gamma}{R^2}(kR)^2 \tag{2}$$

(b) In the long-wave limit,

$$F_m(0, R) = -\frac{J_m(jkR)}{jk\, J_m'(jkR)} = f_m^{-1}(0, R) \tag{3}$$

and in view of Eqs. 28, for $kR << 1$ and $m = 0$

$$F_1(0, R) \rightarrow -R \tag{4}$$

To take the long-wave limit of $f_1(a, R)$, use Eqs. 2.16.24

$$J_1(ju) \rightarrow \frac{1}{2} ju \; ; \; H_1(ju) \rightarrow \frac{2}{j\pi(ju)}$$

$$J_1'(ju) \rightarrow \frac{1}{2} \; ; \; H_1'(ju) \rightarrow \frac{-2}{j\pi(ju)^2} \tag{5}$$

to evaluate

$$f_1(a, R) \rightarrow \frac{R^2 + a^2}{R^2(a - R)} \tag{6}$$

so that Eq. 2 becomes

$$(\omega - kU)^2 \pi \rho R^2 = \pi \epsilon E_o^2 \left[1 - \frac{R^2 + a^2}{R(a-R)}\right] + \pi R k^2 \tag{7}$$

The equivalent "string" equation is

$$\pi\rho R^2 \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial z}\right)^2 \xi = \pi R \gamma \frac{\partial^2 \xi}{\partial z^2} + \pi \epsilon E_o^2 \left[\frac{R^2 + a^2}{R(a-R)} - 1\right]\xi \tag{8}$$

Normalization, as introduced with Eq. 11.11.3, shows that

$$V = \sqrt{\frac{\gamma}{\rho R}} \; ; \; M = \frac{U}{V} \; ; \; P = \frac{\epsilon E_o^2 \Upsilon^2}{\rho R^2}\left[\frac{R^2 + a^2}{R(a-R)} - 1\right] \tag{9}$$

Prob. 11.12.1 The equation of motion is

$$\frac{\partial^2 \xi}{\partial t^2} = V^2 \frac{\partial^2 \xi}{\partial z^2} + f(z,t) \tag{1}$$

and the temporal and spatial transforms are respectively defined as

$$\overset{\blacktriangle}{\xi}(z,\omega) = \int_{-\infty}^{+\infty} \xi(z,t)\, e^{-j\omega t}\, d\omega \Leftrightarrow \xi(z,t) = \int_{-\infty-j\sigma}^{\infty-j\sigma} \overset{\blacktriangle}{\xi}(z,\omega)\, e^{j\omega t} \frac{d\omega}{2\pi} \tag{2}$$

$$\hat{\xi}(k,\omega) = \int_{-\infty}^{+\infty} \overset{\blacktriangle}{\xi}(z,\omega)\, e^{jkz}\, dk \Leftrightarrow \overset{\blacktriangle}{\xi}(z,\omega) = \int_{-\infty}^{+\infty} \hat{\xi}(k,\omega)\, e^{-jkz} \frac{dk}{2\pi} \tag{3}$$

The excitation force is an impulse of width Δz and amplitude f_o in space

and a cosinusoid that is turned on when t=0.

$$f(z,t) = \Delta z\, u_o(z)\, f_o \cos \omega_o t\; u_{-1}(t) \tag{4}$$

It follows from Eq. 2 that

$$\overset{\blacktriangle}{f}(z,\omega) = \Delta z\, u_o(z) f_o \left[\frac{1}{2j(\omega_o - \omega)} - \frac{1}{2j(\omega_o + \omega)} \right] \tag{5}$$

In turn, Eq. 3 transforms this expression to

$$\hat{f}(k,\omega) = \Delta z\, f_o \left[\frac{1}{2j(\omega_o - \omega)} - \frac{1}{2j(\omega_o + \omega)} \right] \tag{6}$$

With the understanding that this is the Fourier-Laplace transform of f(z,t),

it follows from Eq. 1 that the transform of the response is given by

$$\hat{\xi} = \frac{\hat{f}}{V^2 D(\omega,k)} \tag{7}$$

where

$$D(\omega,k) = k^2 - \left(\frac{\omega}{V}\right)^2 = \left(k - \frac{\omega}{V}\right)\left(k + \frac{\omega}{V}\right) \tag{8}$$

Now, to invert this transform, Eq. 3b is used to write

$$\overset{\blacktriangle}{\xi} = \frac{\Delta z f_o}{2 V^2} \left[\frac{1}{j(\omega_o - \omega)} - \frac{1}{j(\omega_o + \omega)} \right] \int_{-\infty}^{\infty} \frac{e^{-jkz}}{D(\omega,k)} \frac{dk}{2\pi} \tag{9}$$

Prob. 11.12.1 (cont.)

This integration is carried out using the residue theorem

$$\oint_C \frac{N(k)}{D(k)}dk = 2\pi j\left[K_1 + K_2 + \cdots\right]; \; K_n = \frac{N(k_n)}{D'(k_n)} \tag{9}$$

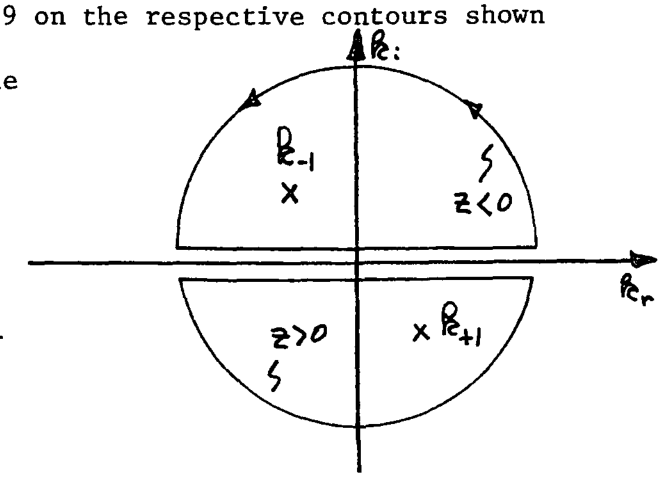
It follows from Eq. 7 that

$$D(\omega, k_n) = 0 \Rightarrow k_n = k_{\substack{1\\-1}} = \pm\frac{\omega}{V} \tag{10}$$

and therefore

$$D'(\omega, k_{\substack{1\\-1}}) = \left(k_{\substack{1\\-1}} + \frac{\omega}{c}\right) + \left(k_{\substack{1\\-1}} - \frac{\omega}{c}\right) = \pm 2\frac{\omega}{V} \tag{11}$$

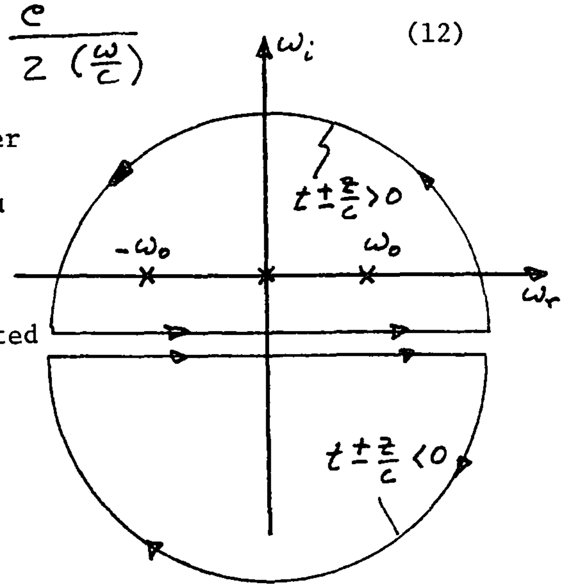
The open integral called for with Eq. 8 is equivalent to the closed contour integral that can be evaluated using Eq. 9 on the respective contours shown in Fig. 11.12.4. Poles, $D(\omega,k)=0$, in the k plane have the locations shown to the right for values of ω on the Laplace contour, because they are given in terms of ω by Eq. 10. The ranges of z associated with the respective contours are those required to make the additional parts of the integral added to make the contours closed ones make zero contribution, Thus, Eq. 8 becomes



$$\hat{\xi} = \frac{\Delta z\, f_0}{2V^2}\left[\frac{1}{\omega_0 - \omega} - \frac{1}{\omega_0 + \omega}\right]\frac{e^{-jk_{\substack{-1\\+1}}z}}{2\left(\frac{\omega}{c}\right)} \tag{12}$$

Here, and in the following discussion, the upper and lower signs respectively refer to $z<0$ and $z>0$.

The Laplace inversion, Eq. 2b, is evaluated using Eq. 12



Prob. 11.12.1 (cont.)

$$\xi(z,t) = \frac{\Delta z f_o}{4V} \int_{-\infty - j\sigma}^{+\infty - j\sigma} \left[\frac{1}{\omega_o - \omega} - \frac{1}{\omega_o + \omega}\right] \frac{e^{\pm j\frac{\omega}{c}t} e^{j\omega t}}{\omega} \frac{d\omega}{2\pi} \tag{13}$$

Choice of the contour used to close the integral is aided by noting that

$$e^{j(\omega t \pm \frac{\omega}{V} z)} = e^{j(\omega_r t \pm \frac{\omega_r}{V} z)} e^{-\omega_i (t \pm \frac{z}{V})} \tag{14}$$

and recognizing that if the addition to the original open integral is to be zero, $t \pm \frac{z}{V} > 0$ on the upper contour and $t \pm \frac{z}{V} < 0$ on the lower one.

The integral on the lower contour encloses no poles (by definition so that causality is preserved) and so the response is zero for

$$t < \mp \frac{z}{V} \tag{15}$$

Conversely, closure in the upper half plane is appropriate for

$$t > \mp \frac{z}{V} \tag{16}$$

By the residue theorem, Eq. 9, Eq. 13 becomes

$$\xi(z,t) = \frac{\Delta z f_o}{4V} \oint_C \Big[\frac{e^{\pm j\frac{\omega}{c}z} e^{j\omega t}}{\underbrace{(\omega_o - \omega)\omega}_{D'(\omega) = -\omega + (\omega_o - \omega)}} - \frac{e^{\pm j\frac{\omega}{c}z} e^{j\omega t}}{\underbrace{(\omega_o + \omega)\omega}_{D'(\omega) = \omega + (\omega_o + \omega)}} \Big] \frac{d\omega}{2\pi}$$

$$= \frac{\Delta z f_o}{4V} j \left[\frac{1}{-\omega_o} e^{j(\omega_o t \pm j\frac{\omega_o}{c} z)} + \frac{1}{\omega_o} - \frac{1}{-\omega_o} e^{-j(\omega_o t \pm \frac{\omega_o}{c} z)} - \frac{1}{\omega_o} \right] \tag{17}$$

This function simplifies to a sinusoidal traveling wave. To encapsulate Eqs. 15 and 16, Eq. 17 is multiplied by the step function

$$\xi(z,t) = \frac{\Delta z f_o}{2V\omega_o} \sin\left[\omega_o \left(t \pm \frac{z}{V}\right) u_{-1}\left(t \pm \frac{z}{V}\right)\right] ; \quad z \lessgtr 0 \tag{18}$$

Prob. 11.12.2 The dispersion equation, without the long-wave approximation, is given by Eq. 8. Solved for ω it gives one root

$$\omega = k + j\frac{k}{U}\tanh k \tag{1}$$

That is, there is only one temporal mode and it is stable. This is sufficient condition to identify all spatial modes as evanescent.

The long-wave limit, if represented by Eq. 11, is not self-consistent. This is evident from the fact that the expression is quadratic in ω and it is clear that an extraneous root has been introduced by the polynomial approximation to the transcendental functions. In fact, two higher order terms must be omitted to make the -k relation self-consistent, and Eq. 5.7.11 becomes

$$k_{\mp} = j\frac{U}{2} \pm \sqrt{-\frac{U^2}{4} - j\omega U} \tag{2}$$

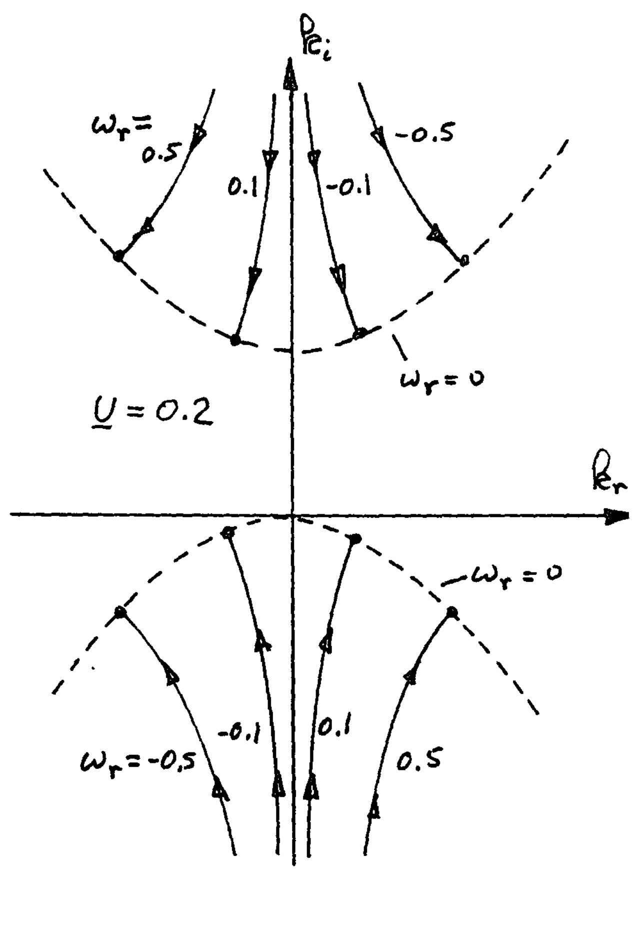
Solved for ω, this expression gives

$$\omega = k\left(1 + \frac{jk}{U}\right) \tag{3}$$

which is directly evident from Eq. 1. To plot the loci of k for fixed values of ω_r as σ goes from ∞ to zero, Eq. 2 is written as

$$k_{\mp} = j\left[\frac{U}{2} \pm \sqrt{\left(\frac{U^2}{4} + \sigma U\right) + j\omega_r U}\right] \tag{4}$$

The loci of k are illustrated by the figure with $U = 0.2$.



Prob.11.13.1 With the understanding that the total solution is the superposition of this solution and one gotten following the prescription of Eq. 11.12.5, the desired limit is

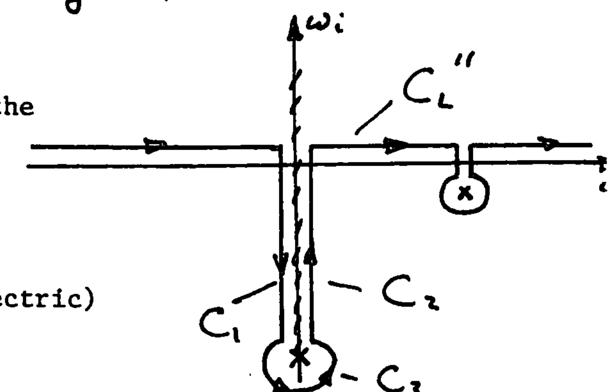
$$\lim_{t\to\infty} \xi(z,t) = \lim_{t\to\infty} \pm \int_{C_L''} \frac{f(\omega)\sum_n j g(k_n)}{D'(\omega,k_n)} e^{j(\omega t - k_n z)} \frac{d\omega}{2\pi} \tag{1}$$

where Eqs. 11.13.8 and 11.13.9 supply

$$f(\omega) = \frac{1}{j(\omega-\omega_o)} \tag{2}$$

$$g(k) = \frac{P\hat{f}_o}{2} \frac{[e^{j(k-\beta)\ell} - 1]}{j(k-\beta)} \tag{3}$$

The contour of integration is shown to the right (Fig. 11.13.4). Calculated here is the response outside the range $z<0,z>\ell$ so that the summation is either n=1 or n=-1. For the particular case where P $>$ 0 and M $<$ 1 (sub-electric) Eq. 11.13.16 is



$$D'(k_\pm) = \mp 2\sqrt{(\omega - j\sigma_s)(\omega + j\sigma_s)}\ ;\ \sigma_s = \sqrt{P(1-M^2)} \tag{4}$$

Note that at the branch point, roots k_n coalesce at k_s in the k plane. From Eq. 11.13.15,

$$k_s = \frac{\omega M}{M^2-1} = \frac{-j\sigma_s M}{M^2-1} \tag{5}$$

as shown graphically by the coalescence of roots in Fig. 11.13.3. As $t\to\infty$, the contributions to the integration on the contour just above the ω_r axis go to zero. ($\omega = \omega_r + j\omega_i$ makes the time dependence of the integrand in Eq. 1 $(\exp j\omega_r t)(\exp -\omega_i t)$ and because $\omega_i > 0$, the integrand goes to zero as $t\to\infty$.) Contributions from the integration around the pole (due to f(ω)) at $\omega = \omega_o$ are finite and hence dominated by the instability now represented by the integration around the half of the branch-cut projecting into the lower half plane.

The integration around the branch-cut is composed of parts C_1 and C_2 paralleling the cut along the imaginary axis and a apart C_3 around the lower branch point. Because D' on C_2 is the negative of that on C_1, and C_1 and C_2

Prob. 11.13.1 (cont.)

are integrations in opposite directions, the contributions on C_1+C_2 are twice that on C_1. Thus, for C_1 and C_2, Eq. 1 is written in terms of $\sigma(\omega = -j\sigma)$

$$-2\int_{\sigma_o}^{\sigma_s} f(-j\sigma)\sum_n \frac{jg(k_n)\, e^{\sigma t}\, e^{-jk_n z}}{2j\sqrt{(\sigma-\sigma_s)(\sigma+\sigma_s)}}\,\frac{(-j\,d\sigma)}{2\pi} \tag{6}$$

In evaluating this expression approximately (for $t\to\infty$) let σ_s be the origin by using $\sigma-\sigma_s$ as a new variable $\sigma^{\#} \equiv -\sigma+\sigma_s \Rightarrow d\sigma = -d\sigma^{\#}$ Then, Eq. 6 becomes

$$\frac{e^{\sigma_s t}}{2\pi}\int_{\sigma_o+\sigma_s}^{0} \frac{f(-j\sigma)\sum_n jg(k_n)e^{-jk_n z}\, e^{-\sigma^{\#}t}\, d\sigma^{\#}}{\sqrt{\sigma^{\#}-2\sigma_s}\,\sqrt{\sigma^{\#}}} \tag{7}$$

Note that $\sigma^{\#} < 0$ as the integration is carried out. Thus, as $t\to\infty$, contributions to the integration are confined to regions where $\sqrt{\sigma^{\#}}\to 0$. The remainder of the integrand, which varies slowly with σ, is approximated by its value at $\sigma=\sigma_s$. Also, σ_o is taken to ∞ so the integral of Eq. 7 becomes ($k_1 \to k_{-1} \to k_s$)

$$\frac{e^{\sigma_s t}\, e^{-jk_s z} f(-j\sigma_s)\, jg(k_s)}{2\pi\sqrt{-2\sigma_s}}\int_{\infty}^{0}\frac{e^{-\sigma^{\#}t}\, d\sigma^{\#}}{\sqrt{\sigma^{\#}}} \tag{8}$$

The definite integration called for here is given in standard tables as

$$-\sqrt{\pi}/\sqrt{t} \tag{9}$$

The integration around the branch point is again in a region where all but the $\sqrt{\omega-j\sigma_s}$ in the denominator is essentially constant. Thus, with $\Omega \equiv \omega + j\sigma_s$, the integration on C_3 of Eq. 1 becomes essentially

$$\lim_{t\to\infty} -\frac{f(j\sigma_s)\, jg(k_s)}{4\pi}\,\frac{e^{-jk_s z}\, e^{-\sigma_s t}}{\sqrt{-2j\sigma_s}}\oint \frac{e^{j\Omega t}}{\sqrt{\Omega}}\, d\Omega \tag{10}$$

Let $\Omega = R\exp j\phi$ and the integral from Eq. 9 becomes

$$\oint_{-3\pi/2}^{\pi/2} \frac{jR\, e^{j\phi}\, e^{j\Omega t}}{\sqrt{R}\, e^{\frac{j\phi}{2}}}\, d\phi = \int_{-3\pi/2}^{\pi/2} j\sqrt{R}\, e^{\frac{j\phi}{2}}\, e^{j\Omega t}\, d\phi \tag{11}$$

In the limit $R\to 0$, this integration gives no contribution. Thus, the asymptotic response is given by the integrations on C_1+C_2 alone.

Prob. 11.13.1 (cont.)

$$\lim_{t\to\infty} \xi(z,t) = -\frac{f(-j\sigma_s)g(k_s)}{2\sqrt{\pi}\sqrt{2\sigma_s}} \frac{e^{\sigma_s t - jk_s z}}{\sqrt{t}} \tag{12}$$

The same solution applies for both $z < 0$ and $\ell < z$. The z dependence in Eq. 11 renders the solution non-symmetric in z. This is the result of the convection, as can be seen from the fact that as $\bar{M} \to 0$, $k_s \to 0$.

Prob. 11.13.2 (a) The dispersion equation is simply

$$(\omega - kU)^2 = V^2k^2 + j\omega\nu \tag{1}$$

Solved for ω, this expression gives the frequency of the temporal modes.

$$\omega = kU + \frac{j\nu}{2} \pm \sqrt{\left(k^2V^2 - \frac{\nu^2}{4}\right) + j\nu kU} \tag{2}$$

Alternatively, Eq. 1 can be normalized such that

$$\underline{\omega} = \omega/\nu,\ M = U/V,\ \underline{k} = kV/\nu \tag{3}$$

and Eqs. 1 and 2 become

$$\begin{gathered}\omega^2 - 2M\omega k + k^2(M^2-1) - j\omega = 0 \\ \omega = Mk + \frac{j}{2} \pm \sqrt{\left(k^2 - \frac{1}{4}\right) + jMk}\end{gathered} \tag{4}$$

To see that $U > V$ ($M > 1$) implies instability, observe that for "small" ν, Eq. 2 becomes

$$\omega = k(U \pm V) + \frac{j\nu}{2}(1 \pm M) \tag{5}$$

Thus, there is an $\omega_i < 0$ if $M > 1$. Another examination of Eq.5 is based on an expansion of M about M=1, showing that instability depends on having $|M|>1$.

Prob. 11.13.2 (cont.)

Complex ω as a function of real k are illustrated in Fig. 11.3.2a.

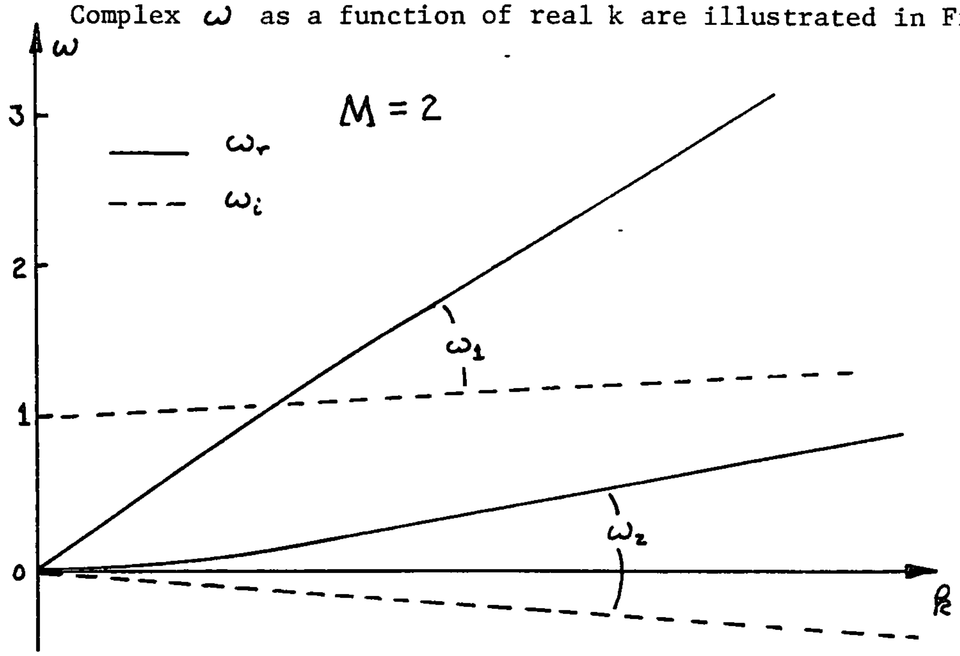


Fig. 11.3.2a

(b) To determine the nature of the instability, Eq. 4 is solved for complex k as a function of $\omega = \omega_r - j\sigma$.

$$k = \frac{M\omega \pm \sqrt{j\omega(M^2-1) + \omega^2}}{M^2-1} \tag{7}$$

or

$$k = \frac{M(\omega_r - j\sigma) \pm \sqrt{[\omega_r^2 - \sigma^2 + \sigma(M^2-1)] + j[\omega_r(M^2-1) - 2\omega_r\sigma]}}{M^2-1} \tag{8}$$

Note that as $\sigma \rightarrow \infty$

$$k \rightarrow \frac{M(\omega_r - j\sigma) \pm j\sigma}{M^2-1} = \frac{M\omega_r - j\sigma(M \pm 1)}{M^2-1} \tag{9}$$

and for $M > 1$ both roots go to $k_i \rightarrow -\infty$. Thus, the loci of complex k for σ varying from $-\infty$ to zero at fixed ω_r move upward through the lower half plane. The two roots to Eq. 7 pass through the k_r axis where ω reaches the values shown in Fig. 11.3.2a. Thus, one of the roots passes into the upper half plane while the other remains in the lower half plane. There is no possibility that they coalesce to form a saddle point, so the instability is convective.

Prob. 11.14.1 (a) Stress equilibrium at the equilibrium interface

$$p^d - p^e = \frac{1}{2}\epsilon E_o^2 \quad ; \quad E_o \equiv V/a \tag{1}$$

In the stationary state,

$$p = \Pi_a - \frac{1}{2}\rho U^2$$
$$p = \Pi_b \tag{2}$$

and so, Eq. (1) requires that

$$\Pi_a - \frac{1}{2}\rho U^2 - \Pi_b = \frac{1}{2}\epsilon E_o^2 \tag{3}$$

All other boundary conditions and bulk relations are automatically satisfied by the stationary state where $\bar{v} = U\bar{i}_z$ in the upper region, $\bar{v} = 0$ in the lower region and

$$p = \begin{cases} \Pi_a - \frac{1}{2}\rho U^2 \\ \Pi_b \end{cases} \tag{4}$$

(b) The alteration to the derivation in Sec. 11.14 comes from the additional electric stress at the perturbed interface. The mechanical bulk relations are again

$$\begin{bmatrix} \hat{p}^c \\ \hat{p}^d \end{bmatrix} = \frac{j(\omega - k_z U)\rho_a}{k} \begin{bmatrix} -\coth ka & \dfrac{1}{\sinh ka} \\ \dfrac{-1}{\sinh ka} & \coth ka \end{bmatrix} \begin{bmatrix} \hat{v}_x^c \\ \hat{v}_x^d \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} \hat{p}^e \\ \hat{p}^f \end{bmatrix} = \frac{j\omega\rho_b}{k} \begin{bmatrix} -\coth kb & \dfrac{1}{\sinh kb} \\ \dfrac{-1}{\sinh kb} & \coth kb \end{bmatrix} \begin{bmatrix} \hat{v}_x^e \\ \hat{v}_x^f \end{bmatrix} \tag{6}$$

The electric field takes the form $\bar{E} = E_o\bar{i}_x + \bar{e}, \bar{e} = -\nabla\Phi$ and perturbations, $\bar{e}$, are represented by

Prob. 11.14.1 (cont.)

$$\begin{bmatrix} \hat{e}_x^c \\ \hat{e}_x^d \end{bmatrix} = k \begin{bmatrix} -\coth ka & \frac{1}{\sinh ka} \\ \frac{-1}{\sinh ka} & \coth ka \end{bmatrix} \begin{bmatrix} \hat{\Phi}^c \\ \hat{\Phi}^d \end{bmatrix} \tag{7}$$

in the upper region. There is no $\bar{E}$ in the lower region.

Boundary conditions reflect mass conservation,

$$\hat{v}_x^d = j(\omega - k_z U)\hat{\xi}, \ \hat{v}_x^e = j\omega\hat{\xi}, \ \hat{v}_x^c = 0, \ \hat{v}_x^f = 0 \tag{8}$$

that the interface and the upper electrode are equipotentials,

$$\begin{bmatrix} \bar{i}_x & \bar{i}_y & \bar{i}_z \\ 1 & -\frac{\partial \xi}{\partial y} & -\frac{\partial \xi}{\partial z} \\ E_o + e_x & e_y & e_z \end{bmatrix} = 0 \Rightarrow e_z^d = -E_o \frac{\partial \xi}{\partial z} \Rightarrow \hat{\Phi}^d = E_o\hat{\xi}; \ \hat{\Phi}^c = 0 \tag{9}$$

and that stress equilibrium prevail in the x direction at the interface

$$-(\rho_a - \rho_b)g\hat{\xi} + \hat{p}^d - \hat{p}^e - E_o\hat{e}_x^d + \gamma k^2\hat{\xi} = 0 \tag{10}$$

The desired dispersion equation is obtained by substituting Eqs. 8 into Eqs. 5b and 6a, and these expressions for $\hat{p}^d$ and $\hat{p}^e$ into Eq. 10, and Eq. 9 into Eq. 7b and the latter into Eq. 10.

$$\hat{\xi}\Big[-(\rho_a - \rho_b)g - \frac{(\omega - k_z U)^2 \rho_a}{k}\coth ka - \frac{\omega^2 \rho_b}{k}\coth kb - \epsilon E_o^2 k \coth ka + \gamma k^2 = 0 \tag{11}$$

To make $\hat{\xi} \neq 0$, the term in brackets must be zero, so

$$[(\omega - k_z U)^2 \rho_a \coth ka/k] + [\omega^2 \rho_b \coth kb/k] = \gamma k^2 + (\rho_b - \rho_a)g - \epsilon E_o^2 k \coth ka \tag{12}$$

This is simply Eq. 11.14.9 with an added term reflecting the self-field-effect of the electric stress. In solving for ω, group this additional term with those due to surface tension and gravity ($\gamma k^2 + (\rho_b - \rho_a)g \rightarrow \gamma k^2 + (\rho_b - \rho_a)g - \epsilon E_o^2 k \coth ka$). It then follows that instability

Prob. 11.14.1 (cont.)

results if (Eq. 11.14.11)

$$U^2 > \left[\frac{\tanh kb}{\rho_b} + \frac{\tanh ka}{\rho_a}\right]\left[\gamma k^2 + g(\rho_b - \rho_a)k - \epsilon E_o^2 k^2 \coth ka\right]\frac{1}{k_z^2} \tag{13}$$

For short waves $(|kb| \gg 1, |ka| \gg 1)$ this condition becomes

$$U^2 > \left[\frac{1}{\rho_b} + \frac{1}{\rho_a}\right]\left[\gamma k + \frac{g(\rho_b - \rho_a)}{k} - \epsilon E_o^2\right] \tag{14}$$

The electric field contribution has no k dependence in this limit, thus making it clear that the most critical wavelength for instability remains the Taylor wavelength

$$k = k^* \equiv \sqrt{\frac{g(\rho_b - \rho_a)}{\gamma}} \tag{15}$$

Insertion of Eq. 15 for k in Eq. 14 gives the critical velocity

$$U^* = \left(\frac{1}{\rho_b} + \frac{1}{\rho_a}\right)\left(2\sqrt{g\gamma(\rho_b - \rho_a)} - \epsilon E_o^2\right) \tag{16}$$

By making

$$\epsilon E_o^2 = 2\sqrt{g\gamma(\rho_b - \rho_a)} \tag{17}$$

the critical velocity becomes zero because the interface is unstable in the Rayleigh-Taylor sense of Secs. 8.9 and 8.10.

In the long-wave limit $(|ka| \ll 1, |kb| \ll 1)$ the electric field has the same effect as gravity. That is $\gamma k^2 + (\rho_b - \rho_a)g \rightarrow \gamma k^2 + [(\rho_b - \rho_a)g - \epsilon E_o^2/a]$ and the k dependence of the gravity and electric field terms is the same.

(c) Because the long-wave field effect can be lumped with that due to gravity, the discussion of absolute vs. convective instability given in Sec. 11.14 pertains directly.

Prob. 11.14.2 (a) This problem is similar to Prob. 11.14.1. The equilibrium pressure is now less above than below, because the surface force density is now down rather than up.

$$\Pi_a - \frac{1}{2}\rho U^2 - \Pi_b = -\frac{1}{2}\mu H_o^2 \tag{1}$$

The analysis then follows the same format except that at the boundaries of the upper region, the conditions are $(\bar{n}\cdot\mu_o\bar{H} = 0)$

$$\left[\bar{i}_x - \frac{\partial\xi}{\partial y}\bar{i}_y - \frac{\partial\xi}{\partial z}\bar{i}_z\right]\left[h_x\bar{i}_x + h_y\bar{i}_y + (H_o + h_z)\bar{i}_z\right] \Rightarrow h_x^d = H_o\frac{\partial\xi}{\partial z} \Rightarrow \hat{h}_x^d = -jk_zH_o\hat{\xi} \tag{2}$$

and

$$\hat{h}_x^c = 0 \tag{3}$$

Thus, the magnetic transfer relations for the upper region are

$$\begin{bmatrix}\hat{\psi}^c\\ \hat{\psi}^d\end{bmatrix} = \frac{1}{k}\begin{bmatrix}-\coth ka & \dfrac{1}{\sinh ka}\\ \dfrac{-1}{\sinh ka} & \coth ka\end{bmatrix}\begin{bmatrix}0\\ -jk_zH_o\hat{\xi}\end{bmatrix} \tag{4}$$

The stress balance for the perturbed interface requires $(\hat{h}_z = jk\hat{\psi})$

$$-(\rho_a - \rho_b)g\hat{\xi} + \hat{p}^d - \hat{p}^e + j\mu H_o k_z\hat{\psi}^d + \gamma k^2\hat{\xi} = 0 \tag{5}$$

Substitution from the mechanical transfer relations for $\hat{p}^d$ and $\hat{p}^e$ (Eqs. 5, 6 and 8 of Prob. 11.14.1) and for $\hat{\psi}^d$ from Eq. 4 gives the desired dispersion equation.

$$\begin{aligned}-(\rho_a-\rho_b)g - \frac{(\omega - k_zU)^2\rho_a}{k}\coth ka - \frac{\omega^2\rho_b}{k}\coth kb\\ + \frac{\mu H_o^2 k_z^2}{k}\coth ka + \gamma k^2 = 0\end{aligned} \tag{6}$$

Thus, the dispersion equation is Eq. 11.14.9 with $\gamma k^2 + g(\rho_b - \rho_a) \rightarrow \gamma k^2 + g(\rho_b - \rho_a) + \mu_o H_o^2 k_z^2 \coth ka/k$. Because the effect of streaming is on perturbations propagating in the z direction, consider $k = k_z$. Then, the problem is the anti-dual of Prob. 11.14.2 (as discussed in Sec. 8.5) and results from Prob. 11.14.1 carry over directly with the substitution $-\epsilon E_o^2 \rightarrow \mu_o H_o^2$.

Prob. 11.14.3 The analysis parallels that of Sec. 8.12. There is now an appreciable mass density to the initially static fluid surrounding the now streaming plasma column. Thus, the mechanical transfer relations are (Table 7.9.1).

$$\begin{bmatrix} \hat{p}^b \\ \hat{p}^c \end{bmatrix} = j\omega\rho_v \begin{bmatrix} F_m(R,a) & G_m(a,R) \\ G_m(R,a) & F_m(a,R) \end{bmatrix} \begin{bmatrix} 0 \\ j\omega\hat{\xi} \end{bmatrix} \tag{1}$$

$$\hat{p}^d = -(\omega - kU)^2 \rho F_m(0,R)\hat{\xi} \tag{2}$$

where substituted on the right are the relations $\hat{v}_r^c = j\omega\hat{\xi}$ and $\hat{v}_r^d = j(\omega - kU)\hat{\xi}$. The magnetic boundary conditions remain the same with $\mathcal{H} = 0$ (no excitation at exterior boundary). Thus, the stress equilibrium equation (Eq. 8.12.10 with $\hat{p}^c$ included)

$$\hat{p}^c - \hat{p}^d = \frac{\mu_o H_t^2}{R}\hat{\xi} - j\mu_o\left(\frac{m}{R}H_t + kH_a\right)\hat{\psi}^c \tag{3}$$

is evaluated using Eqs. 1b, and 2, for $\hat{p}^c$ and $\hat{p}^d$ and Eqs. 8.12.4b, 8.12.7 and $\hat{h}_r^b = 0$ for $\hat{\psi}^c$ to give

$$\begin{aligned} -\omega^2\rho_v F_m(a,R) + (\omega - kU)^2\rho F_m(0,R) \\ = \frac{\mu_o H_t^2}{R} - \mu_o\left(\frac{m}{R}H_t + kH_a\right)^2 F_m(a,R) \end{aligned} \tag{4}$$

This expression is solved for ω.

$$\omega = \frac{-\rho kU F_m(0,R) \pm \left\{\left[\rho_v F_m(a,R) - \rho F_m(0,R)\right]\left[\mu_o\left(\frac{m}{R}H_t + kH_a\right)^2 F_m(a,R) - \frac{\mu_o H_t^2}{R}\right] + k^2 U^2 \rho_v \rho F_m(0,R) F_m(a,R)\right\}^{1/2}}{\rho_v F_m(a,R) - \rho F_m(0,R)} \tag{5}$$

to give an expression having the same form as Eq. 11.14.10

Prob. 11.14.3 (cont.)

$$(F_m(0,R) < 0,\ F_m(a,R) > 0 \quad .$$

The system is unstable for those wavenumbers making the radicand negative, that is for

$$U^2 > \frac{\left[\rho_v F_m(a,R) - \rho F_m(0,R)\right]\left[\mu_o\left(\frac{m}{R}H_t + kH_a\right)^2 F_m(a,R) - \frac{\mu_o H_t^2}{R}\right]}{-k^2 \rho_v \rho\, F_m(0,R)\, F_m(a,R)} \qquad (6)$$

Prob. 11.14.4 (a) The alteration to the analysis as presented in Sec. 8.14 is in the transfer relations of Eq. 8.14.12, which become

$$\begin{bmatrix} \hat{\pi}^c \\ \\ \hat{\pi}^d \end{bmatrix} = j\frac{(\omega - k_z U)\rho_a}{k} \begin{bmatrix} -\coth ka & \frac{1}{\sinh ka} \\ \\ \frac{-1}{\sinh ka} & \coth ka \end{bmatrix} \begin{bmatrix} 0 \\ \\ j(\omega - k_z U)\hat{\xi} \end{bmatrix} \qquad (1)$$

where boundary conditions inserted on the right require that and $\hat{v}_x^c = 0$, $\hat{v}_x^d = j(\omega - k_z U)\hat{\xi}$. Then evaluation of the interfacial stress equilibrium condition, using Eq. 1, requires that

$$\frac{(\omega - k_z U)^2 \rho_a \coth ka}{k} + \frac{\omega^2 \rho_b \coth kb}{\rho} \qquad (2)$$

$$= g(\rho_b - \rho_a) + E_o(q_a - q_b) + \frac{(q_a - q_b)^2}{\epsilon_o k(\coth ka + \coth kb)}$$

(b) To obtain a temporal mode stability condition, Eq. 2 is solved for ω.

$$\omega = \frac{k_z U \frac{\rho_a \coth ka}{k} \pm \left\{\left[\frac{\rho_a \coth ka}{k} + \frac{\rho_b \coth kb}{k}\right]\left[g(\rho_b - \rho_a) + E_o(q_a - q_b) + \frac{(q_a - q_b)^2}{\epsilon_o k(\coth ka + \coth kb)}\right] - \frac{\rho_a \rho_b}{k^2}\coth kb\, k_z^2 U^2 \coth ka\right\}^{1/2}}{(\rho_a \coth ka + \rho_b \coth kb)/k} \qquad (3)$$

Prob. 11.14.4 (cont.)

Thus, instability results if

$$U^2 > \frac{k\left[\rho_a \coth ka + \rho_b \coth kb\right]\left[g(\rho_b - \rho_a) + E_o(q_a - q_b) + \dfrac{(q_a - q_b)^2}{\epsilon_o k(\coth ka + \coth kb)}\right]}{\rho_a \rho_b k_z^2 \coth kb \coth ka} \tag{4}$$

Prob. 11.15.1 Equations 11.15.1 and 11.15.2 become

$$\left(\frac{\partial}{\partial t} + M\frac{\partial}{\partial z}\right)^2 \xi_1 = \frac{\partial^2 \xi_1}{\partial z^2} + P\xi_1 - \frac{1}{2}P\xi_2 \qquad (1)$$

$$\left(\frac{\partial}{\partial t} - M\frac{\partial}{\partial z}\right)^2 \xi_2 = \frac{\partial^2 \xi_2}{\partial z^2} + P\xi_2 - \frac{1}{2}P\xi_1 \qquad (2)$$

Thus, these relations are written in terms of complex amplitudes as

$$\begin{bmatrix} [-(\omega - Mk)^2 + k^2 - P] & \frac{1}{2}P \\ \frac{1}{2}P & [-(\omega + Mk)^2 + k^2 - P] \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0 \qquad (3)$$

and it follows that the dispersion equation is

$$[(\omega - Mk)^2 - k^2 + P][(\omega + Mk)^2 - k^2 + P] - \frac{P^2}{4} = 0 \qquad (4)$$

Multiplied out and arranged as a polynomial in ω, this expression is

$$\omega^4 + \omega^2[2P - 2k^2(M^2+1)] + [(M^2-1)^2k^4 + 2P(M^2-1)k^2 + P^2\tfrac{3}{4}] = 0 \qquad (5)$$

Similarly, written as a polynomial in k, Eq. 4 is

$$k^4[M^2-1]^2 + k^2[2P(M^2-1) - 2\omega^2(M^2+1)] + [\omega^4 + 2\omega^2P + P^2\tfrac{3}{4}] = 0 \qquad (6)$$

These last two expressions are biquadratic in ω and k respecively, and can be conveniently solved for these variables by using the quadratic formula twice.

$$\omega = \pm\left\{k^2(M^2+1) - P \pm \sqrt{4k^2M^2(k^2-P) + \tfrac{1}{4}P^2}\right\}^{1/2} \qquad (7)$$

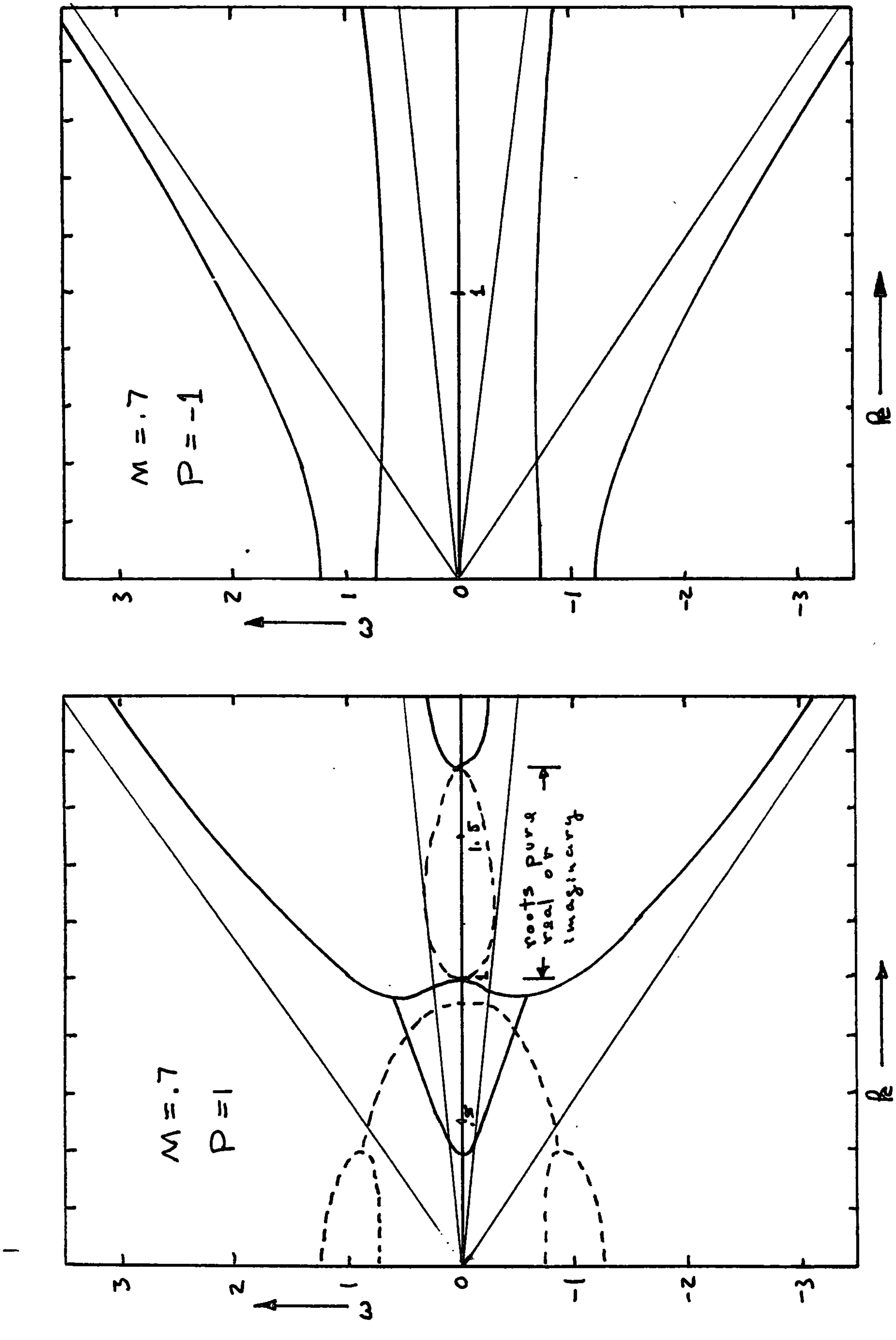
$$k = \pm\left\{\frac{\omega^2(M^2+1) - P(M^2-1) \pm \sqrt{[P(M^2-1) - \omega^2(M^2+1)]^2 - (M^2-1)^2[\omega^4 + 2\omega^2P + \frac{3}{4}P^2]}}{(M^2-1)}\right\}^{\frac{1}{2}} \qquad (8)$$

First, in plotting complex ω for real k, it is helpful to observe that in the limit $k \to \pm\infty$, Eq. 7 takes the asymptotic form

$$\omega \to \pm k(M \pm 1) \qquad (9)$$

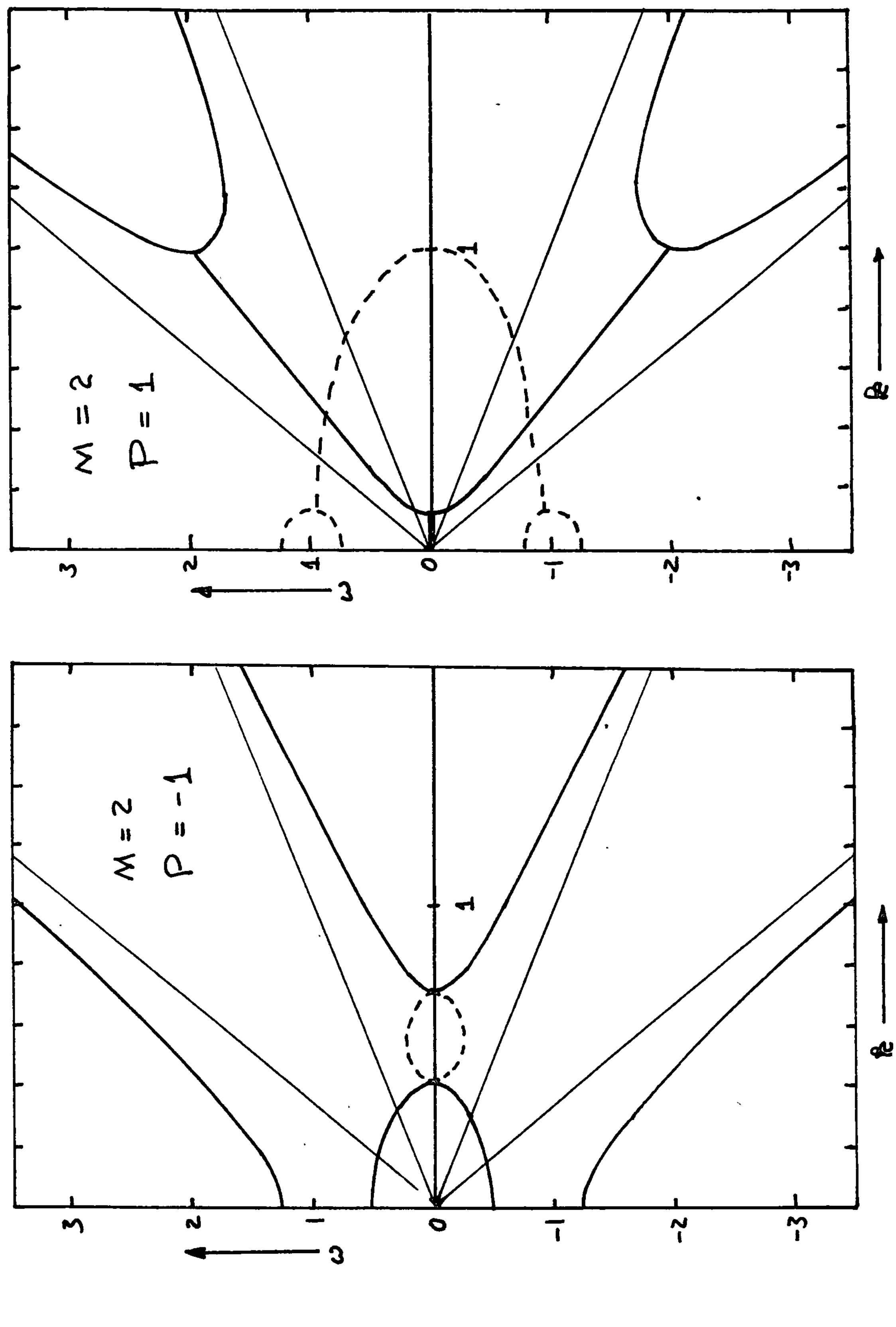
These are shown in the four cases of Fig. 11.15.1a as the light straight lines. Because the dispersion relation is biquadradic in both ω and k, it is clear that for each root given, its negative is also a root. Also, only the complex ω is given as a function of positive k, because the plots must be symmetric in k.

Prob. 11.15.1 (cont.)



Complex ω as a function of real k for subcritical electric-field (P=1) and magnetic-field (P=-1) coupled counterstreaming streams.

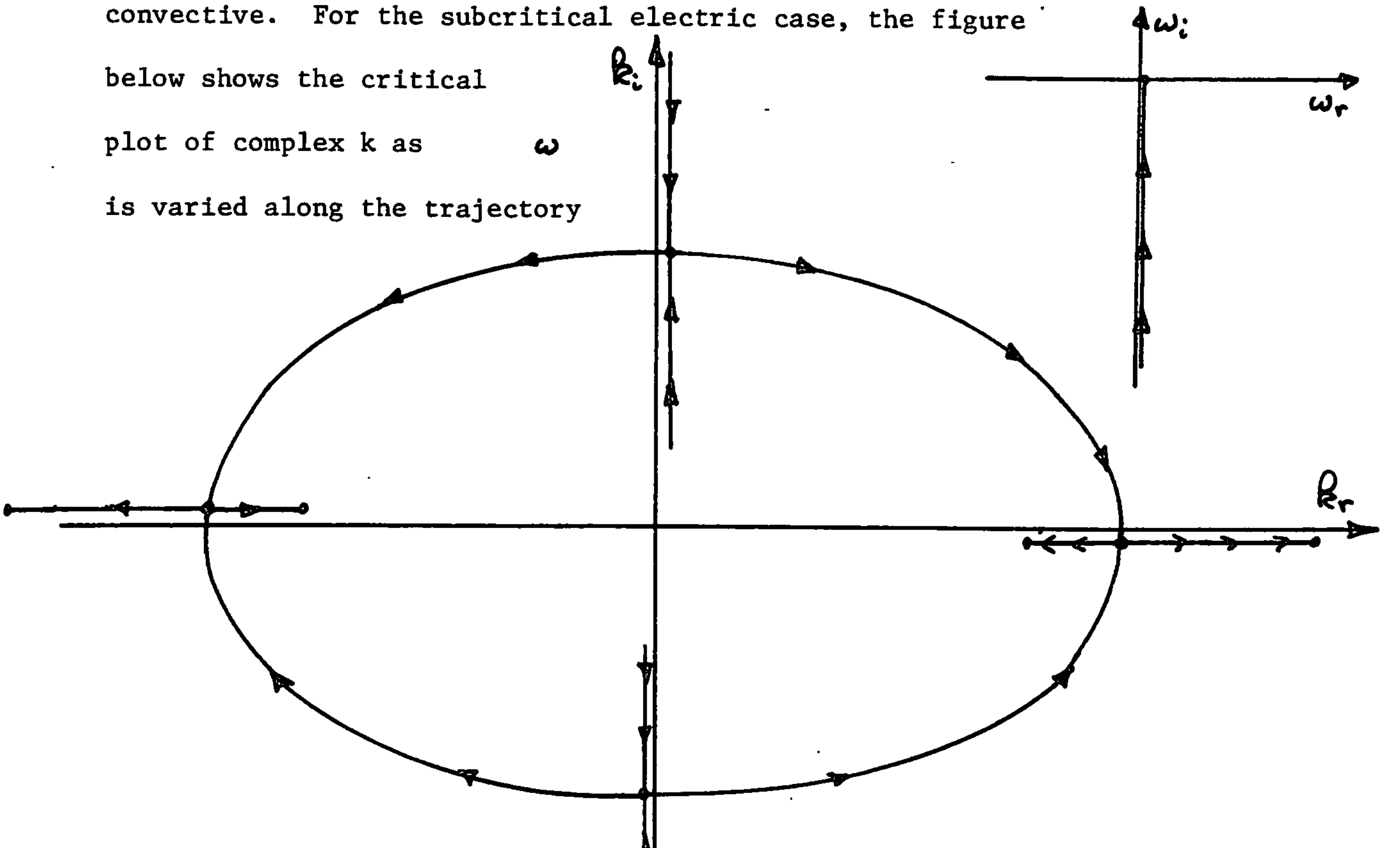
Prob. 11.15.1 (cont.)



Complex ω as a function of real k for supercritical magnetic-field (P=-1) and electric-field (P=1) coupled counterstreaming streams.

Prob. 11.15.1 (cont.)

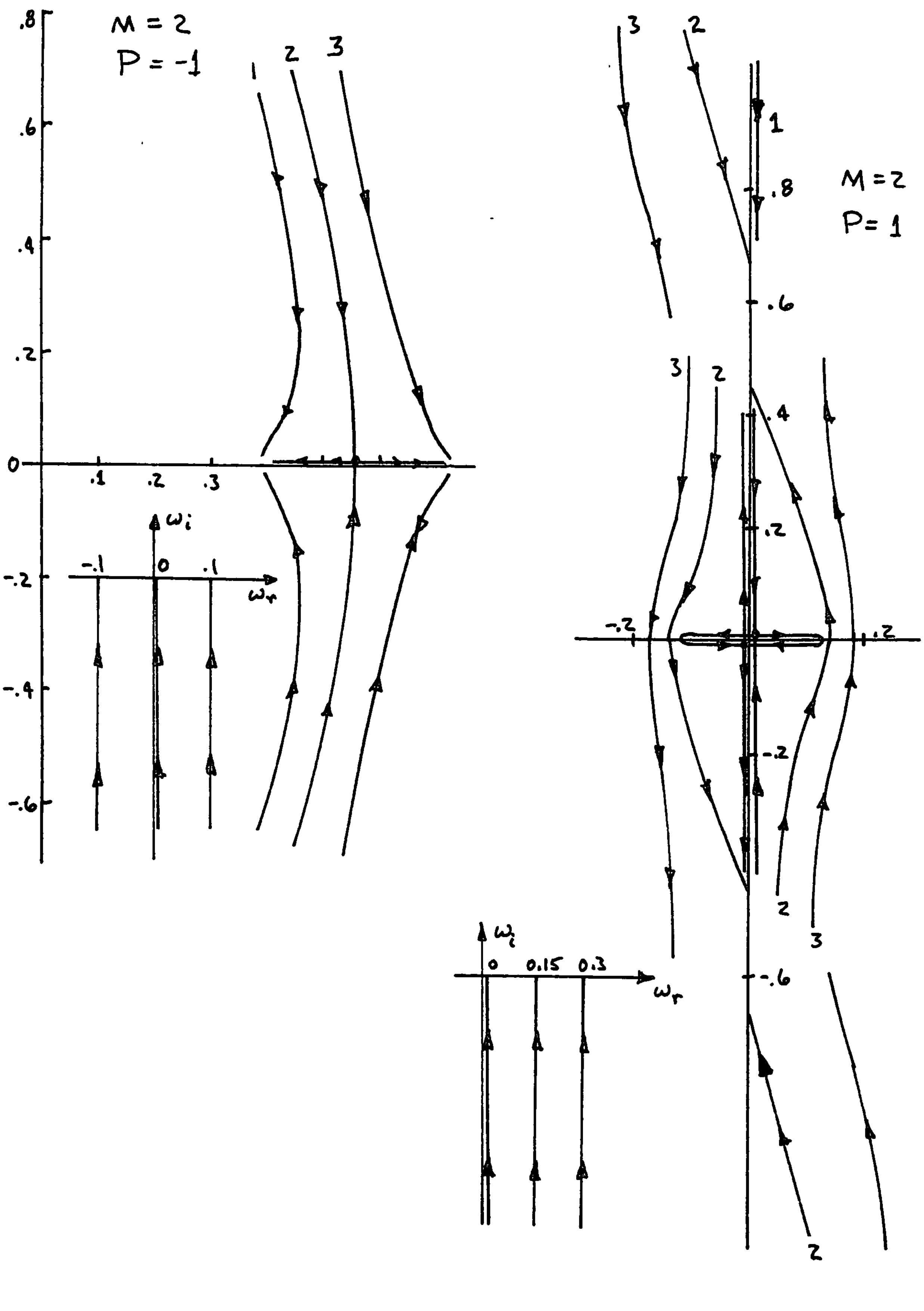
The subcritical magnetic case shows no "unstable" values of ω for real k, so there is no question about whether the instability is absolute or convective. For the subcritical electric case, the figure below shows the critical plot of complex k as ω is varied along the trajectory



shown at the right. The plot makes it clear that the instabilities are absolute, as would be expected from the fact that the streams are subcritical.

Probably the most interesting case is the supercritical magnetic one, because the individual streams then tend to be stable. In the map of complex k shown on the next figure, there are also roots of k that are the negatives of those shown. Thus, there is a branching on the k_r axis at both $k_r \approx .56$ and at $k_r \approx -.56$. Again, the instability is clearly absolute. Finally, the last figure shows the map for a super-electric case. As might be expected, from the fact that the two stable (P=-1) streams become unstable when coupled, this super-electric case is also absolutely unstable.

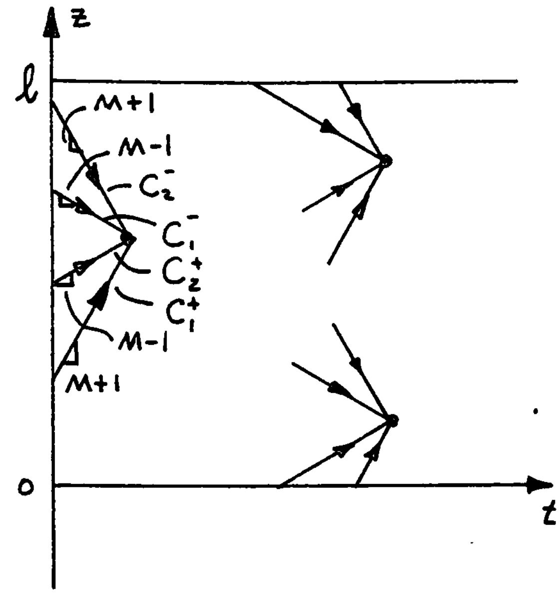
Prob. 11.15.1 (cont.)



Prob. 11.16.1 With homogeneous boundary conditions, the amplitude of an eigenmode is determined by the specific initial conditions. Each eigenmode can be thought of as the response to initial conditions having just the distribution required to excite that mode. To determine that distribution, one of the amplitudes in Eq. 11.16.6 is arbitrarily set. For example, suppose A_1 is given. Then the first three of these equations require that

$$\begin{bmatrix} 1 & 1 & 1 \\ e^{-jk_2 l} & e^{-jk_3 l} & e^{-jk_4 l} \\ Q_2 & Q_3 & Q_4 \end{bmatrix} \begin{bmatrix} A_2 \\ A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} -A_1 \\ -e^{-jk_1 l} A_1 \\ -QA_1 \end{bmatrix} \qquad (1)$$

and the fourth is automatically satisfied because, for each mode, ω is such that the determinant of the coefficients of Eq. 11.16.6 is zero. With A_1 set, A_2, A_3 and A_4 are determined by inverting Eqs. 1. Thus, within a multiplicative factor, namely A_1, the coefficients needed to evaluate Eq. 11.16.2 are determined.



Prob. 11.16.2 (a) With $M_1 = -M_2 = M$ and $|M| < 1$, the characteristic lines are as shown in the figure. Thus, by the arguments given in Sec. 11.10, Causality and Boundary Condition, a point on either boundary has two "incident" characteristics. Thus, two conditions can be imposed at each boundary with the result dynamics that do not require initial conditions implied by subsequent (later) boundary conditions.

The eigenfrequency equation follows from evaluation of the solutions

$$\xi_2 = \mathcal{R}e \sum_{n=1}^{4} A_n e^{-jk_n z} e^{j\omega t} \tag{1}$$

$$\xi_1 = \mathcal{R}e \sum_{n=1}^{4} Q_n A_n e^{-jk_n z} e^{j\omega t} \tag{2}$$

where (from Eq. 11.15.2)

$$Q_n = \frac{2}{\underline{P}}\left[(\omega + Mk)^2 - k^2 + \underline{P}\right] \tag{3}$$

Thus,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ e^{-jk_1 l} & e^{-jk_2 l} & e^{-jk_3 l} & e^{-jk_4 l} \\ Q_1 & Q_2 & Q_3 & Q_4 \\ Q_1 e^{-jk_1 l} & Q_2 e^{-jk_2 l} & Q_3 e^{-jk_3 l} & Q_4 e^{-jk_4 l} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4}$$

Prob. 11.16.2 (cont.)

Given the dispersion equation, $D(\omega, k) \Rightarrow k_n = k_n(\omega)$, this is an eigenfrequency equation.

$$D_e t\ (\omega) = 0 \tag{5}$$

In the limit $M \rightarrow 0$, Eqs. 11.15.1 and 11.15.2 require that

$$\begin{bmatrix} \omega^2 - k^2 + P & -\frac{P}{2} \\ -\frac{P}{2} & \omega^2 - k^2 + P \end{bmatrix} \begin{bmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6}$$

For $\hat{\xi}_1 = \hat{\xi}_2$, both of these equations are satisfied if

$$\omega^2 - k^2 + P\left(1 - \frac{1}{2}\right) = 0 \Rightarrow k_1 = \sqrt{\omega^2 + \frac{P}{2}}\ ,\ k_2 = -\sqrt{\omega^2 + P} \tag{7}$$

and for $\hat{\xi}_1 = -\hat{\xi}_2$,

$$\omega^2 - k^2 + \frac{3}{2}P = 0 \Rightarrow k_3 = \sqrt{\omega^2 + \frac{3}{2}P}\ ,\ k_4 = -\sqrt{\omega^2 + \frac{3}{2}P} \tag{8}$$

and it follows that

$$Q_1 = 1\ ,\ Q_2 = 1\ ,\ Q_3 = -1\ ,\ Q_4 = -1 \tag{9}$$

Thus, in this limit, Eq. 4 becomes

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ e^{-jk_1 l} & e^{jk_1 l} & e^{-jk_2 l} & e^{jk_2 l} \\ 1 & 1 & -1 & -1 \\ e^{-jk_1 l} & e^{jk_1 l} & e^{-jk_2 l} & e^{jk_2 l} \end{bmatrix} = 0 \tag{10}$$

Prob. 11.16.2 (cont.)

and reduces to

$$\sin k_1 \ell \, \sin k_2 \ell = 0 \tag{11}$$

The roots follow from

$$k_1 = \frac{n\pi}{\ell} \quad , \quad k_2 = \frac{m\pi}{\ell} \quad , \quad m = 1,2,3\ldots \tag{12}$$

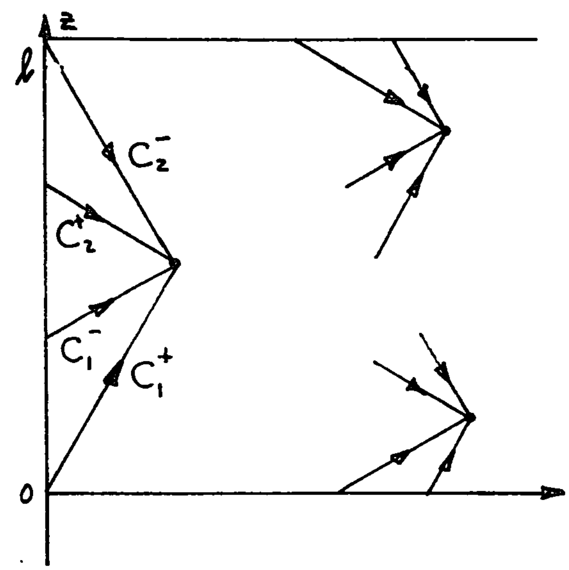
and hence from Eqs. 7 and 8

$$\omega = \pm\sqrt{\left(\frac{n\pi}{\ell}\right)^2 - \frac{P}{2}} \quad ; \quad \omega = \pm\sqrt{\left(\frac{m\pi}{\ell}\right)^2 - \frac{3}{2}P} \tag{13}$$

Instability is incipient in the odd $m=1$ mode when

$$P = \frac{2}{3}\left(\frac{\pi}{\ell}\right)^2 \tag{14}$$

(b) For $M>1$, the characteristics are as shown in the figure. Each boundary has two incident characteristics. Thus, two conditions can be imposed at each boundary. In the limit where $P \to 0$, the streams become uncoupled and it is most likely that conditions would be imposed on the streams where they (and hence their associated characteristics) enter the region of interest.



From Eqs. 11.15.2 and 11.15.5

$$\frac{\partial \xi_2}{\partial z} = Re\sum_{n=1}^{4} -jk_n A_n e^{-jk_n z} e^{j\omega t} \tag{15}$$

$$\frac{\partial \xi_1}{\partial z} = Re\sum_{n=1}^{4} -jk_n Q_n A_n e^{-jk_n z} e^{j\omega t} \tag{16}$$

Prob. 11.16.2 (cont.)

Evaluation of Eqs. 11.15.2, 15, 11.15.5 and 16 at the respective boundaries where the conditions are specified then results in the desired eigen-frequency equation.

$$\begin{bmatrix} Q_1 & Q_2 & Q_3 & Q_4 \\ k_1 Q_1 & k_2 Q_2 & k_3 Q_3 & k_4 Q_4 \\ e^{-jk_1 l} & e^{-jk_2 l} & e^{-jk_3 l} & e^{-jk_4 l} \\ k_1 e^{-jk_1 l} & k_2 e^{-jk_2 l} & k_3 e^{-jk_3 l} & k_4 e^{-jk_4 l} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = 0 \tag{17}$$

Given that $k_n = k_n(\omega)$, the determinant of the coefficients comprises a complex equation in the complex unknown, ω.

Prob. 11.17.1 The voltage and current circuit equations are

$$v(y,t) = \Delta y\, L \frac{\partial i}{\partial t} - n w \Delta y \frac{\partial B_x}{\partial t} + v(y+\Delta y, t) \tag{1}$$

$$i(y,t) = \Delta y\, C \frac{\partial v}{\partial t} + i(y+\Delta y, t) \tag{2}$$

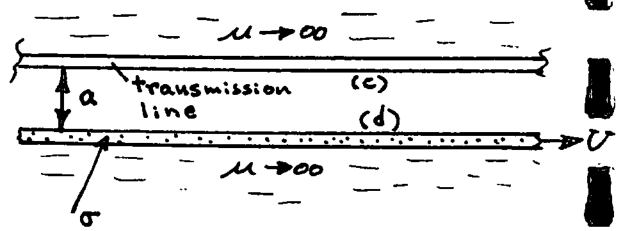
In the limit $\Delta y \to 0$, these become the first two of the given expressions. In addition, the surface current density is given by

$$K_z = \frac{n\, i(y+\Delta y, t) - n\, i(y)}{\Delta y} \tag{3}$$

and in the limit $\Delta y \to 0$, this becomes

$$[\![H_y]\!] = n \frac{\partial i}{\partial y} \tag{4}$$

By Ampere's law, $[\![H_y]\!] = K_z$ and the third expression follows.



Prob. 11.17.2 With amplitudes designated as in the figure, the boundary conditions representing the distributed coils and transmission line (the equations summarized in Prob. 11.17.1) are

$$jk\hat{v} = j\omega L\hat{i} - j\omega n w \hat{B}_x^c \tag{1}$$

$$jk\hat{i} = j\omega C\hat{v} \tag{2}$$

$$-\hat{H}_y^c = -jkn\hat{i} \tag{3}$$

The resistive sheet is represented by the boundary condition of Eq. (a) from Table 6.3.1.

$$-k^2\hat{H}_y^d = -\sigma_s k(\omega - kU)\hat{B}_x^d \tag{4}$$

The air-gap fields are represented by the transfer relations, Eqs. (a), from Table 6.5.1 with $\gamma \to k$.

$$\begin{bmatrix}\hat{H}_y^c\\ \hat{H}_y^d\end{bmatrix} = \frac{j}{\mu_0}\begin{bmatrix}-\coth ka & \dfrac{1}{\sinh ka}\\ \dfrac{-1}{\sinh ka} & \coth ka\end{bmatrix}\begin{bmatrix}\hat{B}_x^c\\ \hat{B}_x^d\end{bmatrix} \tag{5}$$

These expressions are now combined to obtain the dispersion equation. Equations 1 and 2 give the first of the following three equations

$$\begin{bmatrix} j\left(\omega L - \dfrac{k^2}{\omega C}\right) & -jnw\omega & 0\\ -jkn & -\dfrac{j}{\mu_0}\coth ka & \dfrac{j}{\mu_0}\dfrac{1}{\sinh ka}\\ 0 & -\dfrac{j}{\mu_0}\dfrac{1}{\sinh ka} & \dfrac{j}{\mu_0}\coth ka - \dfrac{\sigma_s(\omega - kU)}{k}\end{bmatrix} = \begin{bmatrix}\hat{i}\\ \hat{B}_x^c\\ \hat{B}_x^d\end{bmatrix} \tag{6}$$

The second of these equations is Eq. 5a with $\hat{H}_y^c$ given by Eq. 3. The third is Eq. 5b with Eq. 4 substituted for $\hat{H}_y^d$. The dispersion equation follows from the condition that the determinant of the coefficients vanish.

Prob. 11.17.2 (cont.)

$$(\omega^2 LC - k^2)\left[\frac{\mu_o \sigma_s(\omega - kU)}{k}\coth ka - j\right] + \mu_o k n^2 w \omega^2 C\left[\frac{\mu_o\sigma_s(\omega - kU)}{k} - j\coth ka\right] = 0 \qquad (7)$$

As should be expected, as $n \to 0$ (so that coupling between the transmission line and the resistive moving sheet is removed), the dispersion equations for the transmission line waves and convective diffusion mode are obtained. The coupled system is represented by the cubic obtained by expanding Eq. 7. In terms of characteristic times respectively representing the transite of electromagnetic waves on the line (without the effect of the coupling coils), material transport, magnetic diffusion and coupling,

$$\tau_{em} \equiv a\sqrt{LC},\ \tau_v \equiv \frac{a}{U},\ \tau_m \equiv \mu_o\sigma_s a,\ \tau_c \equiv \sqrt{\mu_o w a C n^2} \qquad (8)$$

and the normalized frequency and wavenumber

$$\omega = \underline{\omega}/\tau_{em},\ k = \underline{k}/a \qquad (9)$$

the dispersion equation is

$$(\omega)^3\left[\frac{\tau_m}{\tau_{em}}\frac{\coth k}{k} + \frac{\tau_c^2\tau_m}{\tau_{em}^3}\right] \qquad (10)$$

$$(\omega)^2\left[-\frac{\tau_m}{\tau_v}\coth k - j - \frac{\tau_c^2\tau_m}{\tau_v\tau_{em}^2}k - j\frac{\tau_c^2}{\tau_{em}^2}k\coth k\right]$$

$$(\omega)\left[-\frac{\tau_m}{\tau_{em}}k\coth k\right] + \left[\frac{\tau_m}{\tau_v}k^2\coth k + j k^2\right] = 0$$

Prob. 11.17.2 (cont.)

The long-wave limit of Eq. 10 is

$$(\omega)^3\left[\frac{\tau_m}{\tau_{em}}+\frac{\tau_c^2\tau_m k^2}{\tau_{em}^3}\right]+(\omega)^2\left[-\frac{\tau_m}{\tau_v}k-jk^2-\frac{\tau_c^2\tau_m}{\tau_v\tau_{em}^2}k^3-j\frac{\tau_c^2k^2}{\tau_{em}^2}\right] \qquad (\underline{11})$$

$$+\omega\left[-\frac{\tau_m}{\tau_{em}}k^2\right]+\left[\frac{\tau_m}{\tau_v}k^3+jk^4\right]=0$$

In the form of a polynomial in k, this is

$$k^4-k^3\left[j\frac{\tau_m}{\tau_v}-j\omega^2\frac{\tau_c^2\tau_m}{\tau_v\tau_{em}^2}\right]$$

$$+k^2\left[\frac{j\omega\tau_m}{\tau_{em}}-\frac{\omega^2\tau_c^2}{\tau_{em}^2}-j\frac{\omega^3\tau_c^2\tau_m}{\tau_{em}^3}-\omega^2\right] \qquad (\underline{12})$$

$$+k\left[j\omega^2\frac{\tau_m}{\tau_v}\right]-\left[j\omega^3\frac{\tau_m}{\tau_{em}}\right]=0$$

where it must be remembered that $\underline{k} << 1$

As would be expected for the coupling of two systems that individually have two spatial modes, the coupled transmission line and convecting sheet are represented by a quartic dispersion equation. The complex values of for real k are shown in Fig. 11.17.2a. One of the three modes is indeed unstable for the parameters used. Note that these are assigned to make the material velocity exceed that of the uncoupled transmission-line wave. It is unfortunate that the system exhibits instability even as k is increased beyond the range of validity for the long-wave approximation $\underline{k}<\underline{1}$. The mapping of complex shown in Fig. 11.17.2b is typical of a convective instability. Note that for $\omega_r=0.5$ the root crosses the k_r axis. Of course, a rigerous proof that there are no absolute instabilities requires considering all possible values of $\sigma>0$.

Prob. 11.17.2a (cont.)

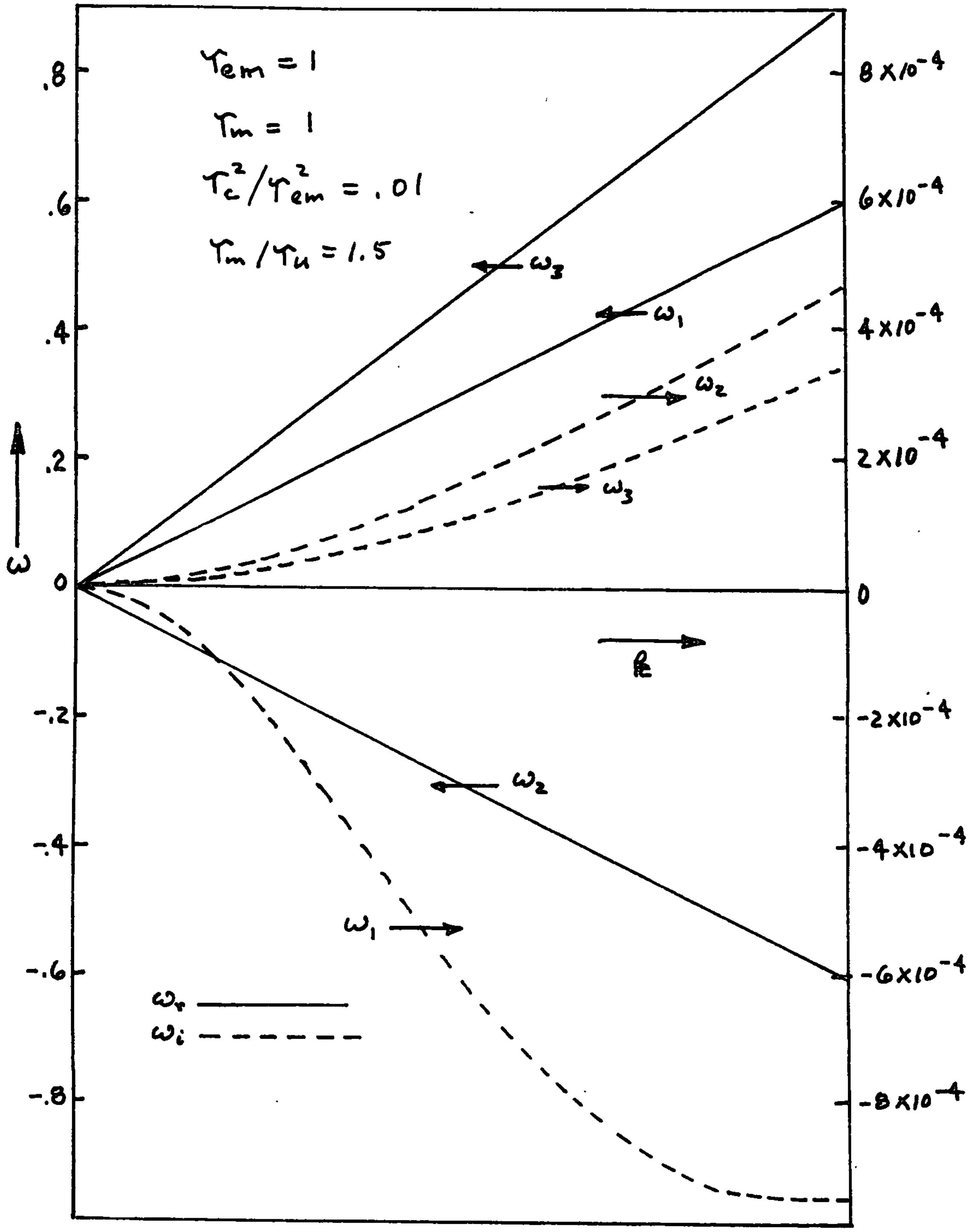


Fig. 11.17.2a Complex ω for real k.

Prob. 11.17.2 (cont.)

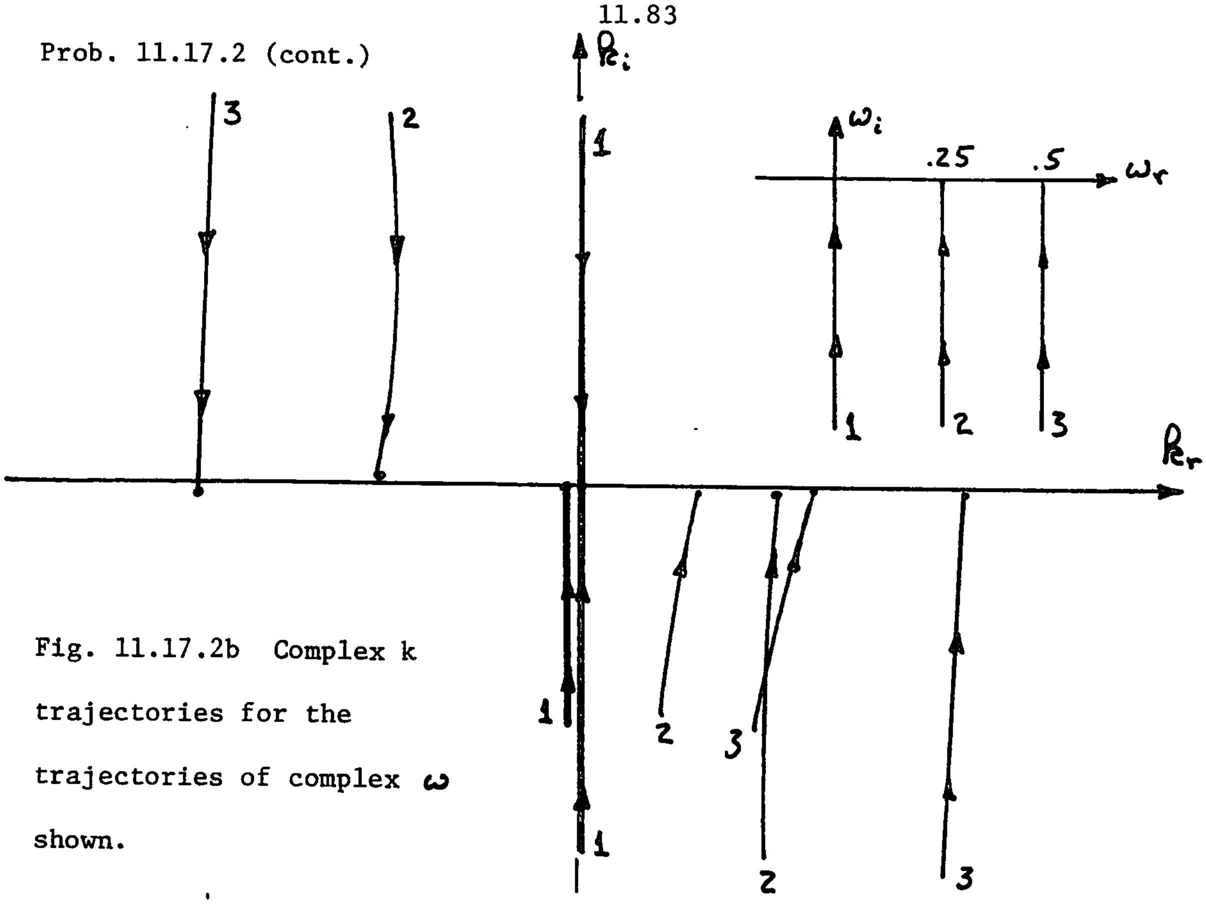


Fig. 11.17.2b Complex k trajectories for the trajectories of complex ω shown.

Prob. 11.17.3 The first relation requires that the drop in voltage across the inductor be

$$v(z) - v(z+\Delta z) = L\,\Delta z\,\frac{\partial i}{\partial t} \tag{1}$$

Divided by Δz and in the limit where $\Delta z \to 0$ this becomes

$$-\frac{\partial v}{\partial z} = L\frac{\partial i}{\partial t} \tag{2}$$

The second requires that the sum of currents into the mode at $z+\Delta z$ be zero.

$$i(z) - i(z+\Delta z) = C\,\Delta z\frac{\partial v}{\partial t} + \frac{\partial}{\partial t}(\sigma_f w\,\Delta z) \tag{3}$$

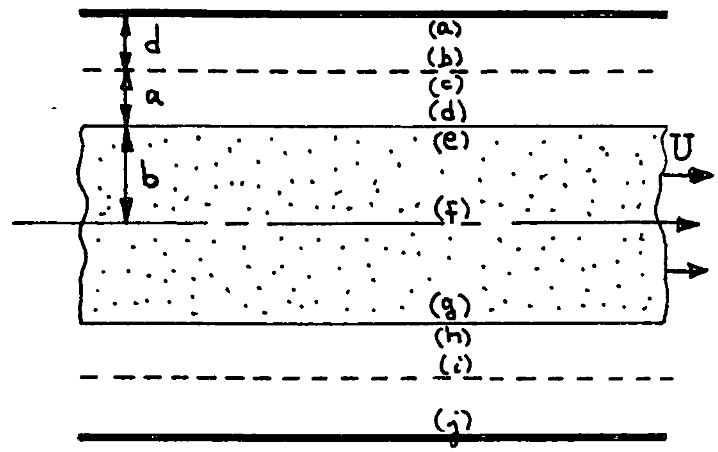
where σ_f is the net charge per unit area on the electrode

$$\sigma_f = [\![D_x]\!] \tag{4}$$

Divided by Δz and in the limit $\Delta z \to 0$, Eq. 3 becomes

$$-\frac{\partial i}{\partial z} = C\frac{\partial v}{\partial t} + w\frac{\partial \sigma_f}{\partial t} \tag{5}$$

Prob. 11.17.4 (a) The beam and air-gaps are represented by Eq. 11.5.11, which is ($k_y = 0$, $k_z = k$)



$$\hat{D}_x^c = \frac{-\epsilon k(k+\gamma \coth ka \tanh \gamma b)\,\hat{\Phi}^c}{k \coth ka + \gamma \tanh \gamma b} \tag{1}$$

$$\gamma^2 \equiv k^2[1-\omega_p^2/(\omega - kU)^2]$$

The transfer relations for the region a-b, with $\hat{\Phi}^a = 0$ require that

$$\hat{D}_x^b = \epsilon k \coth kd\, \hat{\Phi}^b \tag{2}$$

With the recognition that $\hat{v} \to \hat{\Phi}^b = \hat{\Phi}^c$, the traveling-wave structure equations from Prob. 11.17.3 require that

$$jk\hat{\Phi}^c = j\omega L \hat{i} \tag{3}$$

$$jk\hat{i} = j\omega C \hat{\Phi}^c + j\omega w(\hat{D}_x^b - \hat{D}_x^c) \tag{4}$$

The dispersion equation follows from substitution of Eqs. 1 and 2 (for $\hat{D}_x^c$ and $\hat{D}_x^b$) and Eq. 3 (for $\hat{i}$) into Eq. 4.

$$\frac{k^2}{\omega L} = \omega C + \omega w \epsilon k\left[\coth kd + \frac{(k+\gamma \coth ka \tanh \gamma b)}{k \coth ka + \gamma \tanh \gamma b}\right] \tag{5}$$

As a check, in the limit where $L \to \infty$ and $C \to 0$ this expression should be the dispersion relation for the electron beam (D=o in Eq. 11.5.11) with a of that problem replaced by a+d. (This follows by using the identity $(\coth kd + \coth ka)/(\coth ka \coth kd + 1) = \tanh k(a+d)$.)

In taking the long-wave limit of Eq. 5, where $kd \ll 1$, $ka \ll 1$ and $\gamma b \ll 1$,

Prob. 11.17.4 (cont.)

the object is to retain the dominant modes of the uncoupled systems. These are the transmission line and the electron beam. Each of these is represented by a dispersion equation that is quadratic in ω and in k. Thus, the appropriate limit of Eq. 5 should retain terms in ω and k of sufficient order that the resulting dispersion equation for the coupled system is quartic in ω and in k. With $C' \equiv C + w\epsilon/d$, Eq. 5 becomes

$$\left(\frac{k^2}{L} - C'\omega^2\right)\left[\frac{(\omega - kU)^2}{a} - b k^2 \omega_p^2\right] = w\epsilon k^2 \omega^2 \left[(\omega - kU)^2\left(1 + \frac{b}{a}\right) - \frac{b}{a}\omega_p^2\right] \tag{6}$$

With normalization

$$\underline{k} = kb \qquad c^2 = (\omega_p^2 L b^2 C')^{-1}$$

$$\underline{\omega} = \omega/\omega_p$$

$$\underline{U} = U/b\omega_p \qquad K = \frac{w\epsilon}{C'b}$$

this expression becomes

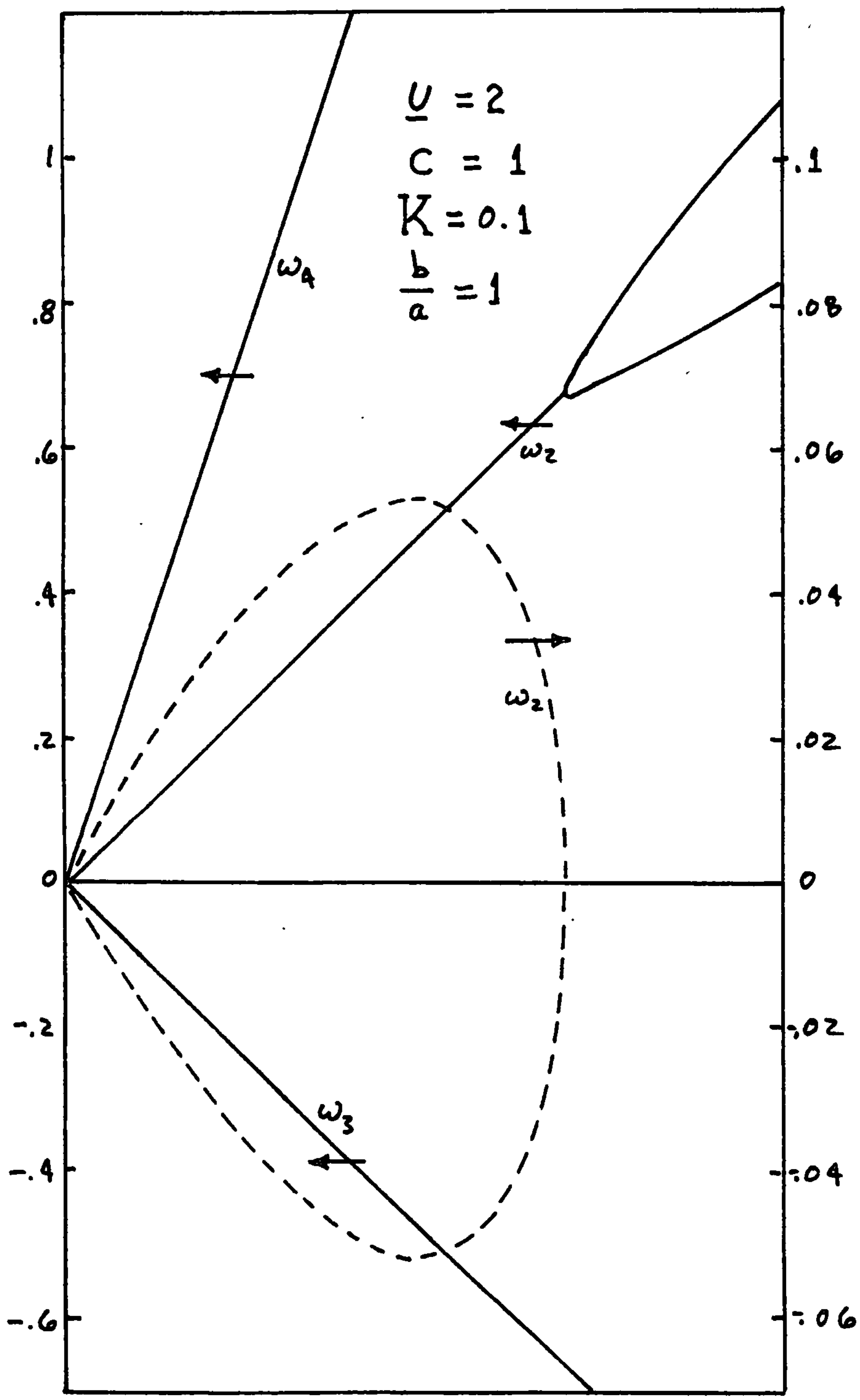
$$(k^2 c^2 - \omega^2)\left[(\omega - kU)^2 \frac{b}{a} - k^2\right] - K k^2 \omega^2 \left[(\omega - kU)^2\left(1 + \frac{b}{a}\right) - \frac{b}{a}\right] = 0 \tag{7}$$

Written as a polynomial in ω, this expression is

$$\left[\frac{b}{a} + K k^2\left(1 + \frac{b}{a}\right)\right]\omega^4 - 2\left[\frac{b}{a}kU + K k^3 U\left(1 + \frac{b}{a}\right)\right]\omega^3 + \left[k^2\frac{b}{a}(U^2 - c^2) - k^2 + K k^4 U\left(1 + \frac{b}{a}\right) - K k^2 \frac{b}{a}\right]\omega^2 + \left[2\frac{b}{a}k^3 U c^2\right]\omega + \left[k^4 c^2\left(1 - U^2 \frac{b}{a}\right)\right] = 0 \tag{8}$$

Prob. 11.17.4 (cont.)

This expression can be numerically solved for ω to determine if the system is unstable, convective or absolute. A typical plot of complex ω for real k, shown in Fig. P11.17.4a, shows that the system is indeed unstable.



Prob. 11.17.4(cont.)

To determine whether the instability is convective or absolute, it is necessary to map the loci of complex k as a function of complex $\omega = \omega_r - j\sigma$. Typical trajectories for the values of ω shown by the inset are shown in Fig. 11.17.4b.

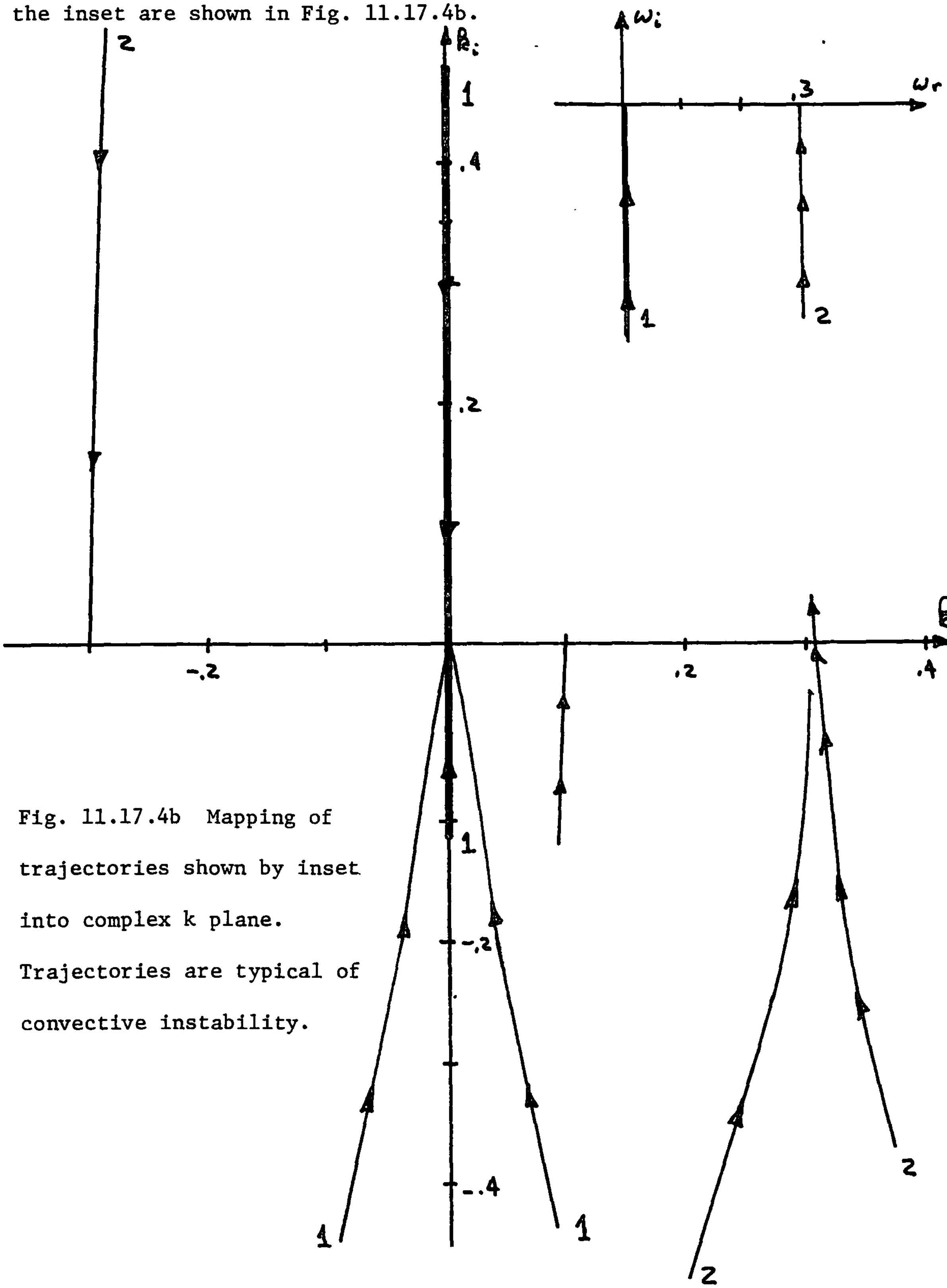
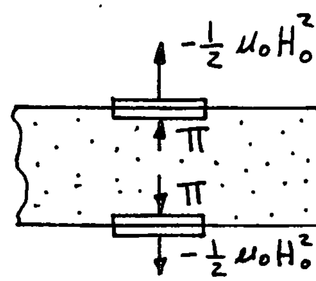


Fig. 11.17.4b Mapping of trajectories shown by inset into complex k plane. Trajectories are typical of convective instability.

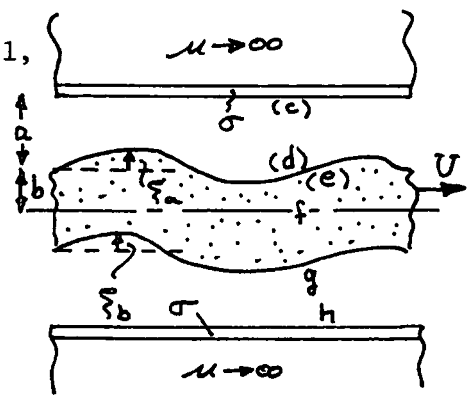
Prob. 11.17.5 (a) In a state of stationary equilibrium, $\bar{v} = U\bar{i}_y$ and $p = \Pi$ = constant, to satisfy mass and momentum conservation conditions in the fluid bulk. Boundary conditions are automatically satisfied, with normal stress equilibrium at the interfaces making

$$\Pi = \frac{1}{2}\mu_o H_o^2 \tag{1}$$



where the pressure in the low mass density media surrounding the jet is taken as zero.

(b) Bulk relations describe the magnetic perturbations in the free-space region and the fluid motion in the stream. From Eqs. (a) of Table 2.16.1, with



$$\bar{H} = H_o\bar{i}_y + \bar{h}\, ; \ \bar{h} = -\nabla\psi \tag{2}$$

$$\begin{bmatrix} \hat{h}_x^c \\ \hat{h}_x^d \end{bmatrix} = k \begin{bmatrix} -\coth ka & \dfrac{1}{\sinh ka} \\ \dfrac{-1}{\sinh ka} & \coth ka \end{bmatrix} \begin{bmatrix} \hat{\psi}^c \\ \hat{\psi}^d \end{bmatrix} \tag{3}$$

and from Table 7.9.1, Eq. (c),

$$\begin{bmatrix} \hat{p}^e \\ \hat{p}^f \end{bmatrix} = j\frac{(\omega - k_y U)}{k} \begin{bmatrix} -\coth kb & \dfrac{1}{\sinh kb} \\ \dfrac{-1}{\sinh kb} & \coth kb \end{bmatrix} \begin{bmatrix} \hat{v}_x^e \\ \hat{v}_x^f \end{bmatrix} \tag{4}$$

Because only the kinking motions are to be described, Eq. 4 has been written with position (f) at the center of the stream. From the symmetry of the system, it can be argued that for the kinking motions the perturbation pressure at the center-plane must valish. Thus, Eq. 4b requires that

$$\hat{v}_x^f = \frac{\hat{v}_x^e}{\sinh kb \ \coth kb} = \frac{\hat{v}_x^e}{\cosh kb} \tag{5}$$

so that Eq. 4a becomes

Prob. 11.17.5 (cont.)

$$\hat{p}^e = j\frac{(\omega - k_y U)\rho}{k}\left(-\coth kb + \frac{1}{\sinh kb \cosh kb}\right)\hat{v}_x^e \tag{6}$$

or

$$\hat{p}^e = -j\frac{(\omega - k_y U)\rho}{k}\tanh kb\,\hat{v}_x^e = \frac{(\omega - kU)^2}{k}\rho \tanh kb\,\hat{\xi}$$

where the last equality introduces the fact that $\hat{v}_x = j(\omega - kU)\hat{\xi}$.

<u>Boundary conditions</u> begin with the resistive sheet, described by Eq. (a) of Table 6.3.1.

$$k_y^2 \hat{h}_y^c = -\sigma_s(-jk_y)(j)\omega\mu_o \hat{h}_x^c \tag{7}$$

which is written in terms of $\hat{\psi}^c$ as $(\hat{h}_y = jk_y\hat{\psi})$.

$$\hat{\psi}^c = \frac{j\mu_o\sigma_s}{k_y^2}\omega \hat{h}_x^c \tag{8}$$

At the perfectly conducting interface, $\left(\bar{n} \simeq \bar{i}_x - \frac{\partial\xi}{\partial y}\bar{i}_y - \frac{\partial\xi}{\partial z}\bar{i}_z\right)$

$$\bar{n}\cdot\bar{H} = 0 \Rightarrow \hat{h}_x^d + jk_y H_o\hat{\xi} = 0 \tag{9}$$

Stress equilibrium for the perturbed interface is written for the x component, with the others identically satisfied to first order because the interface is free of shear stress. From Eq. 7.7.6 with $i \to x$

$$[\![p]\!]n_x = [\![T_{xj}]\!]n_j - \gamma(\nabla\cdot\bar{n})n_x \tag{10}$$

Linearization gives

$$-\hat{p}^e = -\mu_o H_o \hat{h}_y^d - \gamma k^2\hat{\xi} \tag{11}$$

where Eq. (d) of Table 7.6.2 has been used for the surface tension term.

With $\hat{h}_y = jk\hat{\psi}$, Eq. 5 becomes

$$\hat{p}^e = jk_y\mu_o H_o\hat{\psi}^d + \gamma k^2\hat{\xi} \tag{12}$$

Now, to combine the boundary conditions and bulk relations, Eq. 8 is expressed using Eq. 3a as the first of the three relations

Prob. 11.17.5 (cont.)

$$\begin{bmatrix} 1+\dfrac{j\mu_o\sigma_s k}{k_y^2}\omega\coth ka & \dfrac{-j\mu_o k\sigma_s\omega}{k_y^2\sinh ka} & 0 \\ \dfrac{-k}{\sinh ka} & k\coth ka & jk_y H_o \\ 0 & -jk_y\mu_o H_o & \dfrac{(\omega-k_yU)^2}{k}\rho\tanh kb-\gamma k^2 \end{bmatrix}\begin{bmatrix}\hat{\psi}^c\\ \hat{\psi}^d\\ \hat{\xi}\end{bmatrix}=0 \tag{13}$$

The second is Eq. 9 with $\hat{H}_x^d$ expressed using Eq. 3b. The third is Eq. 12 with $\hat{p}^e$ given by Eq. 6.

Expansion by minors gives

$$-k_y^2H_o^2\mu_o\left[1+\frac{j\mu_o\sigma_s k}{k_y^2}\omega\coth ka\right]+$$

$$k\left[\frac{(\omega-k_yU)^2}{k}\rho\tanh kb-\gamma k^2\right]\left[\coth ka+\frac{j\mu_o\sigma_s k\omega}{k_y^2}\right]=0 \tag{14}$$

Some limits of interest are:

$H_o\to 0$ so that mechanics and magnetic diffusion are uncoupled. Then, Eq. 14 factors into dispersion equations for the capillary jet and the magnetic diffusion

$$(\omega-k_yU)^2=\gamma k^3/\rho\tanh kb \tag{15}$$

$$\omega=\frac{jk_y^2}{\mu_o\sigma_s k}\coth ka \tag{16}$$

The latter gives modes similar to those of Sec. 6.10 except that the wall opposite the conducting sheet is now perfectly conducting rather than

Prob. 11.17.5 (cont.)

infinitely permeable.

$\sigma \to \infty$, so that Eq. 14 can be factored into the dispersion equations

$$\omega = \tag{17}$$

$$(\omega - k_y U)^2 \rho \tanh kb = \gamma k^3 + k_y^2 \mu_o H_o^2 \coth ka \tag{18}$$

This last expression agrees with the kink mode dispersion equation (with $\gamma \to 0$) of Prob. 8.12.1.

In the long-wave limit, coth $ka \to 1/ka$, tanh $kb \to kb$ and Eq. 14 becomes

$$-k_y^2 \frac{\mu_o H_o^2}{k}\left(1 + j\frac{\mu_o \sigma_s}{k_y^2 a}\omega\right) + \left[(\omega - k_y U)^2 \rho b - \gamma k^2\right]\left[\frac{1}{ka} + j\frac{\mu_o \sigma_s \omega}{k_y^2}\right] = 0 \tag{19}$$

In general, this expression is cubic in ω. However, with interest limited to frequencies such that

$$ka \Big/ \frac{\mu_o \sigma_s \omega}{k} << 1 \tag{20}$$

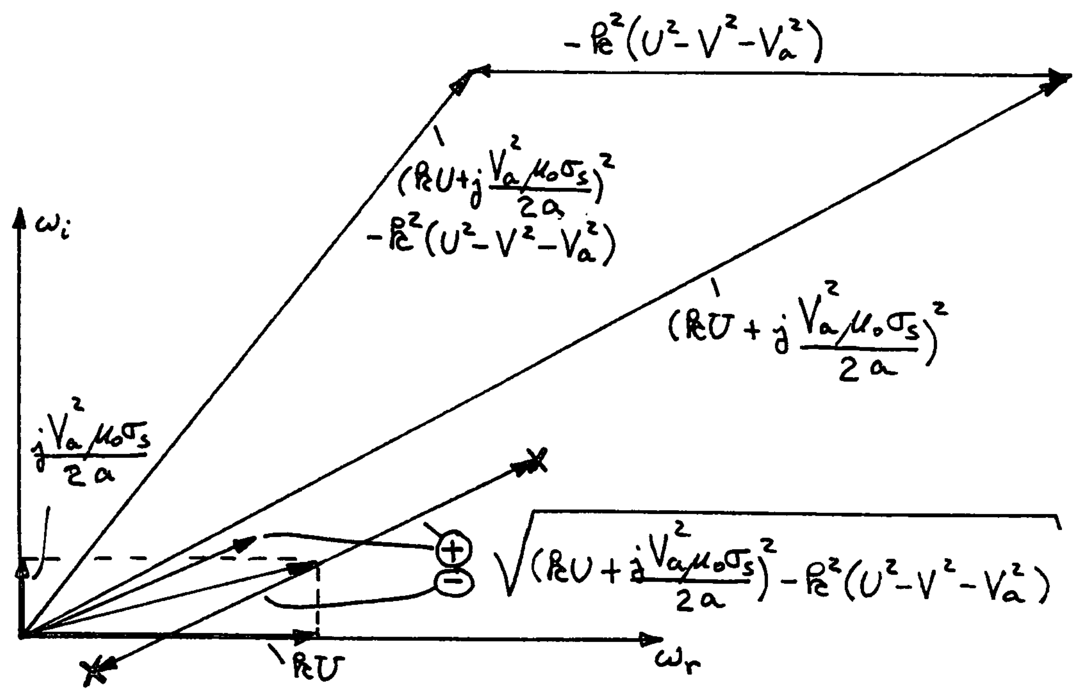
and $k = k_y$, the expression reduces to

$$\omega^2 - \omega\left(2kU + jV_a^2 \frac{\mu_o \sigma_s}{a}\right) + k^2\left(U^2 - V^2 - V_a^2\right) \tag{21}$$

where $V^2 \equiv \gamma/\rho b$ and $V_a^2 \equiv (\mu_o H_o^2/\rho)(a/b)$. Thus, in this long-wave low frequency approximation,

$$\omega = kU + jV_a^2 \frac{\mu_o \sigma_s}{2a} \pm \left\{\left(kU + jV_a^2 \frac{\mu_o \sigma_s}{2a}\right)^2 - k^2\left(U^2 - V^2 - V_a^2\right)\right\}^{1/2} \tag{22}$$

Prob. 11.17.5 (cont.)



It follows from the diagram that if $U>\sqrt{V^2+V_a^2}$, the system is unstable. To explore the nature of the instability, Eq. 21 is written as a polynomial in k.

$$(U^2-V^2-V_a^2)k^2-2\omega Uk+\omega\left(\omega-j\frac{V_a^2\mu_o\sigma_s}{a}\right)=0 \tag{23}$$

This quadratic in k is solved to give

$$k=\frac{\omega U\pm\sqrt{\omega^2(V^2+V_a^2)+j(U^2-V^2-V_a^2)V_a^2\frac{\mu_o\sigma_s\omega}{a}}}{(U^2-V^2-V_a^2)} \tag{24}$$

With $\omega=\omega_r-j\sigma$, this becomes

$$k=\frac{\omega_r U-j\sigma U\pm\sqrt{A+jB}}{U^2-V^2-V_a^2} \tag{25}$$

Prob. 11.17.5 (cont.)

where

$$A \equiv (\omega_r^2 - \sigma^2)(V^2 + V_a^2) + (U^2 - V^2 - V_a^2)\frac{V_a^2 \mu_o \sigma_s}{a}\sigma$$

$$B \equiv \left[(U^2 - V^2 - V_a^2)\frac{V_a^2 \mu_o \sigma_s}{a} - 2\sigma(V^2 + V_a^2)\right]\omega_r$$

The loci of complex k at fixed ω_r as σ is varied from ∞ to 0 for $U^2 > (V^2 + V_a^2)$ could be plotted in detail. However, it is already known that one of these passes through the k_r axis when $\sigma < 0$ (that one temporal mode is unstable). To see that the instability is convective it is only necessary to observe that both families of loci originate at $k_i \to -\infty$. That is, in the limit $\sigma \to \infty$, Eq. 25 gives

$$k \to \frac{-j\sigma U \pm j\sigma\sqrt{V^2 + V_a^2}}{U^2 - V^2 - V_a^2} \tag{26}$$

and if $U^2 > V^2 + V_a^2$ it follows that for both roots $k \to -j\infty$ as $\sigma \to \infty$. Thus, the loci have the character of Fig. 11.12.8. The "unstable" root crosses the k_r axis into the upper half-plane. Because the "stable" root never crosses the k_r axis, these two loci cannot coalesce, as required for an absolute instability.

Note that the same conclusion follows from reverting to a z-t model for the dynamics. The long-wave model represented by Eq. 21 is equivalent to a "string" having the equation of motion

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial z}\right)^2 \xi = (V^2 + V_a^2)\frac{\partial^2 \xi}{\partial z^2} - V_a^2 \frac{\mu_o \sigma_s}{a}\frac{\partial \xi}{\partial t} \tag{27}$$

The characteristics for this expression are

$$\frac{dz}{dt} = U \pm \sqrt{V^2 + V_a^2} \tag{28}$$

and it follows that if $U > \sqrt{V^2 + V_a^2}$, the instability must be convective.

Prob. 11.17.6 (a) With the understanding that the potential represents an electric field that is in common to both beams, the linearized longitudinal force equations for the respective one-dimensional overlapping beams are

$$\frac{\partial v_{z1}}{\partial t} + U_1 \frac{\partial v_{z1}}{\partial z} = \frac{e}{m}\frac{\partial \Phi}{\partial z} \tag{1}$$

$$\frac{\partial v_{z2}}{\partial t} + U_2 \frac{\partial v_{z2}}{\partial z} = \frac{e}{m}\frac{\partial \Phi}{\partial z} \tag{2}$$

To write particle conservation, first observe that the longitudinal current density for the first beam is

$$\bar{J}_1 = -e n_o U_1 \bar{i}_z - e(n_1 U_1 + n_o v_{z1}) \tag{3}$$

and hence particle conservation for that beam is represented by

$$\frac{\partial n_1}{\partial t} + U_1 \frac{\partial n_1}{\partial z} + n_{01}\frac{\partial v_{z1}}{\partial z} = 0 \tag{4}$$

Similarly, the conservation of particles on the second beam is represented by

$$\frac{\partial n_2}{\partial t} + U_2 \frac{\partial n_2}{\partial z} + n_{02}\frac{\partial v_{z2}}{\partial z} = 0 \tag{5}$$

Finally, perturbations of charge density in each of the beams contribute to the electric field, and the one-dimensional form of Gauss' Law is

$$\frac{\partial^2 \Phi}{\partial z^2} = \frac{e}{\epsilon_o}(n_1 + n_2) \tag{6}$$

The five dependent variables $v_{z1}, v_{z2}, \Phi, n_1$, and n_2 are described by Eqs. 1, 2, 4 and 5. In terms of complex amplitudes, these expressions are represented by the five algebraic statements summarized by

$$\begin{bmatrix} \omega - kU_1 & 0 & \frac{ek}{m} & 0 & 0 \\ 0 & \omega - kU_2 & \frac{ek}{m} & 0 & 0 \\ -kn_{01} & 0 & 0 & \omega - kU_1 & 0 \\ 0 & -kn_{02} & 0 & 0 & \omega - kU_2 \\ 0 & 0 & k^2 & \frac{e}{\epsilon_o} & \frac{e}{\epsilon_o} \end{bmatrix} \begin{bmatrix} \hat{v}_{z1} \\ \hat{v}_{z2} \\ \hat{\Phi} \\ \hat{n}_1 \\ \hat{n}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{7}$$

Prob. 11.17.6 (cont.)

The determinant of the coefficients reduces to the desired dispersion equation.

$$1 = \frac{\omega_{p1}^2}{(\omega - kU_1)^2} + \frac{\omega_{p2}^2}{(\omega - kU_2)^2} \qquad (8)$$

where the respective beam plasma frequencies are defined as

$$\omega_{p1} = \sqrt{\frac{n_{01} e^2}{\epsilon_0 m}} \; ; \; \omega_{p2} = \sqrt{\frac{n_{02} e^2}{\epsilon_0 m}} \qquad (9)$$

(b) In the limit where the second "beam" is actually a plasma (formally equivalent to making $U_2=0$), the dispersion equation, Eq. 8, becomes the polynomial,

$$\underline{k}^2 - 2\underline{\omega}\underline{k} + \underline{\omega}^2\left(1 - \frac{r}{\underline{\omega}^2 - 1}\right) = 0 \qquad (10)$$

where $r \equiv (\omega_{p1}/\omega_{p2})^2$, $\underline{\omega} \equiv \omega/\omega_{p2}$ and $\underline{k} \equiv kU_1/\omega_{p2}$. The mapping of complex k as a function of $\omega = \omega_r - j\sigma$, σ varying from $\infty \to 0$ with ω_r held fixed, shown in Fig. P11.17.6a, is that characteristic of a convective instability (Fig. 11.12.8, for example).

(c) In the limit of counter-streaming beams $U_1 = U_2 \equiv U$, Eq. 8 becomes

$$\underline{k}^4 - (2\underline{\omega}^2 + r + 1)\underline{k}^2 + 2\underline{\omega}(1-r)\underline{k} + \underline{\omega}^2[\underline{\omega}^2 - (r+1)] = 0 \qquad (11)$$

where the normalization is as before. This time, the mapping is as illustrated by Fig. P11.17.6b, and it is clear that there is an absolute instability. (The loci are as typified by Fig. 11.13.3.)

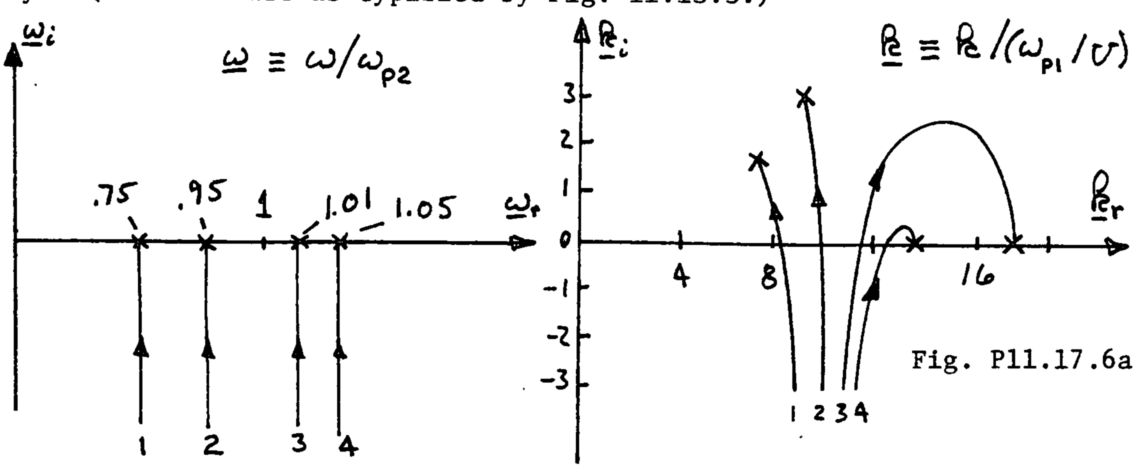
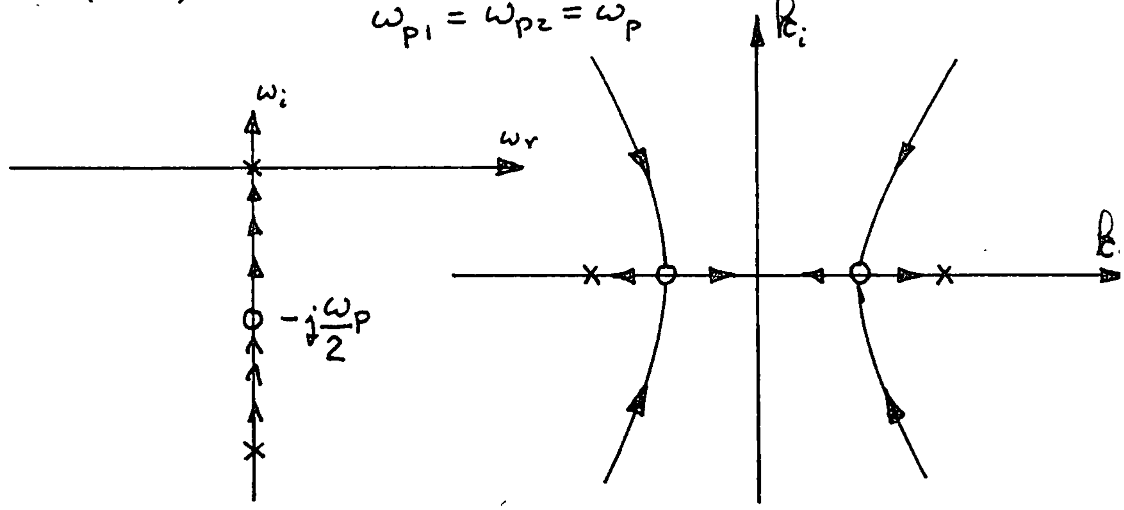


Fig. P11.17.6a

Prob. 11.17.6 (cont.)



See, Briggs, R.J., Electron-Stream Interaction With Plasmas, M.I.T. Press (1964) pp 32-34 and 42-44.