PART 3

INTRODUCTION TO ENGINEERING HEAT TRANSFER

Introduction to Engineering Heat Transfer

These notes provide an introduction to engineering heat transfer. Heat transfer processes set limits to the performance of aerospace components and systems and the subject is one of an enormous range of application. The notes are intended to describe the three types of heat transfer and provide basic tools to enable the readers to estimate the magnitude of heat transfer rates in realistic aerospace applications. There are also a number of excellent texts on the subject; some accessible references which expand the discussion in the notes are listen in the bibliography.

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1.0 Heat Transfer Modes

Heat transfer processes are classified into three types. The first is conduction, which is defined as transfer of heat occurring through intervening matter without bulk motion of the matter. Figure 1.1 shows the process pictorially. A solid (a block of metal, say) has one surface at a high temperature and one at a lower temperature. This type of heat conduction can occur, for example, through a turbine blade in a jet engine. The outside surface, which is exposed to gases from the combustor, is at a higher temperature than the inside surface, which has cooling air next to it. The level of the wall temperature is critical for a turbine blade.

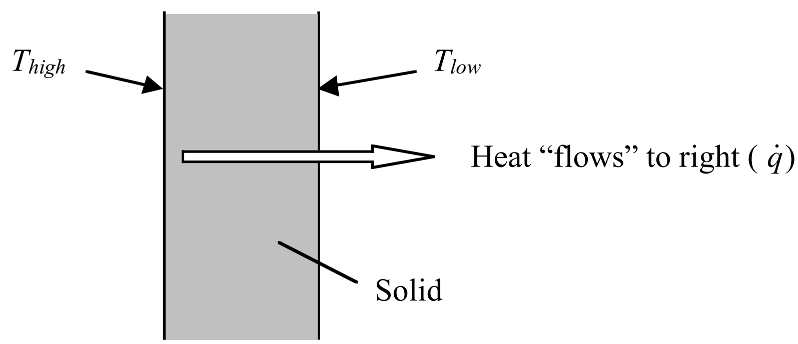


Figure 1.1: Conduction heat transfer

The second heat transfer process is convection, or heat transfer due to a flowing fluid. The fluid can be a gas or a liquid; both have applications in aerospace technology. In convection heat transfer, the heat is moved through bulk transfer of a non-uniform temperature fluid.

The third process is radiation or transmission of energy through space without the necessary presence of matter. Radiation is the only method for heat transfer in space. Radiation can be important even in situations in which there is an intervening medium; a familiar example is the heat transfer from a glowing piece of metal or from a fire.

Muddy points

How do we quantify the contribution of each mode of heat transfer in a given situation? (MP HT.1)

2.0 Conduction Heat Transfer

We will start by examining conduction heat transfer. We must first determine how to relate the heat transfer to other properties (either mechanical, thermal, or geometrical). The answer to this is rooted in experiment, but it can be motivated by considering heat flow along a "bar" between two heat reservoirs at T_A, T_B as shown in Figure 2.1. It is plausible that the heat transfer rate $\dot{Q}$, is a function of the temperature of the two reservoirs, the bar geometry and the bar properties. (Are there other factors that should be considered? If so, what?). This can be expressed as

$$\dot{Q} = f_1\,(T_A\,,\ T_B\,,\ \text{bar geometry, bar properties}) \tag{2.1}$$

It also seems reasonable to postulate that $\dot{Q}$ should depend on the temperature difference T_A - T_B. If $T_A - T_B$ is zero, then the heat transfer should also be zero. The temperature dependence can therefore be expressed as

$$\dot{Q} = f_2\,[\,(T_A - T_B),\ T_A,\ \text{bar geometry, bar properties}] \tag{2.2}$$

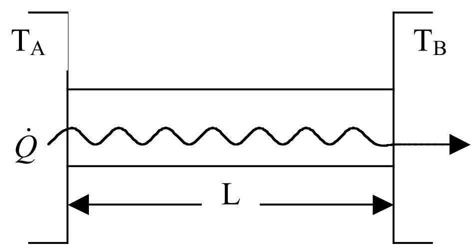


Figure 2.1: Heat transfer along a bar

An argument for the general form of f_2 can be made from physical considerations. One requirement, as said, is $f_2 = 0$ if $T_A = T_B$. Using a MacLaurin series expansion, as follows:

$$f(\Delta T) = f(0) + \left.\frac{\partial f}{\partial(\Delta T)}\right|_0 \Delta T + \cdots \tag{2.3}$$

If we define $\Delta T = T_A - T_B$ and $f = f_2$, we find that (for small $T_A - T_B$),

$$f_2(T_A - T_B) = \dot{Q} = f_2(0) + \left.\frac{\partial f_2}{\partial(T_A - T_B)}\right|_{T_A - T_B = 0} \left(T_A - T_B\right) + \cdots . \tag{2.4}$$

We know that $f_2(0) = 0$. The derivative evaluated at $T_A = T_B$ (thermal equilibrium) is a measurable property of the bar. In addition, we know that $\dot{Q} > 0$ if $T_A > T_B$ or $\frac{\partial f_2}{\partial\left(T_A - T_B\right)} > 0$. It also seems reasonable that if we had two bars of the same area, we would have twice the heat transfer, so that we can postulate that $\dot{Q}$ is proportional to the area. Finally, although the argument is by no means rigorous, experience leads us to believe that as L increases $\dot{Q}$ should get smaller. All of these lead to the generalization (made by Fourier in 1807) that, for the bar, the derivative in equation (2.4) has the form

$$\left.\frac{\partial f_2}{\partial\left(T_A - T_B\right)}\right|_{T_A - T_B = 0} = \frac{kA}{L}. \tag{2.5}$$

In equation (2.5), k is a proportionality factor that is a function of the material and the temperature, A is the cross-sectional area and L is the length of the bar. In the limit for any temperature difference ΔT across a length Δx as both L, $T_A - T_B \rightarrow 0$, we can say

$$\dot{Q} = kA\frac{\left(T_A - T_B\right)}{L} = -kA\frac{\left(T_B - T_A\right)}{L} = -kA\frac{dT}{dx}. \tag{2.6}$$

A more useful quantity to work with is the heat transfer per unit area, defined as

$$\frac{\dot{Q}}{A} = \dot{q}. \tag{2.7}$$

The quantity $\dot{q}$ is called the heat flux and its units are Watts/m^2. The expression in (2.6) can be written in terms of heat flux as

$$\dot{q} = -k\frac{dT}{dx}. \tag{2.8}$$

Equation 2.8 is the one-dimensional form of *Fourier's law of heat conduction*. The proportionality constant k is called the thermal conductivity. Its units are W / m-K. Thermal conductivity is a well-tabulated property for a large number of materials. Some values for familiar materials are given in Table 1; others can be found in the references. The thermal conductivity is a function of temperature and the values shown in Table 1 are for room temperature.

Table 2.1: Thermal conductivity at room temperature for some metals and non-metals

Metals	Ag	Cu	Al	Fe	Steel
k [W/m-K]	420	390	200	70	50

Non-metals	H_20	Air	Engine oil	H_2	Brick	Wood	Cork
k [W/m-K]	0.6	0.026	0.15	0.18	0.4 -0 .5	0.2	0.04

2.1 Steady-State One-Dimensional Conduction

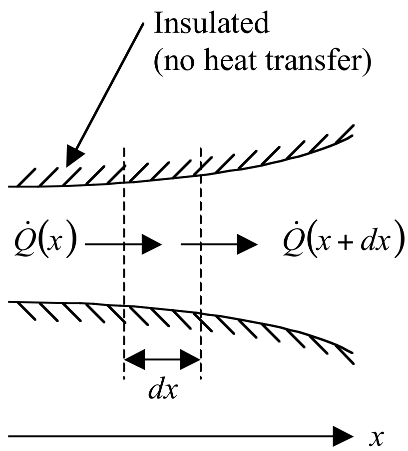


Figure 2.2: One-dimensional heat conduction

For one-dimensional heat conduction (temperature depending on one variable only), we can devise a basic description of the process. The first law in control volume form (steady flow energy equation) with no shaft work and no mass flow reduces to the statement that $\Sigma\dot{Q}$ for all surfaces = 0 (no heat transfer on top or bottom of figure 2.2). From equation (2.8), the heat transfer rate in at the left (at x) is

$$\dot{Q}(x) = -k\left(A\frac{dT}{dx}\right)_x . \tag{2.9}$$

The heat transfer rate on the right is

$$\dot{Q}(x+dx) = \dot{Q}(x) + \left.\frac{d\dot{Q}}{dx}\right|_x dx + \cdots . \tag{2.10}$$

Using the conditions on the overall heat flow and the expressions in (2.9) and (2.10)

$$\dot{Q}(x) - \left(\dot{Q}(x) + \frac{d\dot{Q}}{dx}(x)dx + \cdots\right) = 0. \tag{2.11}$$

Taking the limit as dx approaches zero we obtain

$$\frac{d\dot{Q}(x)}{dx} = 0, \tag{2.12a}$$

or

$$\frac{d}{dx}\left(kA\frac{dT}{dx}\right) = 0. \tag{2.12b}$$

If k is constant (i.e. if the properties of the bar are independent of temperature), this reduces to

$$\frac{d}{dx}\left(A\frac{dT}{dx}\right) = 0 \tag{2.13a}$$

or (using the chain rule)

$$\frac{d^2T}{dx^2} + \left(\frac{1}{A}\frac{dA}{dx}\right)\frac{dT}{dx} = 0. \tag{2.13b}$$

Equations (2.13a) or (2.13b) describe the temperature field for quasi-one-dimensional steady state (no time dependence) heat transfer. We now apply this to some examples.

Example 2.1: Heat transfer through a plane slab

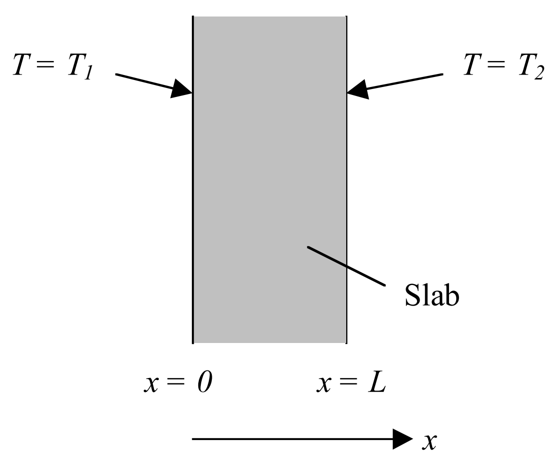


Figure 2.3: Temperature boundary conditions for a slab

For this configuration, the area is not a function of x, i.e. A = constant. Equation (2.13) thus became

$$\frac{d^2T}{dx^2} = 0 \cdot \tag{2.14}$$

Equation (2.14) can be integrated immediately to yield

$$\frac{dT}{dx} = a \tag{2.15}$$

and $$T = ax + b\,. \tag{2.16}$$

Equation (2.16) is an expression for the temperature field where a and b are constants of integration. For a second order equation, such as (2.14), we need two boundary conditions to determine a and b. One such set of boundary conditions can be the specification of the temperatures at both sides of the slab as shown in Figure 2.3, say $T(0) = T_1$; $T(L) = T_2$.

The condition $T(0) = T_1$ implies that $b = T_1$. The condition $T_2 = T(L)$ implies that $T_2 = aL + T_1$, or $a = \dfrac{T_2 - T_1}{L}$.

With these expressions for a and b the temperature distribution can be written as

$$T(x) = T_1 + \left(\frac{T_2 - T_1}{L}\right)x\,. \tag{2.17}$$

This linear variation in temperature is shown in Figure 2.4 for a situation in which $T_1 > T_2$.

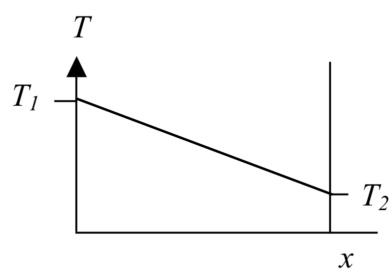


Figure 2.4: Temperature distribution through a slab

The heat flux $\dot{q}$ is also of interest. This is given by

$$\dot{q} = -k\frac{dT}{dx} = -k\frac{(T_2 - T_1)}{L} = \text{constant}\,. \tag{2.18}$$

Muddy points

How specific do we need to be about when the one-dimensional assumption is valid? Is it enough to say that dA/dx is small? (MP HT.2)

Why is the thermal conductivity of light gases such as helium (monoatomic) or hydrogen (diatomic) much higher than heavier gases such as argon (monoatomic) or nitrogen (diatomic)? (MP HT.3)

2.2 Thermal Resistance Circuits

There is an electrical analogy with conduction heat transfer that can be exploited in problem solving. The analog of $\dot{Q}$ is current, and the analog of the temperature difference, T_1 - T_2, is voltage difference. From this perspective the slab is a pure resistance to heat transfer and we can define

$$\dot{Q} = \frac{T_1 - T_2}{R} \tag{2.19}$$

where $R = L/kA$, the thermal resistance. The thermal resistance R increases as L increases, as A decreases, and as k decreases.

The concept of a thermal resistance circuit allows ready analysis of problems such as a composite slab (composite planar heat transfer surface). In the composite slab shown in Figure 2.5, the heat flux is constant with x. The resistances are in series and sum to $R = R_1 + R_2$. If T_L is the temperature at the left, and T_R is the temperature at the right, the heat transfer rate is given by

$$\dot{Q} = \frac{T_L - T_R}{R} = \frac{T_L - T_R}{R_1 + R_2}. \tag{2.20}$$

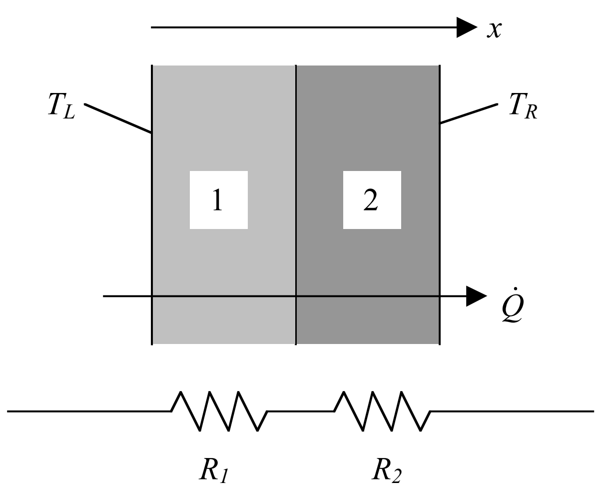


Figure 2.5: Heat transfer across a composite slab (series thermal resistance)

Another example is a wall with a dissimilar material such as a bolt in an insulating layer. In this case, the heat transfer resistances are in parallel. Figure 2.6 shows the physical configuration, the heat transfer paths and the thermal resistance circuit.

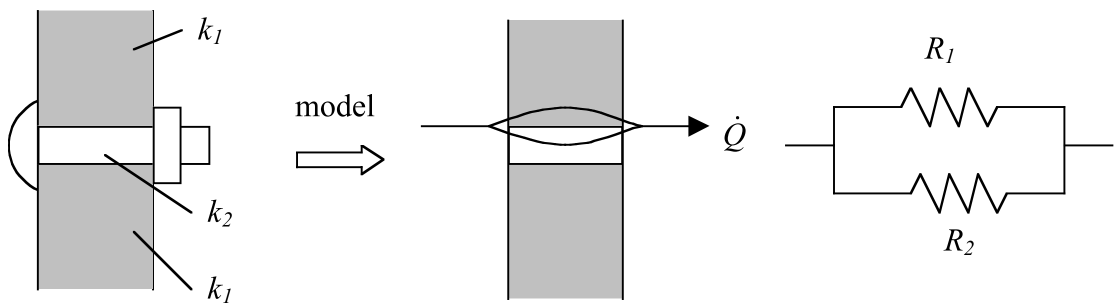


Figure 2.6: Heat transfer for a wall with dissimilar materials (Parallel thermal resistance)

For this situation, the total heat flux $\dot{Q}$ is made up of the heat flux in the two parallel paths: $\dot{Q} = \dot{Q}_1 + \dot{Q}_2$ with the total resistance given by:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \ . \tag{2.21}$$

More complex configurations can also be examined; for example, a brick wall with insulation on both sides.

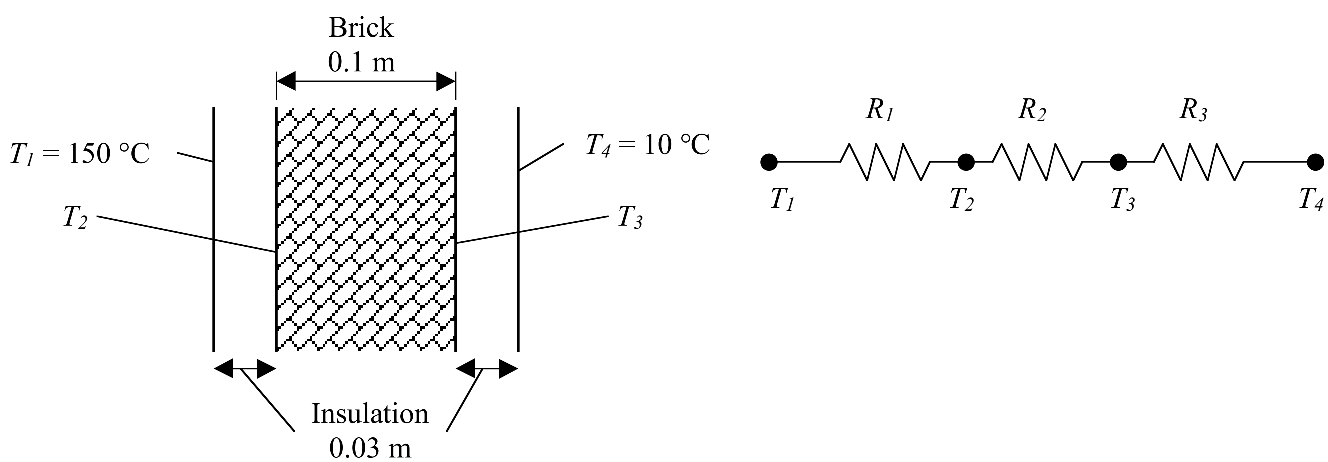


Figure 2.7: Heat transfer through an insulated wall

The overall thermal resistance is given by

$$R = R_1 + R_2 + R_3 = \frac{L_1}{k_1 A_1} + \frac{L_2}{k_2 A_2} + \frac{L_3}{k_3 A_3} . \tag{2.22}$$

Some representative values for the brick and insulation thermal conductivity are:

$k_{brick} = k_2 = 0.7$ W/m-K
$k_{insulation} = k_1 = k_3 = 0.07$ W/m-K

Using these values, and noting that $A_1 = A_2 = A_3 = A$, we obtain:

$$AR_1 = AR_3 = \frac{L_1}{k_1} = \frac{0.03\text{ m}}{0.07\text{ W/m K}} = 0.42\text{ m}^2\text{ K/W}$$

$$AR_2 = \frac{L_2}{k_2} = \frac{0.1\text{ m}}{0.7\text{ W/m K}} = 0.14\text{ m}^2\text{ K/W}.$$

This is a series circuit so

$$\dot{q} = \frac{\dot{Q}}{A} = \text{constant throughout} = \frac{T_1 - T_4}{RA} = \frac{140\text{ K}}{0.98\text{ m}^2\text{ K/W}} = 142\text{ W/m}^2$$

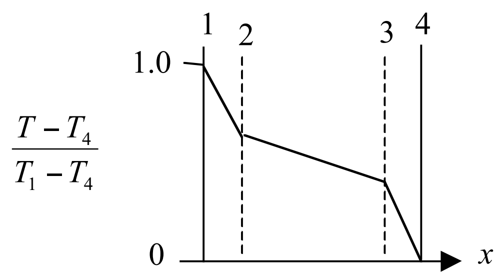


Figure 2.8: Temperature distribution through an insulated wall

The temperature is continuous in the wall and the intermediate temperatures can be found from applying the resistance equation across each slab, since $\dot{Q}$ is constant across the slab. For example, to find T_2:

$$\dot{q} = \frac{T_1 - T_2}{R_1 A} = 142\text{ W/m}^2$$

This yields $T_1 - T_2 = 60$ K or $T_2 = 90$ °C.

The same procedure gives $T_3 = 70$ °C. As sketched in Figure 2.8, the larger drop is across the insulating layer even though the brick layer is much thicker.

Muddy points

What do you mean by continuous? (MP HT.4)
Why is temperature continuous in the composite wall problem? Why is it continuous at the interface between two materials? (MP HT.5)

Why is the temperature gradient dT/dx not continuous? (MP HT.6)
Why is ΔT the same for the two elements in a parallel thermal circuit? Doesn't the relative area of the bolt to the wood matter? (MP HT.7)

2.3 Steady Quasi-One-Dimensional Heat Flow in Non-Planar Geometry

The quasi one-dimensional equation that has been developed can also be applied to non-planar geometries. An important case is a cylindrical shell, a geometry often encountered in situations where fluids are pumped and heat is transferred. The configuration is shown in Figure 2.9.

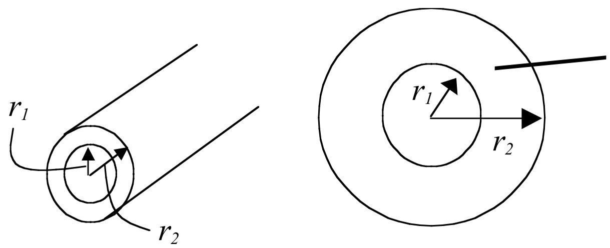


Figure 2.9: Cylindrical shell geometry notation

For a steady axisymmetric configuration, the temperature depends only on a single coordinate *(r)* and Equation (2.12b) can be written as

$$k\frac{d}{dr}\left(A(r)\frac{dT}{dr}\right) = 0 \tag{2.23}$$

or, since $A = 2\pi r$,

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0. \tag{2.24}$$

The steady-flow energy equation (no flow, no work) tells us that $\dot{Q}_{in} = \dot{Q}_{out}$ or

$$\frac{d\dot{Q}}{dr} = 0$$
(2.25)

The heat transfer rate per unit length is given by

$$\dot{Q} = -k \cdot 2\pi r\frac{dT}{dr}.$$

Equation (2.24) is a second order differential equation for T. Integrating this equation once gives

$$r\frac{dT}{dr} = a \ . \tag{2.26}$$

where a is a constant of integration. Equation (2.26) can be written as

$$dT = a\frac{dr}{r} \tag{2.27}$$

where both sides of equation (2.27) are exact differentials. It is useful to cast this equation in terms of a dimensionless normalized spatial variable so we can deal with quantities of order unity. To do this, divide through by the inner radius, r_1

$$dT = a\frac{d(r/r_1)}{(r/r_1)} \tag{2.28}$$

Integrating (2.28) yields

$$T = a\ln\left(\frac{r}{r_1}\right) + b. \tag{2.29}$$

To find the constants of integration a and b, boundary conditions are needed. These will be taken to be known temperatures T_1 and T_2 at r_1 and r_2 respectively. Applying $T = T_1$ at $r = r_1$ gives $T_1 = b$. Applying $T = T_2$ at $r = r_2$ yields

$$T_2 = a\ln\frac{r_2}{r_1} + T_1,$$

or

$$a = \frac{T_2 - T_1}{\ln(r_2/r_1)}.$$

The temperature distribution is thus

$$T = (T_2 - T_1)\frac{\ln(r/r_1)}{\ln(r_2/r_1)} + T_1 . \tag{2.30}$$

As said, it is generally useful to put expressions such as (2.30) into non-dimensional and normalized form so that we can deal with numbers of order unity (this also helps in checking whether results are consistent). If convenient, having an answer that goes to zero at one limit is also useful from the perspective of ensuring the answer makes sense. Equation (2.30) can be put in non-dimensional form as

$$\frac{T-T_1}{T_2-T_1}=\frac{\ln(r/r_1)}{\ln(r_2/r_1)}. \tag{2.31}$$

The heat transfer rate, $\dot{Q}$, is given by

$$\dot{Q}=-kA\frac{dT}{dr}=-2\pi r_1 k\frac{(T_2-T_1)}{\ln(r_2/r_1)}\frac{1}{r_1}=2\pi\frac{k(T_1-T_2)}{\ln(r_2/r_1)}$$

per unit length. The thermal resistance R is given by

$$R=\frac{\ln(r_2/r_1)}{2\pi k} \tag{2.32}$$

$$\dot{Q}=\frac{T_1-T_2}{R}.$$

The cylindrical geometry can be viewed as a limiting case of the planar slab problem. To make the connection, consider the case when $\frac{r_2-r_1}{r_1}<<1$. From the series expansion for ln *(1 + x)* we recall that

$$\ln(1+x)\approx x-\frac{x^2}{2}+\frac{x^3}{3}+\ldots \tag{2.33}$$

(Look it up, try it numerically, or use the binomial theorem on the series below and integrate term by term. $\frac{1}{1+x}=1-x+x^2+\ldots$)

The logarithms in Equation (2.31) can thus be written as

$$\ln\left(1+\frac{r-r_1}{r_1}\right)\cong\frac{r-r_1}{r_1} \quad \text{and} \quad \ln\frac{r_2}{r_1}\cong\frac{r_2-r_1}{r_1} \tag{2.34}$$

in the limit of $(r_2-r_1)<<r_1$. Using these expressions in equation (2.30) gives

$$T=(T_2-T_1)\frac{(r-r_1)}{(r_2-r_1)}+T_1. \tag{2.35}$$

With the substitution of $r-r_1=x$, and $r_2-r_1=L$ we obtain

$$T=T_1+(T_2-T_1)\frac{x}{L} \tag{2.36}$$

which is the same as equation (2.17). The plane slab is thus the limiting case of the cylinder if $(r - r_1) / r << 1$, where the heat transfer can be regarded as taking place in (approximately) a planar slab. To see when this is appropriate, consider the expansion $\frac{\ln(1+x)}{x}$, which is the ratio of heat flux for a cylinder and a plane slab.

Table 2.2: Utility of plane slab approximation

x	.1	.2	.3	.4	.5
$\frac{\ln(1+x)}{x}$	.95	.91	.87	.84	.81

For < 10% error, the ratio of thickness to inner radius should be less than 0.2, and for 20% error, the thickness to inner radius should be less than 0.5.

A second example is the spherical shell with specified temperatures $T(r_1) = T_1$ and $T(r_2) = T_2$, as sketched in Figure 2.10.

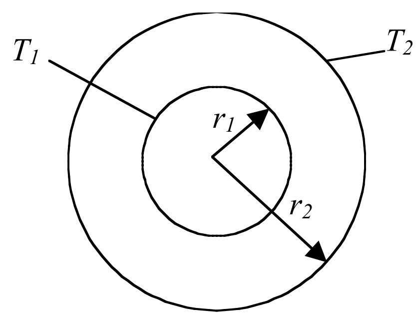


Figure 2.10: Spherical shell

The area is now $A(r) = 4\pi r^2$, so the equation for the temperature field is

$$\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = 0. \tag{2.37}$$

Integrating equation (2.37) once yields

$$\frac{dT}{dr} = a / r^2. \tag{2.38}$$

Integrating again gives

$$T = -\frac{a}{r} + b$$

or, normalizing the spatial variable

$$T = \frac{a'}{(r/r_1)} + b \tag{2.39}$$

where a' and b are constants of integration. As before, we specify the temperatures at $r = r_1$ and $r = r_2$. Use of the first boundary condition gives $T(r_1) = T_1 = a' + b$. Applying the second boundary condition gives

$$T(r_2) = T_2 = \frac{a'}{(r_2/r_1)} + b$$

Solving for a' and b,

$$\begin{aligned} a' &= \frac{T_1 - T_2}{1 - r_1/r_2} \\ b &= T_1 - \frac{T_1 - T_2}{1 - r_1/r_2}. \end{aligned} \tag{2.40}$$

In non-dimensional form the temperature distribution is thus:

$$\frac{T_1 - T}{T_1 - T_2} = \frac{1 - (r_1/r)}{1 - (r_1/r_2)} \tag{2.41}$$

3.0 Convective Heat Transfer

The second type of heat transfer to be examined is convection, where a key problem is determining the boundary conditions at a surface exposed to a flowing fluid. An example is the wall temperature in a turbine blade because turbine temperatures are critical as far as creep (and thus blade) life. A view of the problem is given in Figure 3.1, which shows a cross-sectional view of a turbine blade. There are three different types of cooling indicated, all meant to ensure that the metal is kept at a temperature much lower than that of the combustor exit flow in which the turbine blade operates. In this case, the turbine wall temperature is not known and must be found as part of the solution to the problem.

Figure 3.1: Turbine blade heat transfer configuration

To find the turbine wall temperature, we need to analyze convective heat transfer, which means we need to examine some features of the fluid motion near a surface. The conditions near a surface are illustrated schematically in Figure 3.2.

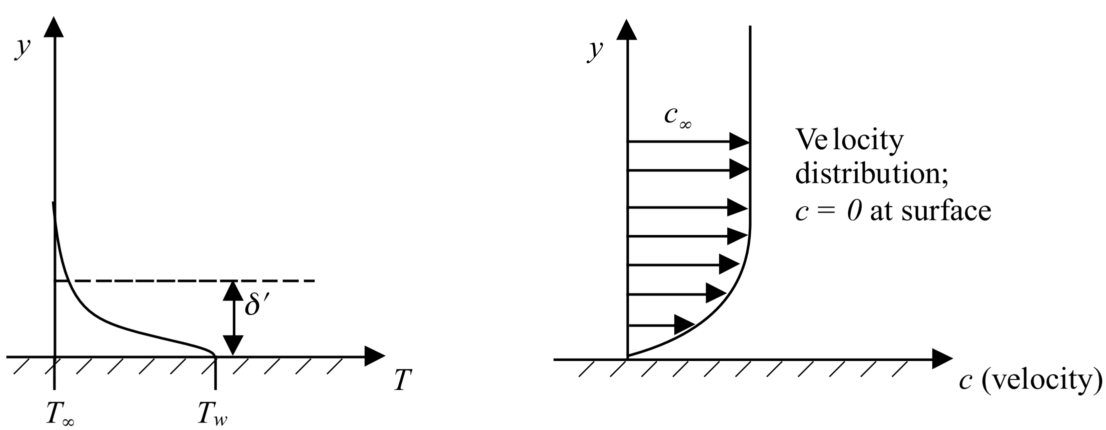


Figure 3.2: Temperature and velocity distributions near a surface.

In a region of thickness δ', there is a thin "film" of slowly moving fluid through which most of the temperature difference occurs. Outside this layer, T is roughly uniform (this defines δ'). The heat flux can thus be expressed as

$$\dot{q} = \frac{\dot{Q}}{A} = \frac{k\left(T_w - T_\infty\right)}{d'} \tag{3.1}$$

It cannot be emphasized enough that this is a very crude picture. The general concept, however, is correct, in that close to the wall, there is a thin layer in which heat is transferred basically by conduction. Outside of this region is high mixing. The difficulty is that the thickness of the layer is not a fluid property. It depends on velocity (Reynolds number), structure of the wall surface, pressure gradient and Mach number. Generally δ' is not known and needs to be found and it is customary to calculate the heat transfer using $[k_{fluid} / \delta']$. This quantity has the symbol h and is known as the convective heat transfer coefficient. The units of h are W/m^2K. The convective heat transfer coefficient is defined by

$$\dot{q} = \frac{\dot{Q}}{A} = h\left(T_w - T_\infty\right) \tag{3.2}$$

Equation 3.2 is often called Newton's Law of Cooling. For many situations of practical interest, the quantity h is still known mainly through experiments.

Muddy points

How do we know that δ' is not a fluid property? (MP HT.8)

3.1 The Reynolds Analogy

We describe the physical mechanism for the heat transfer coefficient in a *turbulent boundary layer* because most aerospace vehicle applications have turbulent boundary layers. The treatment closely follows that in Eckert and Drake (1959). Very near the wall, the fluid motion is smooth and laminar, and molecular conduction and shear are important. The shear stress, τ, at a plane is given by $\mu \frac{dc}{dy} = \tau$ (where μ is the dynamic viscosity), and the heat flux by $\dot{q} = -k\frac{dT}{dy}$. The latter is the same expression that was used for a solid. The boundary layer is a region in which the velocity is lower than the free stream as shown in Figures 3.2 and 3.3. In a turbulent boundary layer, the dominant mechanisms of shear stress and heat transfer change in nature as one moves away from the wall.

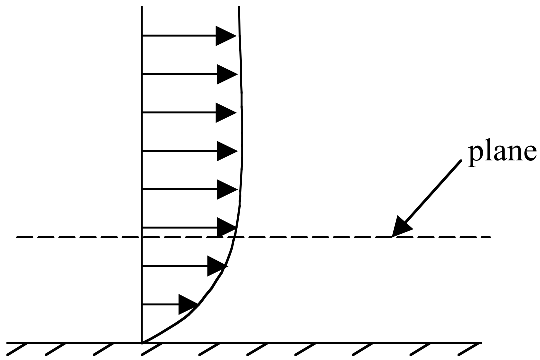


Figure 3.3: Velocity profile near a surface

As one moves away from the wall (but still in the boundary layer), the flow is turbulent. The fluid particles move in random directions and the transfer of momentum and energy is mainly through interchange of fluid particles, shown schematically in Figure 3.4.

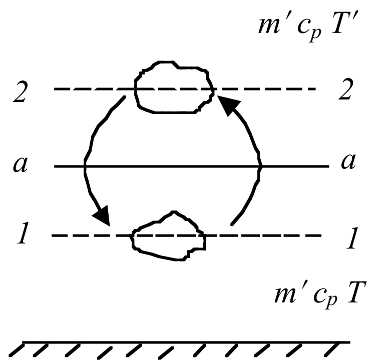


Figure 3.4: Momentum and energy exchanges in turbulent flow.

With reference to Figure 3.4, because of the turbulent velocity field, a fluid mass m' penetrates the plane *aa* per unit time and unit area. In steady flow, the same amount crosses *aa* from the other side. Fluid moving up transports heat $m' c_p T$. Fluid moving down transports $m' c_p T'$ downwards. If $T > T'$, there is a turbulent downwards heat flow, $\dot{q}_t$, given by $\dot{q}_t = m'c_p(T' - T)$ that results.

Fluid moving up also has momentum $m' c$ and fluid moving down has momentum $m' c'$. The net flux of momentum down per unit area and time is therefore $m' (c' - c)$. This net flux of momentum per unit area and time is a force per unit area or stress, given by

$$t_t = m'\left(c' - c\right) \tag{3.3}$$

Based on these considerations, the relation between heat flux and shear stress at plane *aa* is

$$\dot{q}_t = \tau_t c_p \left(\frac{T' - T}{c' - c} \right) \tag{3.4}$$

or (again approximately)

$$\dot{q}_t = \tau_t c_p \frac{dT}{dc} \tag{3.5}$$

since the locations of planes 1-1 and 2-2 are arbitrary. For the laminar region, the heat flux towards the wall is

$$\dot{q} = \tau \frac{k}{\mu} \frac{dT}{dc}$$

The same relationship is applicable in laminar or turbulent flow if $\frac{k}{\mu} = c_p$ or, expressed slightly differently,

$$\frac{c_p}{k} = \frac{\mu / \rho}{k / \rho c_p} = \frac{\upsilon}{\alpha} = 1$$

where υ is the kinematic viscosity, and α is the thermal diffusivity.

The quantity $\mu c_p/k$ is known as the Prandtl number (Pr), after the man who first presented the idea of the boundary layer and was one of the pioneers of modern fluid mechanics. For gases, Prandtl numbers are in fact close to unity and for air $Pr = 0.71$ at room temperature. The Prandtl number varies little over a wide range of temperatures; approximately 3% from 300-2000 K.

We want a relation between the values at the wall (at which $T = T_w$ and $c = 0$) and those in the free stream. To get this, we integrate the expression for dT from the wall to the free stream

$$dT = \frac{1}{c_p} \frac{\dot{q}}{\tau} dc \tag{3.6}$$

where the relation between heat transfer and shear stress has been taken as the same for both the laminar and the turbulent portions of the boundary layer. The assumption being made is that the mechanisms of heat and momentum transfer are similar. Equation (3.6) can be integrated from the wall to the freestream (conditions "at ∞"):

$$\int_{w}^{\infty} dT = \frac{1}{c_p} \int_{w}^{\infty} \left(\frac{\dot{q}}{\tau} \right) dc \tag{3.7}$$

where $\frac{\dot{q}}{\tau}$ and c_p are assumed constant.

Carrying out the integration yields

$$T_\infty - T_w = \frac{\dot{q}_w}{\tau_w}\frac{c_\infty}{c_p} \tag{3.8}$$

where c_∞ is the velocity and c_p is the specific heat. In equation (3.8), $\dot{q}_w$ is the heat flux to the wall and τ_w is the shear stress at the wall. The relation between skin friction (shear stress) at the wall and heat transfer is thus

$$\frac{\dot{q}_w}{\rho_\infty c_p (T_\infty - T_w) c_\infty} = \frac{\tau_w}{\rho_\infty {c_\infty}^2}. \tag{3.9}$$

The quantity $\dfrac{\tau_w}{1/2\rho_\infty {c_\infty}^2}$ is known as the skin friction coefficient and is denoted by C_f. The skin friction coefficient has been tabulated (or computed) for a large number of situations. If we define a non-dimensional quantity

$$\frac{\dot{q}_w}{\rho_\infty c_p (T_\infty - T_w) c_\infty} = \frac{h(T_\infty - T_w)}{\rho_\infty c_p (T_\infty - T_w) c_\infty} = \frac{h}{\rho_\infty c_p c_\infty} = \text{St},$$

known as the Stanton Number, we can write an expression for the heat transfer coefficient, h as

$$h \approx \rho_\infty c_p c_\infty \frac{C_f}{2}. \tag{3.10}$$

Equation (3.10) provides a useful estimate of h, or $\dot{q}_w$, based on knowing the skin friction, or drag. The direct relationship between the Stanton Number and the skin friction coefficient is

$$\text{St} = \frac{C_f}{2}$$

The relation between the heat transfer and the skin friction coefficient

$$\dot{q}_w \approx \frac{\tau_w c_p (T_w - T_\infty)}{c_\infty}$$

is known as the Reynolds analogy between shear stress and heat transfer. The Reynolds analogy is extremely useful in obtaining a first approximation for heat transfer in situations in which the shear stress is "known".

An example of the use of the Reynolds analogy is in analysis of a heat exchanger. One type of heat exchanger has an array of tubes with one fluid flowing inside and another fluid flowing outside, with the objective of transferring heat between them. To begin, we need to examine the flow resistance of a tube. For fully developed flow in a tube, it is more appropriate to use an average velocity $\bar{c}$ and a bulk temperature T_B. Thus, an approximate relation for the heat transfer is

$$\dot{q}_W \approx \tau_W c_p \frac{T_B - T_W}{\bar{c}}. \tag{3.11}$$

The fluid resistance (drag) is all due to shear forces and is given by $\tau_w A_w = D$, where A_w is the tube "wetted" area (perimeter *x* length). The total heat transfer, $\dot{Q}$, is $\dot{q}_w A_w$, so that

$$\dot{Q} = Dc_p \frac{T_B - T_W}{\bar{c}} \tag{3.12}$$

The power, *P*, to drive the flow through a resistance is given by the product of the drag and the velocity, $D\bar{c}$, so that

$$\frac{\dot{Q}}{P} = \frac{c_p(T_B - T_w)}{\bar{c}^{\,2}} \tag{3.13}$$

The mass flow rate is given by $\dot{m} = \rho\bar{c}A$ where *A* is the cross sectional area. For given mass flow rate and overall heat transfer rate, the power scales as c^2 or as $1/A^2$, *i.e.*

$$P \propto \frac{\dot{Q}\dot{m}^2}{\rho^2 c_p(T_B - T_w)} \frac{1}{A^2} \tag{3.14}$$

Equations (3.13) and (3.14) show that to decrease the power dissipated, we need to decrease $\bar{c}$, which can be accomplished by increasing the cross-sectional area. Two possible heat exchanger configurations are sketched in Figure 3.5; the one on the right will have a lower loss.

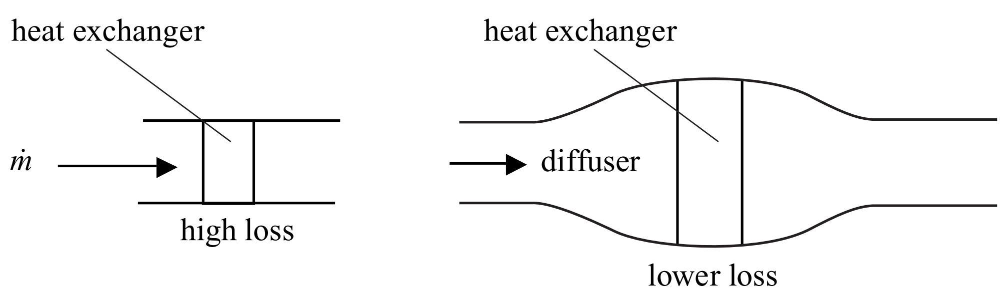


Figure 3.5: Heat exchanger configurations

To recap, there is an approximate relation between skin friction (momentum flux to the wall) and heat transfer called the Reynolds analogy that provides a useful way to estimate heat transfer rates in situations in which the skin friction is known. The relation is expressed by

$$\text{St} = \frac{C_f}{2} \tag{3.15a}$$

or

$$\frac{\text{heat flux to wall}}{\text{convected heat flux}} = \frac{\text{momentum flux to wall}}{\text{convected momentum flux}} \tag{3.15b}$$

or

$$\frac{\dot{q}_w}{\rho_\infty c_\infty c_p (T_\infty - T_w)} \approx \frac{\tau_w}{\rho_\infty c_\infty{}^2} \tag{3.15c}$$

The Reynolds analogy can be used to give information about scaling of various effects as well as initial estimates for heat transfer. It is emphasized that it is a useful tool based on a hypothesis about the mechanism of heat transfer and shear stress and not a physical law.

Muddy points

What is the "analogy" that we are discussing? Is it that the equations are similar? (MP HT.9)
In what situations does the Reynolds analogy "not work"? (MP HT.10)

3.2 Combined Conduction and Convection

We can now analyze problems in which both conduction and convection occur, starting with a wall cooled by flowing fluid on each side. As discussed, a description of the convective heat transfer can be given explicitly as

$$\frac{\dot{Q}}{A} = \dot{q} = h(T_w - T_\infty) \tag{3.16}$$

This could represent a model of a turbine blade with internal cooling. Figure 3.6 shows the configuration.

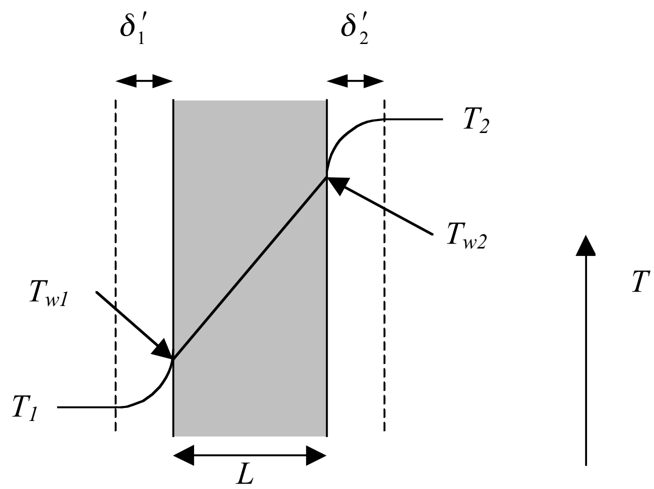


Figure 3.6: Wall with convective heat transfer

The heat transfer in fluid 1 is given by

$$\frac{\dot{Q}}{A} = h_1\left(T_{w_1} - T_1\right),$$

which is the heat transfer per unit area to the fluid. The heat transfer in fluid (2) is similarly given by

$$\frac{\dot{Q}}{A} = h_2(T_2 - T_{w2}).$$

Across the wall, we have

$$\frac{\dot{Q}}{A} = \frac{k}{L}(T_{w2} - T_{w1}).$$

The quantity $\dot{Q}/A$ is the same in all of these expressions. Putting them all together to write the known overall temperature drop yields a relation between heat transfer and overall temperature drop, $T_2 - T_1$:

$$T_2 - T_1 = (T_2 - T_{w2}) + (T_{w2} - T_{w1}) + (T_{w1} - T_1) = \frac{\dot{Q}}{A}\left[\frac{1}{h_1} + \frac{L}{k} + \frac{1}{h_2}\right]. \qquad (3.17)$$

We can define a thermal resistance, R, as before, such that

$$\dot{Q} = \frac{(T_2 - T_1)}{R},$$

where R is given by

$$R = \frac{1}{h_1 A} + \frac{L}{Ak} + \frac{1}{h_2 A}. \tag{3.18}$$

Equation (3.18) is the thermal resistance for a solid wall with convection heat transfer on each side.

For a turbine blade in a gas turbine engine, cooling is a critical consideration. In terms of Figure 3.6, T_2 is the combustor exit (turbine inlet) temperature and T_1 is the temperature at the compressor exit. We wish to find T_{w_2} because this is the highest metal temperature. From (3.17), the wall temperature can be written as

$$T_{w2} = T_2 - \frac{\dot{Q}}{Ah_2} = T_2 - \frac{T_2 - T_1}{R}\frac{1}{Ah_2} \tag{3.19}$$

Using the expression for the thermal resistance, the wall temperatures can be expressed in terms of heat transfer coefficients and wall properties as

$$T_{w2} = T_2 - \frac{T_2 - T_1}{\frac{h_2}{h_1} + \frac{Lh_2}{k} + 1} \tag{3.20}$$

Equation (3.20) provides some basic design guidelines. The goal is to have a low value of T_{w_2}. This means h_1/h_2 should be large, k should be large (but we may not have much flexibility in choice of material) and L should be small. One way to achieve the first of these is to have h_2 low (for example, to flow cooling air out as in Figure 3.1 to shield the surface).

A second example of combined conduction and convection is given by a cylinder exposed to a flowing fluid. The geometry is shown in Figure 3.7.

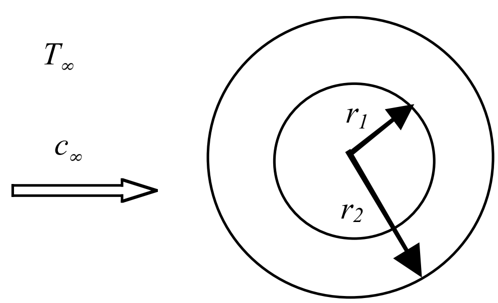


Figure 3.7: Cylinder in a flowing fluid

For the cylinder the heat flux at the outer surface is given by

$$\dot{q} = \frac{\dot{Q}}{A} = h(T_w - T_\infty) \text{ at } r = r_2$$

The boundary condition at the inner surface could either be a heat flux condition or a temperature specification; we use the latter to simplify the algebra. Thus, $T = T_1$ at $r = r_1$. This is a model for the heat transfer in a pipe of radius r_1 surrounded by insulation of thickness $r_2 - r_1$. The solution for a cylindrical region was given in Section 2.3 as

$$T(r) = a \ln \frac{r}{r_1} + b$$

Use of the boundary condition $T(r_1) = T_1$ yields $b = T_1$.

At the interface between the cylinder and the fluid, $r = r_2$, the temperature and the heat flow are continuous. (Question: Why is this? How would you argue the point?)

$$\dot{q} = -k\frac{dT}{dr} = -k\frac{a}{r_2} = h\left[\left(a\ln\frac{r_2}{r_1} + T_1\right) - T_\infty\right] \tag{3.21}$$

heat flux just inside cylinder surface heat flux to fluid

Plugging the form of the temperature distribution in the cylinder into Equation (3.21) yields

$$-a\left(\frac{k}{r_2} + h\ln\frac{r_2}{r_1}\right) = h(T_1 - T_\infty).$$

The constant of integration, a, is

$$a = \frac{-h(T_1 - T_\infty)}{\dfrac{k}{r_2} + h\ln\dfrac{r_2}{r_1}} = -\frac{(T_1 - T_\infty)}{\dfrac{k}{hr_2} + \ln\dfrac{r_2}{r_1}},$$

and the expression for the temperature is, in normalized non-dimensional form

$$\frac{T_1 - T}{T_1 - T_\infty} = \frac{\ln(r/r_1)}{\dfrac{k}{hr_2} + \ln(r_2/r_1)}. \tag{3.22}$$

The heat flow per unit length, $\dot{Q}$, is given by

$$\dot{Q} = \frac{2\pi(T_1 - T_\infty)k}{\frac{k}{hr_2} + \ln(r_2/r_1)} \tag{3.23}$$

The units in Equation (3.23) are W / m-s.

A problem of interest is choosing the thickness of insulation to minimize the heat loss for a fixed temperature difference T_1 - T_∞ between the inside of the pipe and the flowing fluid far away from the pipe. (T_1 - T_∞ is the driving temperature distribution for the pipe). To understand the behavior of the heat transfer we examine the denominator in Equation (3.23) as r_2 varies. The thickness of insulation that gives maximum heat transfer is given by

$$\frac{d}{dr_2}\left(\frac{k}{hr_2} + \ln\frac{r_2}{r_1}\right) = 0 \tag{3.24}$$

(Question: How do we know this is a maximum?)

From Equation (3.24), the value of r_2 for maximum $\dot{Q}$ is thus

$$(r_2)_{maximum\ heat\ transfer} = k/h. \tag{3.25}$$

If r_2 is less than this, we can add insulation and increase heat loss. To understand why this occurs, consider Figure 3.8, which shows a schematic of the thermal resistance and the heat transfer. As r_2 increases from a value less than $r_2 = k/h$, two effects take place. First, the thickness of the insulation increases, tending to drop the heat transfer because the temperature gradient decreases. Secondly, the area of the outside surface of the insulation increases, tending to increase the heat transfer. The second of these is (loosely) associated with the k/hr_2 term, the first with the $\ln(r_2/r_1)$ term. There are thus two competing effects which combine to give a maximum $\dot{Q}$ at $r_2 = k/h$.

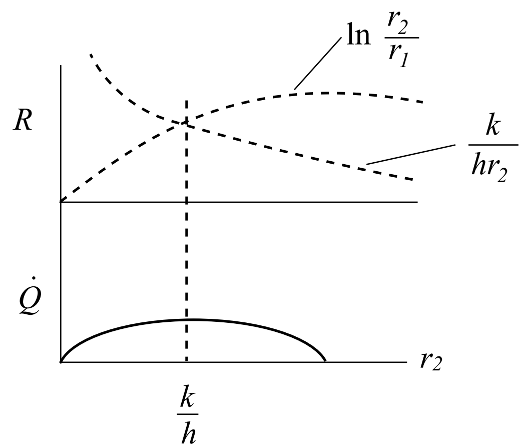


Figure 3.8: Critical radius of insulation

Muddy points

In the expression $\frac{1}{h.A}$, what is A? (MP HT.11)

It seems that we have simplified convection a lot. Is finding the heat transfer coefficient, h, really difficult? (MP HT.12)

What does the "K" in the contact resistance formula stand for? (MP HT.13)

In the equation for the temperature in a cylinder (3.22), what is "r"? (MP HT.14)

3.3 Dimensionless Numbers and Analysis of Results

Phenomena in fluid flow and heat transfer depend on dimensionless parameters. The Mach number and the Reynolds number are two you have already seen. These parameters give information as to the relevant flow regimes of a given solution. Casting equations in dimensionless form helps show the generality of application to a broad class of situations (rather than just one set of dimensional parameters). It is generally good practice to use non-dimensional numbers, forms of equations, and results presentation whenever possible. The results for heat transfer from the cylinder are already in dimensionless form but we can carry the idea even further. For the cylinder:

$$\frac{T - T_1}{T_\infty - T_1} = \frac{\ln(r/r_1)}{k/hr_2 + \ln(r_2/r_1)} \qquad (3.26)$$

The parameter $\frac{hr_2}{k}$ or $\frac{hL}{k}$, where L is a relevant length for the particular problem of interest, is called the *Biot number* denoted by *Bi*. In terms of this parameter,

$$\frac{T - T_1}{T_\infty - T_1} = \frac{\ln(r/r_1)}{\frac{1}{Bi} + \ln(r_2/r_1)} \tag{3.27}$$

The size of the Biot number gives a key to the regimes in which different features are dominant. For $Bi >> 1$ the convection heat transfer process offers little resistance to heat transfer. There is thus only a small ΔT outside (i.e. $T(r_2)$ close to T_∞) compared to the ΔT through the solid with a limiting behavior of

$$\frac{T - T_1}{T_\infty - T} = \frac{\ln r/r_1}{\ln r_2/r_1}$$

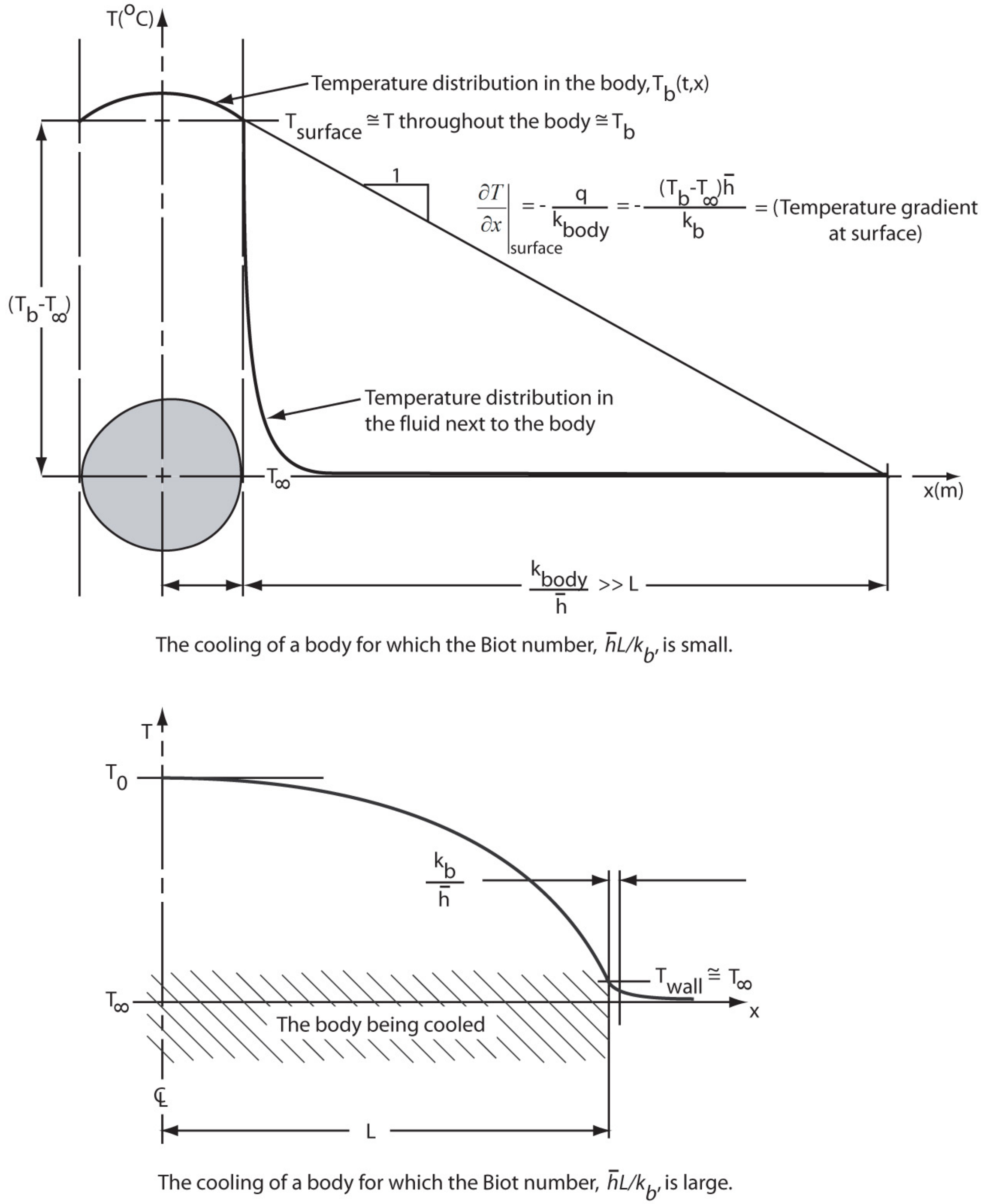
as Bi goes to infinity. This is much like the situation with an external temperature specified.

For $Bi << 1$ the conduction heat transfer process offers little resistance to heat transfer. The temperature difference in the body (i.e. from r_1 to r_2) is small compared to the external temperature difference, $T_1 - T_\infty$. In this situation, the limiting case is

$$\frac{T - T_1}{T_\infty - T_1} = Bi \ln(r/r_1)$$

In this regime there is approximately uniform temperature in the cylinder. The size of the Biot number thus indicates the regimes where the different effects become important.

Figure 3.9 shows the general effect of Biot number on temperature distribution. Figure 3.10 is a plot of the temperature distribution in the cylinder for values of Bi = 0.1, 1.0 and 10.0.



The cooling of a body for which the Biot number, $\bar{h}L/k_b$, is small.

The cooling of a body for which the Biot number, $\bar{h}L/k_b$, is large.

Figure 3.9: Effect of the Biot Number [hL / k_{body}] on the temperature distributions in the solid and in the fluid for convective cooling of a body. Note that k_{body} is the thermal conductivity of the body, not of the fluid.

[adapted from: *A Heat Transfer Textbook,* John H. Lienhard, Prentice-Hall Publishers, 1980]

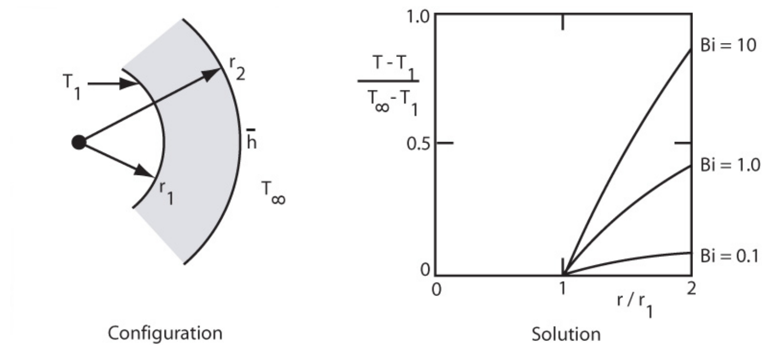


Figure 3.10: Temperature distribution in a convectively cooled cylinder for different values of Biot number, *Bi; $r_2 / r_1 = 2$* [from: *A Heat Transfer Textbook,* John H. Lienhard]

4.0 Temperature Distributions in the Presence of Heat Sources

There are a number of situations in which there are sources of heat in the domain of interest. Examples are:

1) Electrical heaters where electrical energy is converted resistively into heat
2) Nuclear power supplies
3) Propellants where chemical energy is the source

These situations can be analyzed by looking at a model problem of a slab with heat sources α *(W/m^3)* distributed throughout. We take the outside walls to be at temperature T_w. and we will determine the maximum internal temperature.

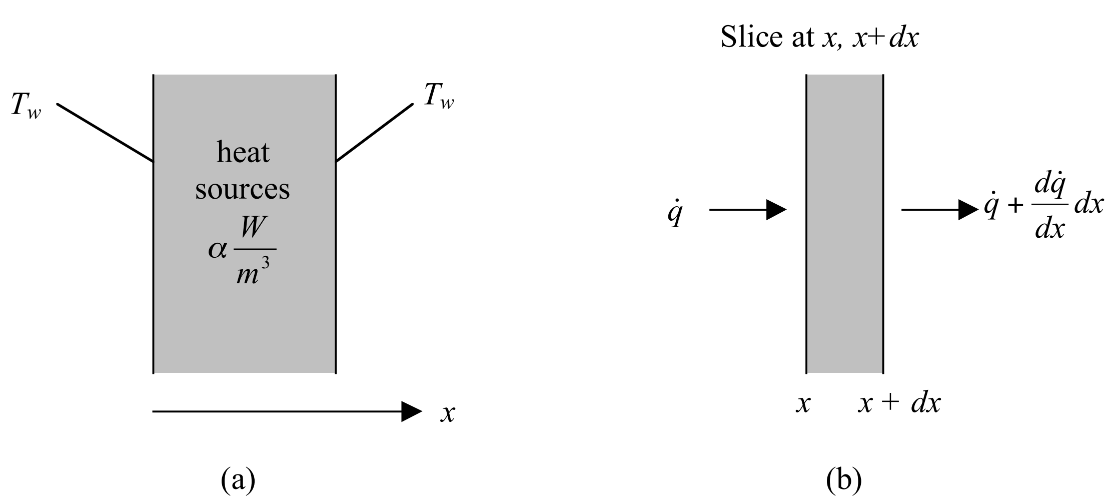


Figure 4.1: Slab with heat sources (a) overall configuration, (b) elementary slice

With reference to Figure 4.1(b), a steady-state energy balance yields an equation for the heat flux, $\dot{q}$.

$$\dot{q} + a dx - \left(\dot{q} + \frac{d\dot{q}}{dx} dx \right) = 0 \tag{4.1}$$

or
$$\frac{d\dot{q}}{dx} = \alpha . \tag{4.2}$$

There is a change in heat flux with x due to the presence of the heat sources. The equation for the temperature is

$$\frac{d^2T}{dx^2} + \alpha / k = 0 \tag{4.3}$$

Equation (4.3) can be integrated once,

$$\frac{dT}{dx} = -\frac{\alpha}{k} x + a \tag{4.4}$$

and again to give

$$T = -\frac{\alpha}{2k} x^2 + ax + b \tag{4.5}$$

where a and b are constants of integration. The boundary conditions imposed are $T(0) = T(L) = T_w$. Substituting these into Equation (4.5) gives $b = T_w$ and $a = \frac{\alpha L}{2k}$. The temperature distribution is thus

$$T = -\frac{\alpha x^2}{2k} + \frac{\alpha}{2k} Lx + T_W . \tag{4.6}$$

Writing (4.6) in a normalized, non-dimensional fashion gives a form that exhibits in a more useful manner the way in which the different parameters enter the problem:

$$\frac{T - T_W}{\alpha L^2 / k} = \frac{1}{2} \left(\frac{x}{L} - \frac{x^2}{L^2} \right) \tag{4.7}$$

This distribution is sketched in Figure 4.2. It is symmetric about the mid-plane at $x = \frac{L}{2}$, with half the energy due to the sources exiting the slab on each side.

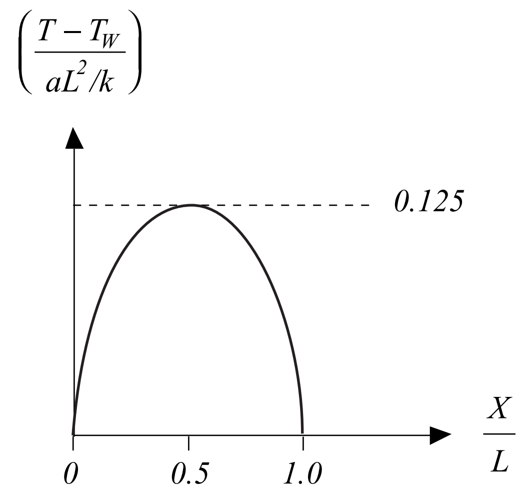


Figure 4.2:Temperature distribution for slab with distributed heat sources

The heat flux at the side of the slab, $x = 0$, can be found by differentiating the temperature distribution and evaluating at $x = 0$: $-k\left.\frac{dT}{dx}\right|_{x=0} = -k\frac{\alpha L^2}{2K}\left(\frac{1}{L}\right) = -\alpha L/2$.

This is half of the total heat generated within the slab. The magnitude of the heat flux is the same at $x = L$, although the direction is opposite.

A related problem would be one in which there were heat flux (rather than temperature) boundary conditions at $x = 0$ and $x = L$, so that T_w is not known. We again determine the maximum temperature. At $x = 0$ and L, the heat flux and temperature are continuous so

$$-k\frac{dT}{dx} = h\left(T - T_\infty\right) \text{ at } x = 0, L. \tag{4.8}$$

Referring to the temperature distribution of Equation (4.6) gives for the two terms in Equation (4.8),

$$k\frac{dT}{dx} = k\left(-\frac{\alpha x}{k} + a\right) = \left(-\alpha x + ka\right) \tag{4.9}$$

$$h\left(T - T_\infty\right) = h\left(-\frac{\alpha x^2}{k} + \alpha x + b - T_\infty\right) \tag{4.10}$$

Evaluating (4.10) at $x = 0$ and L allows determination of the two constants a and b. This is left as an exercise for the reader.

Muddy points

For an electric heated strip embedded between two layers, what would the temperature distribution be if the two side temperatures were not equal? (MP HT.15)

5.0 Heat Transfer From a Fin

Fins are used in a large number of applications to increase the heat transfer from surfaces. Typically, the fin material has a high thermal conductivity. The fin is exposed to a flowing fluid, which cools or heats it, with the high thermal conductivity allowing increased heat being conducted from the wall through the fin. The design of cooling fins is encountered in many situations and we thus examine heat transfer in a fin as a way of defining some criteria for design.

A model configuration is shown in Figure 5.1. The fin is of length *L*. The other parameters of the problem are indicated. The fluid has velocity c_∞ and temperature T_∞.

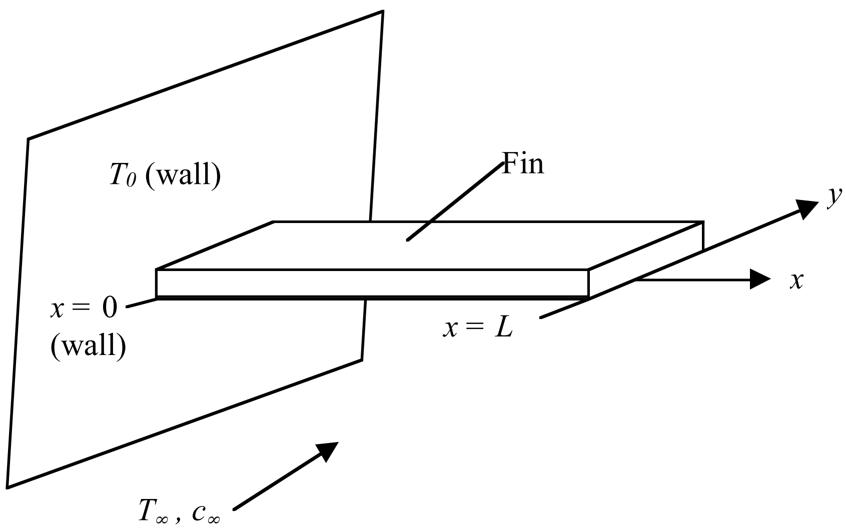


Figure 5.1: Geometry of heat transfer fin

We assume (using the Reynolds analogy or other approach) that the heat transfer coefficient for the fin is known and has the value *h*. The end of the fin can have a different heat transfer coefficient, which we can call h_L.

The approach taken will be quasi-one-dimensional, in that the temperature in the fin will be assumed to be a function of *x* only. This may seem a drastic simplification, and it needs some explanation. With a fin cross-section equal to *A* and a perimeter *P,* the characteristic dimension in the transverse direction is A / P (For a circular fin, for example, $A / P = r / 2$). The regime of interest will be taken to be that for which the Biot number is much less than unity, $Bi = \frac{h(A/P)}{k} << 1$, which is a realistic approximation in practice.

The physical content of this approximation can be seen from the following. Heat transfer per unit area out of the fin to the fluid is roughly of magnitude $\sim h(T_w - T_\infty)$ per unit area . The heat transfer per unit area *within the* fin in the transverse direction is (again in the same approximate terms)

$k\dfrac{(T_1 - T_w)}{A/P}$, where T_I is an internal temperature. These two quantities must be of the same magnitude. If $h\dfrac{A/P}{k} << 1$, then $\dfrac{T_I - T_w}{T_w - T_\infty} << 1$. In other words, if $Bi << 1$, there is a much larger capability for heat transfer per unit area across the fin than there is between the fin and the fluid, and thus little variation in temperature inside the fin in the transverse direction. To emphasize the point, consider the limiting case of zero heat transfer to the fluid i.e., an insulated fin. Under these conditions, the temperature within the fin would be uniform and equal to the wall temperature.

If there *is* little variation in temperature *across* the fin, an appropriate model is to say that the temperature within the fin is a function of x only, $T = T(x)$, and use a quasi-one-dimensional approach. To do this, consider an element, dx, of the fin as shown in Figure 5.2. There is heat flow of magnitude $\dot{Q}_{in}$ at the left-hand side and heat flow out of magnitude $\dot{Q}_{out} = \dot{Q}_{in} + \dfrac{d\dot{Q}}{dx}dx$ at the right hand side. There is also heat transfer around the perimeter on the top, bottom, and sides of the fin. From a quasi-one-dimensional point of view, this is a situation similar to that with internal heat sources, but here, for a cooling fin, in each elemental slice of thickness dx there is essentially a heat sink of magnitude $Pdxh\left(T - T_\infty\right)$, where Pdx is the area for heat transfer to the fluid.

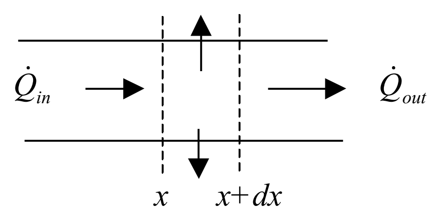


Figure 5.2:Element of fin showing heat transfer

The heat balance for the element in Figure 5.2 can be written in terms of the heat flux using $\dot{Q} = \dot{q}A$, for a fin of constant area:

$$\dot{q}A = Ph(T - T_\infty)dx + \left(\dot{q}A + \frac{d\dot{q}}{dx}dxA\right) \quad . \tag{5.1}$$

From Equation (5.1) we obtain

$$A\frac{d\dot{q}}{dx} + Ph(T - T_\infty) = 0. \tag{5.2}$$

In terms of the temperature distribution, $T(x)$:

$$\frac{d^2T}{dx^2} - \frac{Ph}{Ak}\left(T - T_\infty\right) = 0. \tag{5.3}$$

The quantity of interest is the temperature difference $(T - T_\infty)$, and we can change variables to put Equation (5.3) in terms of this quantity using the substitution

$$\frac{d}{dx}(T - T_\infty) = \frac{dT}{dx}. \tag{5.4}$$

Equation (5.3) can therefore be written as

$$\frac{d^2}{dx^2}(T - T_\infty) - \frac{hP}{Ak}(T - T_\infty) = 0. \tag{5.5}$$

Equation (5.5) describes the temperature variation along the fin. It is a second order equation and needs two boundary conditions. The first of these is that the temperature at the end of the fin that joins the wall is equal to the wall temperature. (Does this sound plausible? Why or why not?)

$$(T - T_\infty)_{x=0} = T_0 - T_\infty. \tag{5.6}$$

The second boundary condition is at the other end of the fin. We will assume that the heat transfer from this end is negligible[1]. The boundary condition at $x = L$ is

$$\frac{d}{dx}\left(T - T_\infty\right)\Big|_{x=L} = 0. \tag{5.7}$$

The last step is to work in terms of non-dimensional variables to obtain a more compact description. In this we define $\frac{T - T_\infty}{T_0 - T_\infty}$ as $\Delta\tilde{T}$, where the values of $\Delta\tilde{T}$ range from zero to one and $\xi = x/L$, where ξ also ranges over zero to one. The relation between derivatives that is needed to cast the equation in terms of ξ is

$$\frac{d}{dx} = \frac{d\xi}{dx}\frac{d}{d\xi} = \frac{1}{L}\frac{d}{d\xi}.$$

Equation (5.5) can be written in this dimensionless form as

$$\frac{d^2\Delta\tilde{T}}{d\xi^2} - \left(\frac{hP}{kA}L^2\right)\Delta\tilde{T} = 0. \tag{5.8}$$

There is one non-dimensional parameter in Equation (5.8), which we will call m and define by

[1] Note: We don't need to make this assumption, and if we were looking at the problem in detail we would solve it numerically and not worry about whether an analytic solution existed. In the present case, developing the analytic solution is useful in presenting the structure of the solution as well as the numbers, so we resort to the mild fiction of no heat transfer at the fin end. We need to assess, after all is said and done, whether this is appropriate or not.

$$m^2L^2 = \frac{hPL^2}{kA}\ . \tag{5.9}$$

The equation for the temperature distribution we have obtained is

$$\frac{d^2\Delta\tilde{T}}{d\xi^2} - m^2L^2\Delta\tilde{T} = 0\,. \tag{5.10}$$

This second order equation has the solution

$$\Delta\tilde{T} = ae^{mL\xi} + be^{-mL\xi}\ . \tag{5.11}$$

(Try it and see). The boundary condition at $\xi = 0$ is

$$\Delta\tilde{T}(0) = a + b = 1\,. \tag{5.12a}$$

The boundary condition at $\xi = 1$ is that the temperature gradient is zero or

$$\frac{d\Delta\tilde{T}}{d\xi}(L) = mLae^{m} - mLbe^{-m} = 0\,. \tag{5.12b}$$

Solving the two equations given by the boundary conditions for *a* and *b* gives an expression for $\Delta\tilde{T}$ in terms of the hyperbolic cosine or cosh: $\left(\cosh x = \frac{e^x + e^{-x}}{2}\right)$

$$\Delta\tilde{T} = \frac{\cosh mL(1-\xi)}{\cosh mL} \tag{5.13}$$

This is the solution to Equation (5.8) for a fin with no heat transfer at the tip. In terms of the actual heat transfer parameters it is written as

$$\frac{T - T_\infty}{T_0 - T_\infty} = \frac{\cosh\left(\left(1 - \frac{x}{L}\right)\sqrt{\frac{hP}{kA}}L\right)}{\cosh\left(\sqrt{\frac{hP}{kA}}L\right)}. \tag{5.14}$$

The amount of heat removed from wall due to the fin, which is the quantity of interest, can be found by differentiating the temperature and evaluating the derivative at the wall, $x = 0$:

$$\dot{Q} = -kA\frac{d}{dx}\left(T - T_\infty\right)\Big|_{x=0} \tag{5.15}$$

or

$$\frac{\dot{Q}L}{kA\left(T_0 - T_\infty\right)} = -\frac{dD\tilde{T}}{d\xi}\Big|_{\xi = 0} = \frac{mL\sinh mL}{\cosh mL} = mL\tanh mL \tag{5.16}$$

$$\frac{\dot{Q}}{\sqrt{kAhP}\left(T_0 - T_\infty\right)} = \tanh mL \tag{5.16a}$$

The solution is plotted in Figure 5.3, which is taken from the book by Lienhard. Several features of the solution should be noted. First, one does not need fins which have a length such that *m* is much greater than 3. Second, the assumption about no heat transfer at the end begins to be inappropriate as *m* gets smaller than 3, so for very short fins the simple expression above would not be a good estimate. We will see below how large the error is.

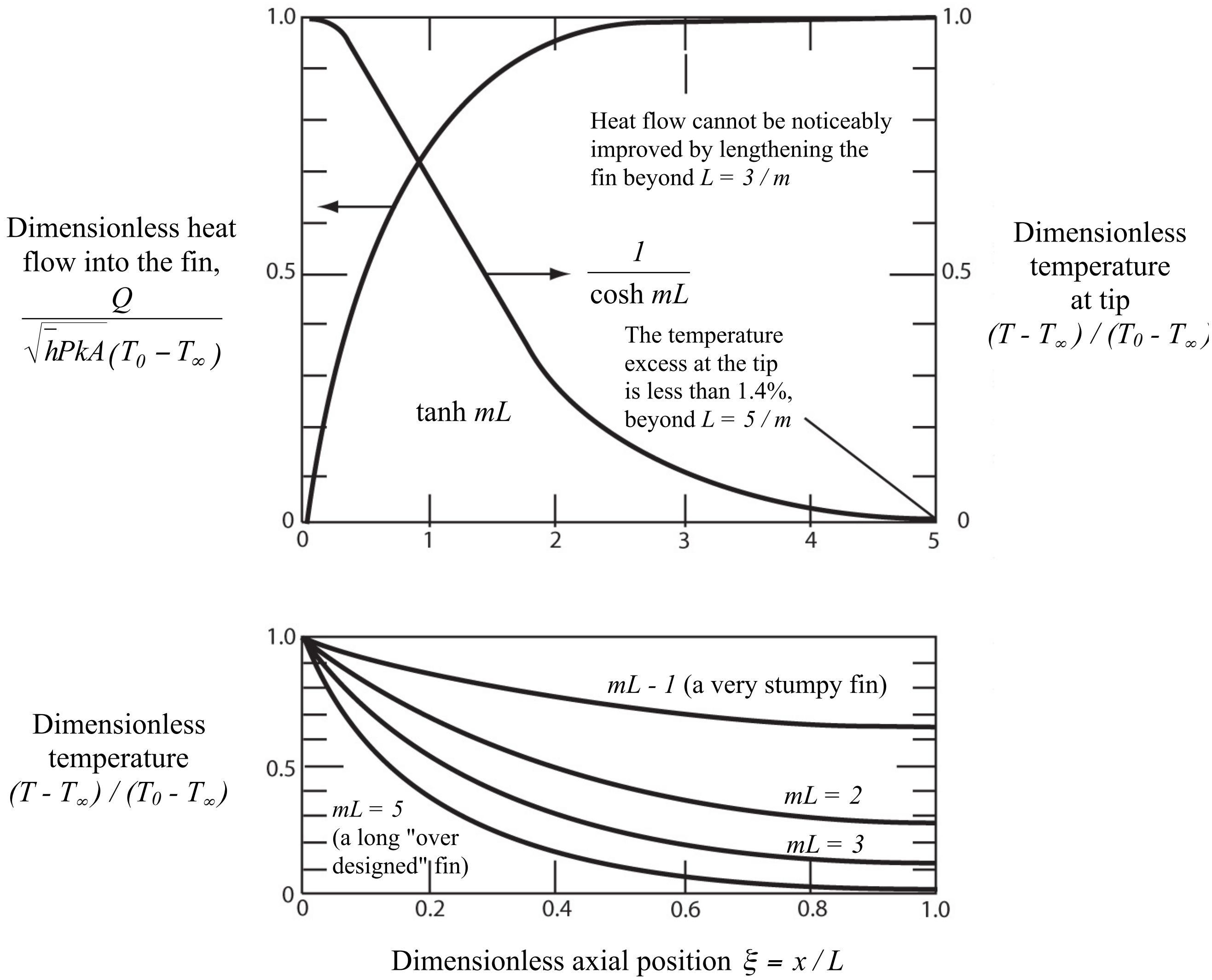


Figure 5.3:The temperature distribution, tip temperature, and heat flux in a straight one-dimensional fin with the tip insulated. [Adapted from: Lienhard, *A Heat Transfer Textbook,* Prentice-Hall publishers]

Muddy points

Why did you change the variable and write the derivative $\frac{d^2T}{dx^2}$ as $\frac{d^2(T - T_\infty)}{dx^2}$ in the equation for heat transfer in the fin? (MP HT.16)

What types of devices use heat transfer fins? (MP HT.17)

Why did the Stegosaurus have cooling fins? Could the stegosaurus have "heating fins"? (MP HT.18)

6.0 Transient Heat Transfer (Convective Cooling or Heating)

All the heat transfer problems we have examined have been steady state, but there are often circumstances in which the *transient* response to heat transfer is critical. An example is the heating up of gas turbine compressors as they are brought up to speed during take-off. The disks that hold the blades are large and take a long time to come to temperature, while the casing is thin and in the path of high velocity compressor flow and thus comes to temperature rapidly. The result is that the case expands away from the blade tips, sometimes enough to cause serious difficulties with aerodynamic performance.

To introduce the topic as well as to increase familiarity with modeling of heat transfer problems, we examine a *lumped parameter* analysis of an object cooled by a stream. This will allow us to see what the relevant non-dimensional parameters are and, at least in a quantitative fashion, how more complex heat transfer objects will behave. We want to view the object as a "lump" described by a single parameter. We need to determine when this type of analysis would be appropriate. To address this, consider the temperature difference T_1 - T_w between two locations in the object, as shown in Figure 6.1.

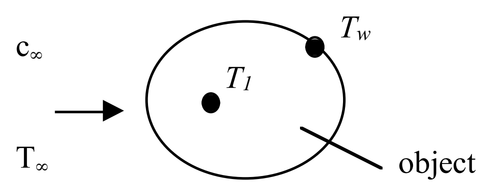


Figure 6.1:Temperature variation in an object cooled by a flowing fluid

If the heat transfer within the body and from the body to the fluid are of the same magnitude,

$$h\left(T_w - T_\infty\right) \approx \frac{k}{L}\left(T_1 - T_w\right) \tag{6.1}$$

where L is a relevant length scale, say half the thickness of the object. The ratio of the temperature difference is

$$\frac{T_1 - T_w}{T_w - T_\infty} \approx \frac{hL}{k} \tag{6.2}$$

If the Biot number is small the ratio of temperature differences described in Equation (6.2) is also $\frac{\left(T_1 - T_w\right)}{\left(T_w - T_\infty\right)} << 1$. We can thus say $\left(T_1 - T_w\right) << T_w - T_\infty$ and neglect the temperature non-uniformity *within* the object.

The approximation made is to view the object as having a spatially uniform temperature that is a function of time only. Explicitly, $T = T(t)$. The first law applied to the object is (using the fact that for solids $c_p = c_v = c$)

$$\dot{Q}_{in} = \rho V c \frac{dT}{dt} \tag{6.3}$$

where ρ is the density of the object and V is its volume. In terms of heat transferred to the fluid, $\dot{Q}_{out} = -\rho V c \frac{dT}{dt}$. The rate of heat transfer to the fluid is $Ah\,(T - T_\infty)$, so the expression for the time evolution of the temperature is

$$Ah\left(T - T_\infty\right) = -\rho V c \frac{dT}{dt}. \tag{6.4}$$

The initial temperature, $T(0)$, is equal to some known value, which we can call T_i. Using this, Equation (6.4) can be written in terms of a non-dimensional temperature difference $\left(\frac{T - T_\infty}{T_i - T_\infty}\right)$,

$$\frac{d}{dt}\left(\frac{T - T_\infty}{T_i - T_\infty}\right) + \frac{hA}{\rho V c}\left(\frac{T - T_\infty}{T_i - T_\infty}\right) = 0 \tag{6.5}$$

At time $t = 0$, this non-dimensional quantity is equal to one. Equation (6.5) is an equation you have seen before, $\left(\frac{dx}{dt} + \frac{x}{\tau} = 0\right)$ which has the solution $x = ae^{-t/\tau}$. For the present problem the form is

$$\frac{T - T_\infty}{T_i - T_\infty} = ae^{-hAt/pVc}. \tag{6.6}$$

The constant a can be seen to be equal to unity to satisfy the initial condition. This form of equation implies that the solution has a heat transfer "time constant" given by $\tau = \frac{\rho V c}{hA}$.

The time constant, τ, is in accord with our "intuition"; high density, large volume, or high specific heat all tend to increase the time constant, while high heat transfer coefficient and large area will tend to decrease the time constant. This is the same form of equation and the same behavior you have seen for the R-C circuit, as shown schematically below. The time dependence of the voltage in the R-C circuit when the switch is opened suddenly is given by the equation $\frac{dE}{dt} + \frac{E}{RC} = 0$. There are, in fact, a number of physical processes which have (or can be modeled as having) this type of exponentially decaying behavior.

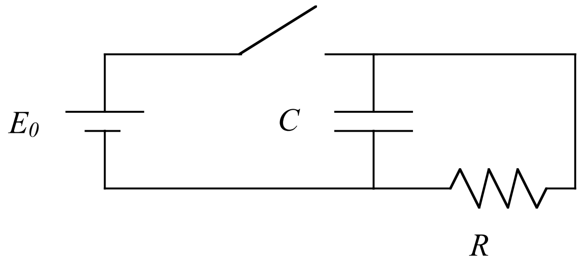


Figure 6.2: Voltage change in an R-C circuit

Muddy points

In equation $\dot{Q}_{in} = \rho.V.c.\frac{dT}{dt}$ (6.3), what is c? (MP HT.19)

In the lumped parameter transient heat transfer problem, does a high density "slow down" heat transfer? (MP HT.20)

7.0 Some Considerations in Modeling Complex Physical Processes

In Sections 5 and 6, a number of assumptions were made about the processes that we were attempting to describe. These are all part of the general approach to modeling of physical systems. The main idea is that for engineering systems, one almost always cannot compute the process exactly, especially for fluid flow problems. At some level of detail, one generally needs to *model*, i.e. to define some plausible behavior for attributes of the system that will not be computed. Modeling can span an enormous range from the level of our assumption of uniform temperature within the solid object to a complex model for the small scale turbulent eddies in the flow past a compressor blade. In carrying out such modeling, it is critical to have a clear idea of just what the assumptions really mean, as well as the fidelity that we ascribe to the descriptions of actual physical phenomena, and we thus look at the statements we have made in this context.

One assertion made was that because $\frac{hL}{k} << 1$ and on the basis of a heat balance,

$$\frac{k}{L}\left(T_c - T_w\right) \approx h\left(T_W - T_\infty\right)$$

we could assume

$$\frac{(T_{\text{body interior}} - T_w)}{(T_w - T_\infty)} << 1$$

Based on this, we said that T_{body} is approximately uniform and $T_w \approx T_{\text{body interior}}$. Another aspect is that setting $\frac{hA}{Pk} << 1$ erases any geometrical detail of the fin cross section. The only place where P and A enter the problem is in a non-dimensional combination. A third assumption, made in the fin problem, was that the heat transfer at the far end can be neglected. The solution including this effect, where $\frac{h_L L}{k}$ is an axial Biot number is given as Equation (7.1). If the quantity $\frac{h_L L}{k}$ is small, you can see that Equation (7.1) reduces to the previous result (5.13).

$$\frac{T - T_\infty}{T_0 - T_\infty} = \frac{\cosh mL(1-\xi) + (Bi_{\text{axial}}/m)\sinh mL(1-\xi)}{\cosh mL + (Bi_{\text{axal}}/mL)\sinh mL} \tag{7.1}$$

and

$$\frac{\dot{Q}}{\sqrt{kAhP}(T_0 - T_\infty)} = \frac{(Bi_{axial}/mL) + \tanh mL}{1 + \frac{Bi_{axial}}{mL}\tanh mL} \tag{7.2}$$

8.0 Heat Exchangers

The general function of a heat exchanger is to transfer heat from one fluid to another. The basic component of a heat exchanger can be viewed as a tube with one fluid running through it and another fluid flowing by on the outside. There are thus three heat transfer operations that need to be described:

1) Convective heat transfer from fluid to the inner wall of the tube
2) Conductive heat transfer through the tube wall
3) Convective heat transfer from the outer tube wall to the outside fluid

Heat exchangers are typically classified according to flow arrangement and type of construction. The simplest heat exchanger is one for which the hot and cold fluids move in the same or opposite directions in a concentric tube (or double-pipe) construction. In the parallel-flow arrangement of Figure 8.1a, the hot and cold fluids enter at the same end, flow in the same direction, and leave at the same end. In the counterflow arrangement of Figure 8.1b, the fluids enter at opposite ends, flow in opposite directions, and leave at opposite ends.

Alternatively, the fluids may be in cross flow (perpendicular to each other), as shown by the finned and unfinned tubular heat exchangers of Figure 8.2. The two configurations differ according to whether the fluid moving over the tubes is unmixed or mixed. In Figure 8.2a, the fluid is said to be unmixed because the fins prevent motion in a direction (y) that is transverse to the main-flow direction (x). In this case the fluid temperature varies with x and y. In contrast, for the unfinned tube bundle of Figure 8.2b, fluid motion, hence mixing, in the transverse direction is possible, and temperature variations are primarily in the main-flow direction. Since the tube flow is unmixed, both fluids are unmixed in the finned exchanger, while one fluid is mixed and the other unmixed in the unfinned exchanger.

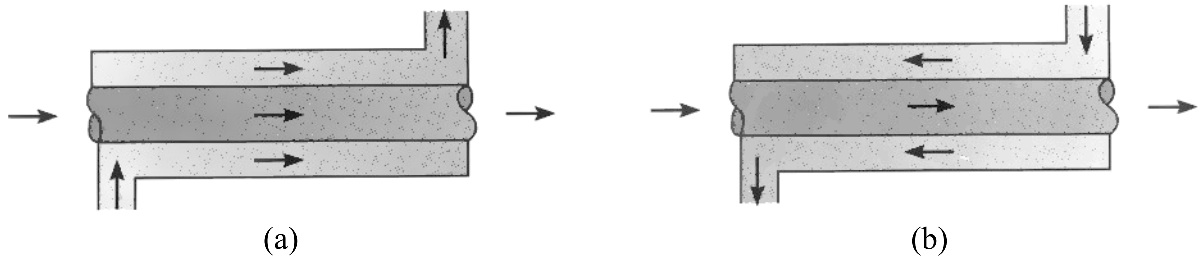


Figure 8.1: Concentric tubes heat exchangers. (a) Parallel flow. (b) Counterflow

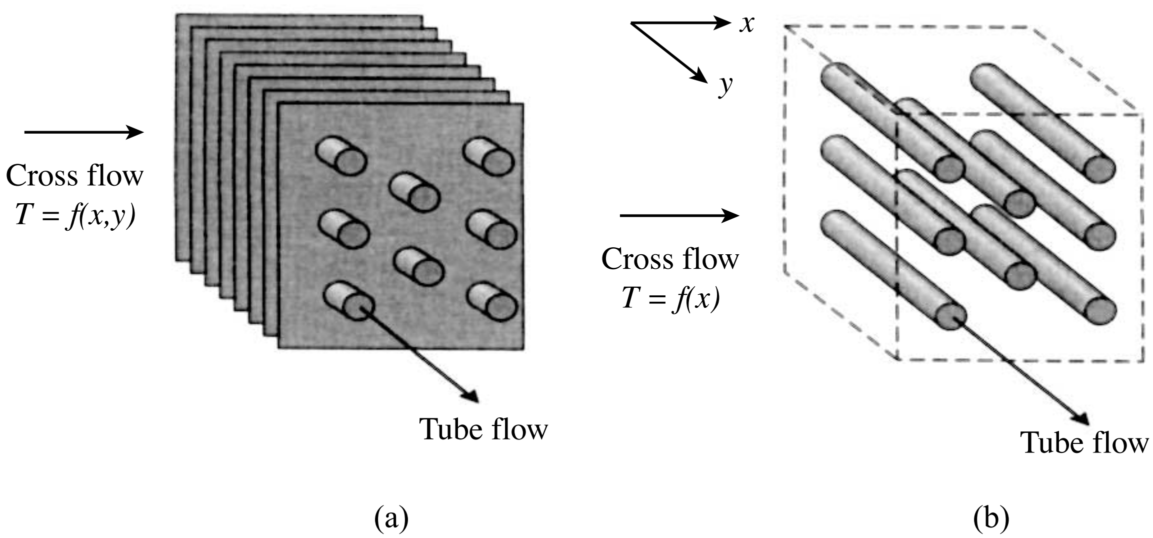


Figure 8.2: Cross-flow heat exchangers. *(a)* Finned with both fluids unmixed. *(b)* Unfinned with one fluid mixed and the other unmixed

To develop the methodology for heat exchanger analysis and design, we look at the problem of heat transfer from a fluid inside a tube to another fluid outside.

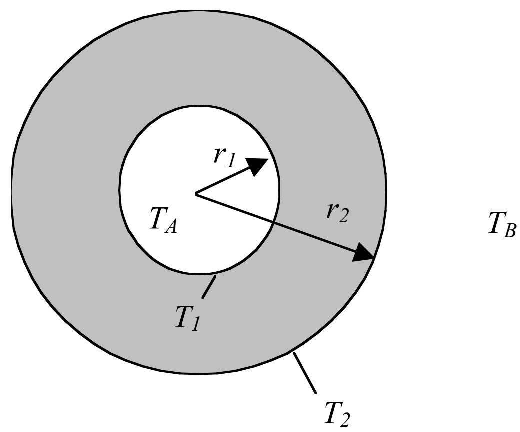


Figure 8.3: Geometry for heat transfer between two fluids

We examined this problem before in Section 3.2 and found that the heat transfer rate per unit length is given by

$$\dot{Q} = \frac{2\pi k(T_A - T_B)}{\frac{k}{r_1 h_1} + \ln\frac{r_2}{r_1} + \frac{k}{r_2 h_2}} \tag{8.1}$$

It is useful to define an overall heat transfer coefficient h_0 per unit length as

$$\dot{Q} = 2\pi r_2 h_0 (T_A - T_B). \tag{8.2}$$

From (8.1) and (8.2) the overall heat transfer coefficient, h_0 , is

$$\frac{1}{h_0} = \frac{r_2}{r_1 h_1} + \frac{r_2}{k}\ln\frac{r_2}{r_1} + \frac{1}{h_2}. \tag{8.3}$$

We will make use of this in what follows.

A schematic of a counterflow heat exchanger is shown in Figure 8.4. We wish to know the temperature distribution along the tube and the amount of heat transferred.

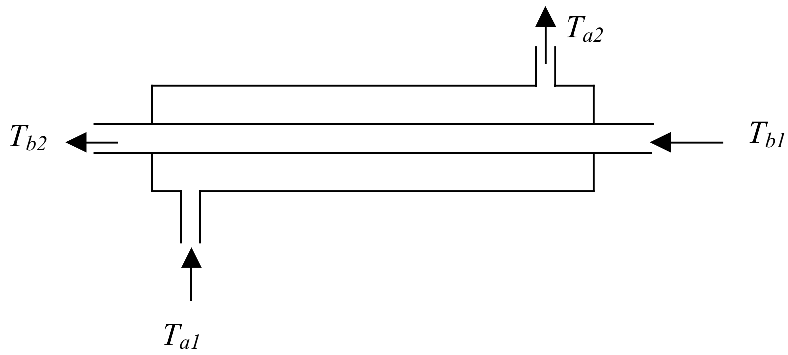


Figure 8.4: Counterflow heat exchanger

To address this we start by considering the general case of axial variation of temperature in a tube with wall at uniform temperature T_0 and a fluid flowing inside the tube.

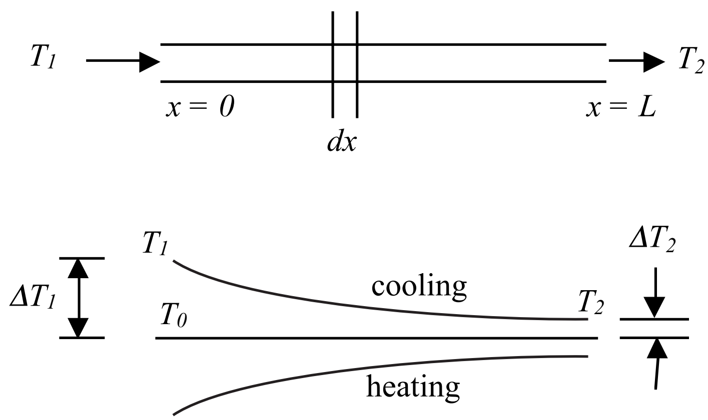


Figure 8.5: Fluid temperature distribution along the tube with uniform wall temperature

The objective is to find the mean temperature of the fluid at x, $T(x)$, in the case where fluid comes in at $x = 0$ with temperature T_1 and leaves at $x = L$ with temperature T_2. The expected distribution for heating and cooling are sketched in Figure 8.5.

For heating ($T_0 > T$), the heat flow from the pipe wall in a length dx is

$$\dot{q}\,\pi D dx = h\pi D(T_0 - T)dx$$

where D is the pipe diameter. The heat given to the fluid (the change in enthalpy) is given by

$$\rho u_m c_p \frac{\pi D^2}{4}\mathrm{dT} = \dot{\mathrm{m}} c_p dT$$

where ρ is the density of the fluid, u_m is the mean velocity of the fluid, c_p is the specific heat of the fluid and $\dot{m}$ is the mass flow rate of the fluid.

Setting the last two expressions equal and integrating from the start of the pipe, we find

$$\int_{T_1}^{T} \frac{dT}{T_0 - T} = \int_{0}^{x} \frac{4h}{\rho u_m c_p D} dx. \tag{8.4}$$

Carrying out the integration,

$$\frac{4hx}{\rho u_m c_p D} = \int_{T_1}^{T} \frac{dT}{T_0 - T} = -\int_{T_1}^{T} \frac{d(T_0 - T)}{T_0 - T} = -\ln(T_0 - T)\Big|_{T_1}^{T}, \tag{8.5}$$

i.e.

$$\ln\left(\frac{T_0 - T}{T_0 - T_1}\right) = -\frac{4hx}{\rho u_m c_p D}. \tag{8.6}$$

Equation (8.6) can be written as

$$\frac{T_0 - T}{T_0 - T_1} = e^{-\beta x} \tag{8.7}$$

where

$$\beta = \frac{4h}{\rho u_m c_p D} = \frac{\pi h D}{\dot{m} c_p} \quad . \tag{8.8}$$

This is temperature distribution along the pipe. The exit temperature at $x = L$ is

$$\frac{T_0 - T_2}{T_0 - T_1} = e^{-\frac{\pi h D L}{\dot{m} c_p}} \tag{8.9}$$

The total heat transfer to the wall all along the pipe is

$$\dot{Q} = \dot{m} c_p (T_1 - T_2).$$

From Equation (8.9),

$$\dot{m}c_p = \frac{h\pi DL}{\ln\left(\dfrac{T_0 - T_1}{T_0 - T_2}\right)}.$$

The total rate of heat transfer is therefore

$$\dot{Q} = \frac{h\pi DL\left(T_1 - T_2\right)}{\ln\dfrac{T_1 - T_0}{T_2 - T_0}}, \tag{8.10}$$

$$\text{or} \quad \dot{Q} = h\pi DL\Delta T_{LM}$$

where ΔT_{LM} is the logarithmic mean temperature difference, defined as

$$\Delta T_{LM} = \frac{T_2 - T_1}{\ln\dfrac{T_0 - T_2}{T_0 - T_1}} = \frac{\Delta T_1 - \Delta T_2}{\ln\left(\dfrac{\Delta T_1}{\Delta T_2}\right)}. \tag{8.11}$$

The concept of a logarithmic mean temperature difference is useful in the analysis of heat exchangers. We can define a logarithmic mean temperature difference for a counterflow heat exchanger as follows. With reference to Figure 8.4, an overall heat balance between the two counterflowing streams is

$$\dot{Q} = \dot{m}_a c_{pa}\left(T_{a1} - T_{a2}\right) = \dot{m}_b c_{pb}\left(T_{b2} - T_{b1}\right). \tag{8.12}$$

From a local heat balance, the heat given up by stream a in length dx is $-\dot{m}_a c_{pa} dT_a$. (There is a negative sign since T_a decreases). The heat taken up by stream b is $-\dot{m}_b c_{pb} dT_b$. (There is a negative sign because T_b decreases as x increases). The local heat balance is:

$$-\dot{m}_a c_{pa} dT_a = -\dot{m}_b c_{pb} dT_b = \dot{q}\, dA = \dot{q}\pi D dx \tag{8.13}$$

Solving (8.13) for dT_a and dT_b, we find:

$$dT_a = -\frac{\dot{q}\, dA}{\dot{m}_a c_{pa}}; \quad dT_b = -\frac{\dot{q}\, dA}{\dot{m}_b c_{pb}} \tag{8.14}$$

$$d\left(T_a - T_b\right) = d\Delta T = -\left(\frac{1}{\dot{m}_a c_{pa}} - \frac{1}{\dot{m}_b c_{pb}}\right)\dot{q}\, dA = -\left(\frac{1}{W_a} - \frac{1}{W_b}\right)\dot{q}\,\pi D dx \qquad (8.15)$$

where $W = \dot{m}c_p$. Also, $\dot{q} = h_0\, \Delta T$ where h_0 is the overall heat transfer coefficient. We can then say:

$$\frac{d\Delta T}{\Delta T} = -h_0 \pi D\left(\frac{1}{W_a} - \frac{1}{W_b}\right)dx\,.$$

Integrating from $x = 0$ to $x = L$ gives

$$\ln\left(\frac{T_{a2} - T_{b1}}{T_{a1} - T_{b2}}\right) = -h_0 \pi D L\left(\frac{1}{W_a} - \frac{1}{W_b}\right). \qquad (8.16)$$

Equation (8.16) can also be written as:

$$\frac{T_{a2} - T_{b1}}{T_{a1} - T_{b2}} = e^{-\alpha} \qquad (8.17)$$

where

$$\alpha = h_0 \pi D L\left(\frac{1}{W_a} - \frac{1}{W_b}\right)$$

We know that

$$T_{a1} - T_{a2} = \dot{Q}/W_a \qquad T_{b2} - T_{b1} = \dot{Q}/W_b\,. \qquad (8.18)$$

Thus

$$\left(T_{a1} - T_{b2}\right) - \left(T_{a2} - T_{b1}\right) = \dot{Q}\left(\frac{1}{W_a} - \frac{1}{W_b}\right). \qquad (8.19)$$

Solving for the total heat transfer:

$$\dot{Q} = \frac{\left(T_{a1} - T_{b2}\right) - \left(T_{a2} - T_{b1}\right)}{\left(\frac{1}{W_a} - \frac{1}{W_b}\right)} \qquad (8.20)$$

Rearranging (8.16) allows us to express $\left(\frac{1}{W_a} - \frac{1}{W_b}\right)$ in terms of other parameters as

$$\left(\frac{1}{W_a} - \frac{1}{W_b}\right) = -\frac{\ln\left(\frac{T_{a2} - T_{b2}}{T_{a1} - T_{b2}}\right)}{h_0 \pi D L}. \tag{8.21}$$

Substituting (8.21) into (8.20) we obtain a final expression for the total heat transfer for a counterflow heat exchanger:

$$\dot{Q} = h_0 \pi D L \frac{\left(T_{a1} - T_{b2}\right) - \left(T_{a2} - T_{b1}\right)}{\ln\left(\frac{T_{a1} - T_{b2}}{T_{a2} - T_{b1}}\right)} \tag{8.22}$$

or

$$\dot{Q} = h_0 \pi D L \Delta T_{LM} \tag{8.23}$$

8.1 Efficiency of a Counterflow Heat Exchanger

Suppose we know the two inlet temperatures T_{a1}, T_{b1}, and we need to find the outlet temperatures.

From (8.17)

$$T_{a2} - T_{b1} = (T_{a1} - T_{b2})\, e^{-\alpha}$$

$$T_{a2} - T_{a1} = T_{b1} - T_{a1} + (T_{a1} - T_{b2})\, e^{-\alpha}$$

Rearranging (8.18),

$$T_{b2} = T_{b1} + \frac{W_a}{W_b}\left(T_{a1} - T_{a2}\right)$$

Thus

$$T_{a2} - T_{a1} = T_{b1} - T_{a1} + (T_{a1} - T_{b1})\, e^{-\alpha} - \frac{W_a}{W_b}\left(T_{a1} - T_{a2}\right) e^{-\alpha}$$

$$\left(T_{a1} - T_{a2}\right)\left(1 - \frac{W_a}{W_b} e^{-\alpha}\right) = \left(T_{a1} - T_{b1}\right)\left(1 - e^{-\alpha}\right)$$

or

$$\left(T_{a1} - T_{a2}\right) = \eta\left(T_{a1} - T_{b1}\right) \tag{8.24}$$

where η is the efficiency of a counterflow heat exchanger:

$$\eta = \frac{1-e^{-\alpha}}{1-\frac{W_a}{W_b}e^{-\alpha}} = \frac{1-e^{-\alpha}}{1-\frac{\dot{m}_a}{\dot{m}_b}\frac{c_{pa}}{c_{pb}}e^{-\alpha}} \tag{8.25}$$

From (8.24) and (8.25) we can find outlet temperatures T_{a2} and T_{b2}:

$$T_{b2} - T_{b1} = \frac{\dot{m}_a c_{pa}}{\dot{m}_b c_{pb}}\left(T_{a1} - T_{a2}\right) = \frac{\dot{m}_a c_{pa}}{\dot{m}_b c_{pb}}\eta\left(T_{a1} - T_{b2}\right)$$

We examine three examples.

i) $\dot{m}_b c_{pb} > \dot{m}_a c_{pa}$

ΔT can approach zero at cold end

$\eta \to 1$ as h_0, surface area , $\pi D L h_0\left(\frac{1}{\dot{m}_a c_{pa}} - \frac{1}{\dot{m}_b c_{pb}}\right)$

Maximum value of ratio $\frac{T_{a1} - T_{a2}}{T_{a1} - T_{b1}} = 1$

Maximum value of ratio $\frac{T_{b2} - T_{b1}}{T_{a1} - T_{b1}} = \frac{\dot{m}_a c_{pa}}{\dot{m}_b c_{pb}}$

ii) $\dot{m}_b c_{pb} < \dot{m}_a c_{pa}$

α is negative , $\eta \to \frac{\dot{m}_b c_{pb}}{\dot{m}_a c_{pa}}$ as $[\ ] \to \infty \; (W_b < W_a)$

Maximum value of $\frac{T_{a1} - T_{a2}}{T_{a1} - T_{b1}} = \frac{\dot{m}_b c_{pb}}{\dot{m}_a c_{pa}}$

Maximum value of $\frac{T_{b2} - T_{b1}}{T_{a1} - T_{b2}} = 1$

iii) $\dot{m}_a c_{pa} = \dot{m}_b c_{pb}$

$d(T_a - T_b) = 0$

temperature difference remains uniform, $\eta = 1$

9.0 Radiation Heat Transfer (Heat transfer by thermal radiation)

All bodies radiate energy in the form of photons moving in a random direction, with random phase and frequency. When radiated photons reach another surface, they may either be absorbed, reflected or transmitted. The behavior of a surface with radiation incident upon it can be described by the following quantities:

α = absorptance - fraction of incident radiation <u>absorbed</u>
ρ = reflectance - fraction of incident radiation <u>reflected</u>
τ = transmittance – fraction of incident radiation <u>transmitted</u>

Figure 9.1 shows these processes graphically.

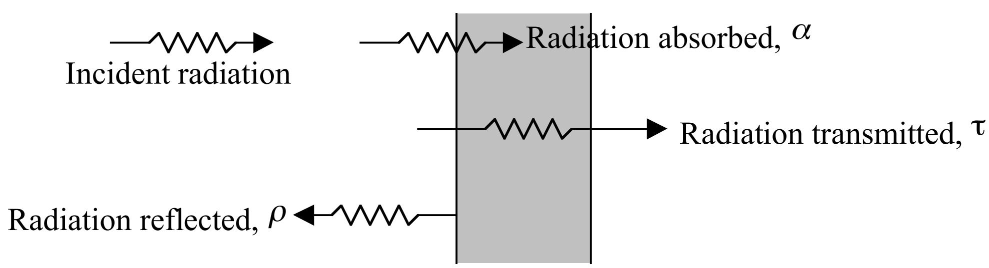


Figure 9.1: Radiation Surface Properties

From energy considerations the three coefficients must sum to unity

$$\alpha + \rho + \tau = 1 \tag{9.1}$$

Reflective energy may be either diffuse or specular (mirror-like). Diffuse reflections are independent of the incident radiation angle. For specular reflections, the reflection angle equals the angle of incidence.

9.1 Ideal Radiators

An ideal thermal radiator is called a "black body". It has several properties:

1) It has $\alpha = 1$, and absorbs all radiation incident on it.

2) The energy radiated per unit area is $E_b = \sigma T^4$ where σ is the Stefan-Boltzmann constant,

$$\sigma = 5.67 \text{ x } 10^{-8} \frac{W}{m^2 K^4} \tag{9.2}$$

The units of E_b are therefore $\frac{W}{m^2}$.

The energy of a black body, E_b, is distributed over a range of wavelengths of radiation. We can define $e_\lambda = \frac{dE_b}{d\lambda} \approx \frac{\Delta E_b}{\Delta\lambda}$, the energy radiated per unit area for a range of wavelengths of width $\Delta\lambda$. The behavior of e_λ is given in Figure 9.2.

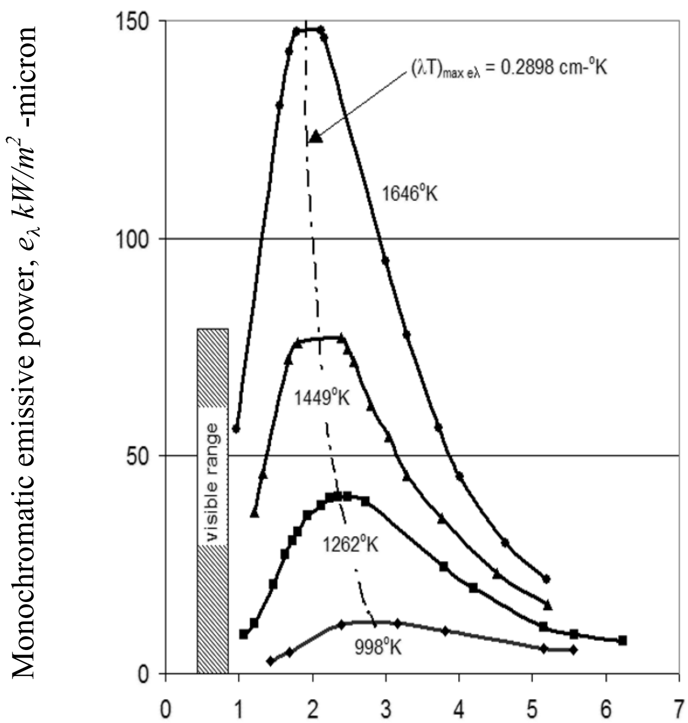


Figure 9.2: Emissive power of a black body at several temperatures - predicted and observed

$(\lambda T)_{e\lambda_{max}}$ = 0.2898 cm K [adapted from: *A Heat Transfer Textbook* by Lienhard, J.]

The distribution of e_λ varies with temperature. The quantity λT at the condition where e_λ is a maximum is given by $(\lambda T)_{e\lambda_{max}}$ = 0.2898 cm K. As T increases, the wavelength for maximum energy emission shifts to shorter values. The frequency of the radiation, f, is given by $f = c/\lambda$ so high energy means short wavelengths and high frequency.

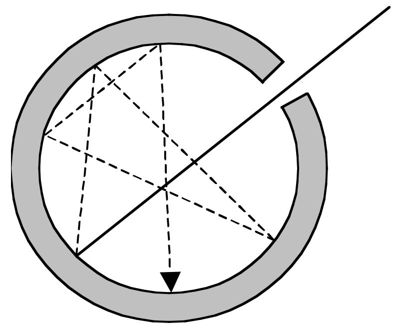


Figure 9.3: A cavity with a small hole (approximates a black body)

A physical realization of a black body is a cavity with a small hole. There are many reflections and absorptions. Very few entering photons (light rays) will get out. The inside of the cavity has radiation which is homogeneous and isotropic (the same in any direction, uniform everywhere).

Suppose we put a small black body inside the cavity as seen in Figure 9.4. The cavity and the black body are both at the same temperature.

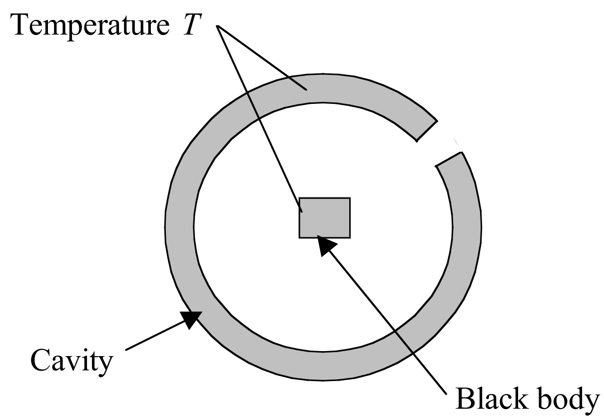


Figure 9.4: A small black body inside a cavity

The radiant energy absorbed by the black body per second and per m^2 is $\alpha_B H$, where H is the irradiance, the radiant energy falling on any surface inside the cavity. The radiant energy emitted by the black body is E_B. Since $\alpha_B = 1$ for a black body, $H = E_B$. The irradiance within a cavity whose walls are at temperature T is therefore equal to the radiant emittance of a black body at the same temperature and irradiance is a function of temperature only.

9.2 Kirchhoff's Law and "Real Bodies"

Real bodies radiate less effectively than black bodies. The measurement of this is the emittance, ε, defined by

Emittance: $\varepsilon = \dfrac{E}{E_b}$ $\begin{matrix}\leftarrow \text{radiation from real body at } T \\ \leftarrow \text{radiation from black body at } T\end{matrix}$

Values of emittance vary greatly for different materials. They are near unity for rough surfaces such as ceramics or oxidized metals, and roughly 0.02 for polished metals or silvered reflectors. A table of emittances for different substances is given at the end of this section as Table 9.1, taken from the book by Lienhard.

The level of the emittance can be related to the absorptance using the following arguments. Suppose we have a small non-black body in the cavity. The power absorbed per unit area is equal to αH. The power emitted is equal to E. An energy balance gives $E = E_b\varepsilon = \alpha H = \alpha E_b$. Thus

$$\frac{E}{E_b} = \alpha = \varepsilon \qquad (9.3)$$

Equation (9.3), the relation $\alpha = \varepsilon$, is known as Kirchhoff's Law. It implies that good radiators are good absorbers. It was derived for the case when $T_{body} = T_{surroundings}$ (cavity) and is not strictly true for all circumstances when the temperature of the body and the cavity are different, but it is true if $\alpha_\lambda = \alpha, \varepsilon_\lambda = \varepsilon$, so the absorptance and emittance are not functions of λ. This situation describes a "gray body". Also, since α_λ, ε_λ are properties of the surface, $\alpha_\lambda = \varepsilon_\lambda$.

9.3 Radiation Heat Transfer Between Planar Surfaces

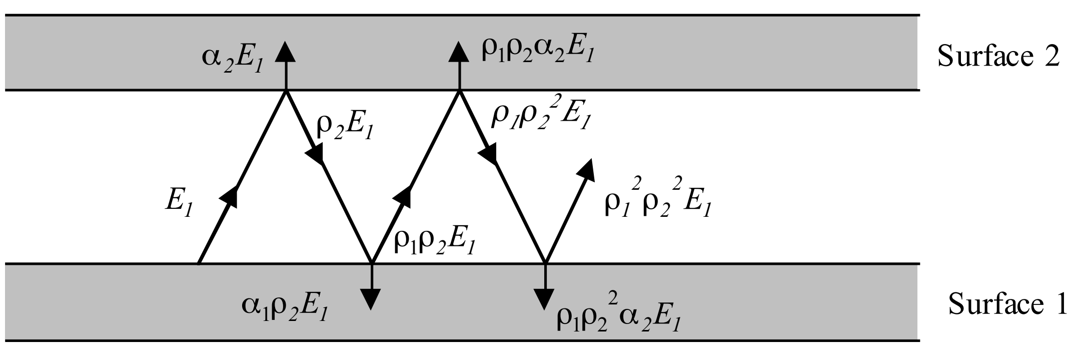


Figure 9.5: Path of a photon between two gray surfaces

Consider the two infinite gray surfaces shown in Figure 9.5. We suppose that the surfaces are thick enough so that, $\alpha + \rho = 1$ (no radiation transmitted so transmittance = 0). Consider a photon emitted from Surface 1 (remembering that the reflectance $\rho = 1 - \alpha$):

Surface 1 emits	E_1
Surface 2 absorbs	$E_1\alpha_2$

Surface 2 reflects $E_1(1-\alpha_2)$
Surface 1 absorbs $E_1(1-\alpha_2)\alpha_1$
Surface 1 reflects $E_1(1-\alpha_2)(1-\alpha_1)$
Surface 2 absorbs $E_1(1-\alpha_2)(1-\alpha_1)\alpha_2$
Surface 2 reflects $E_1(1-\alpha_2)(1-\alpha_1)(1-\alpha_2)$
Surface 1 absorbs $E_1(1-\alpha_2)(1-\alpha_1)(1-\alpha_2)\alpha_1$

The same can be said for a photon emitted from Surface 2:

Surface 2 emits E_2
Surface 1 absorbs $E_2\alpha_1$
Surface 1 reflects $E_2(1-\alpha_1)$
Surface 2 absorbs $E_2(1-\alpha_1)\alpha_2$
Surface 2 reflects $E_2(1-\alpha_1)(1-\alpha_2)$
etc……

We can add up all the energy E_1 absorbed in 1 and all the energy E_2 absorbed in 2. In doing the bookkeeping, it is helpful to define $\beta=(1-\alpha_1)(1-\alpha_2)$. The energy E_1 absorbed in 1 is

$$E_1(1-\alpha_2)\alpha_1+E_1(1-\alpha_2)\alpha_1(1-\alpha_2)(1-\alpha_1)+\ldots$$

This is equal to

$$E_1(1-\alpha_2)\alpha_1\left(1+\beta+\beta^2+\ldots\right).$$

However

$$\frac{1}{1-\beta}=(1-\beta)^{-1}=1+\beta+\beta^2+\ldots.$$

We thus observe that the radiation absorbed by surface 1 can be written as $\dfrac{E_1(1-\alpha_2)\alpha_1}{1-\beta}$.

Likewise $\dfrac{E_2(1-\alpha_1)\alpha_2}{1-\beta}$ is the radiation generated at 2 and absorbed there as well.
Putting this all together we find that

$$E_2-\left(\frac{E_2(1-\alpha_1)\alpha_2}{1-\beta}\right)=\frac{E_2\alpha_1}{1-\beta}$$

is absorbed by 1. The net heat flux from 1 to 2 is

$$\dot{q}_{net\ 1\ \ to\ 2} = E_1 - \frac{E_1(1-\alpha_2)\alpha_1}{1-\beta} - \frac{E_2\alpha_1}{1-\beta} = \frac{E_1 - E_1(1-\alpha_1-\alpha_2+\alpha_1\alpha_2) - E_1\alpha_1 + E_1\alpha_1\alpha_2 - E_2\alpha_1}{1-(1-\alpha_1-\alpha_2+\alpha_1\alpha_2)}$$

or

$$\dot{q}_{\text{net 1 to 2}} = \frac{E_1\alpha_2 - E_2\alpha_1}{\alpha_1 + \alpha_2 - \alpha_1\alpha_2}. \tag{9.4}$$

If $T_1 = T_2$, we would have $\dot{q} = 0$, so from Equation (9.4)

$$\frac{E_1}{\alpha_1} = \frac{E_2}{\alpha_2} = f(T).$$

If body 2 is black, $\alpha_2 = 1$, and $E_2 = \sigma T^4$.

$$\frac{E_1}{\alpha_1} = \sigma T^4$$

$$\frac{\varepsilon_1 \sigma T^4}{\alpha_1} = \sigma T^4$$

Therefore, again, $\varepsilon_1 = \alpha_1$ for any gray surface (Kirchhoff's Law).

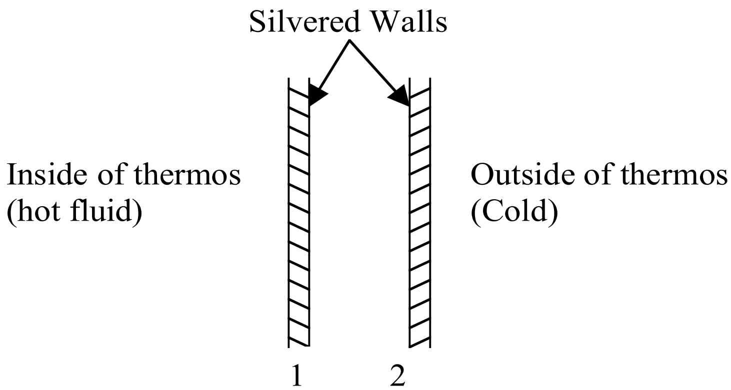
Using Kirchhoff's Law we find,

$$\dot{q}_{\text{net 1 to 2}} = \frac{\varepsilon_1\sigma T_1^4\varepsilon_2 - \varepsilon_2\sigma T_2^4\varepsilon_1}{\varepsilon_1 + \varepsilon_2 - \varepsilon_1\varepsilon_2}$$

or, as the final expression for heat transfer between gray, planar, surfaces:

$$\dot{q}_{\text{net 1 to 2}} = \frac{\sigma\left(T_1^4 - T_2^4\right)}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1}. \tag{9.5}$$

Example 1: Use of a thermos bottle to reduce heat transfer



$\varepsilon_1 = \varepsilon_2 = 0.02$ for silvered walls

$T_1 = 100\,^{\circ}\text{C} = 373$ K
$T_2 = 20\,^{\circ}\text{C} = 293$ K

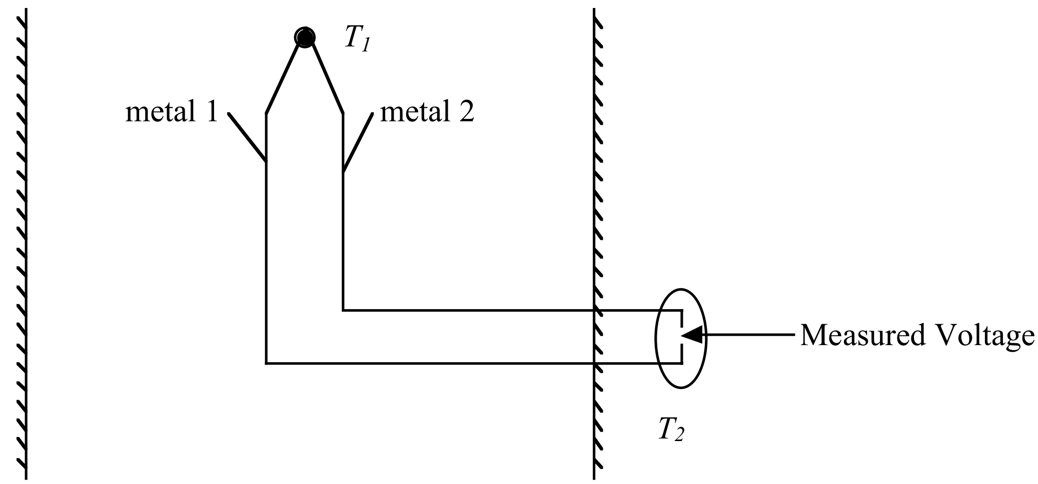
$$\dot{q}_{1\text{ to }2} = \frac{\sigma\left(T_1^4 - T_2^4\right)}{\frac{1}{0.02} + \frac{1}{0.02} - 1} = \frac{1100 - 420}{99} = 6.9 \frac{W}{m^2}$$

For the same ΔT, if we had cork insulation with $k = 0.04$, what thickness would be needed?

$\dot{q} = \frac{k\Delta T}{L}$ so a thickness $L = \frac{k\Delta T}{\dot{q}} = \frac{(0.04)(80)}{6.9} = 0.47\ m$ would be needed. The thermos is indeed a good insulator.

Example 2: Temperature measurement error due to radiation heat transfer

Thermocouples (see Figure 9.6) are commonly used to measure temperature. There can be errors due to heat transfer by radiation. Consider a black thermocouple in a chamber with black walls.



Note: The measured voltage is related to the difference between T_1 and T_2
(the latter is a known temperature).

Figure 9.6: Thermocouple used to measure temperature

Suppose the air is at 20 °C, the walls are at 100 °C, and the convective heat transfer coefficient is $h = 15\frac{W}{m^2K}$.

What temperature does the thermocouple read?

We use a heat (energy) balance on the control surface shown in Figure 9.7. The heat balance states that heat convected away is equal to heat radiated into the thermocouple in steady state. (Conduction heat transfer along the thermocouple wires is neglected here, although it would be included for accurate measurements.)

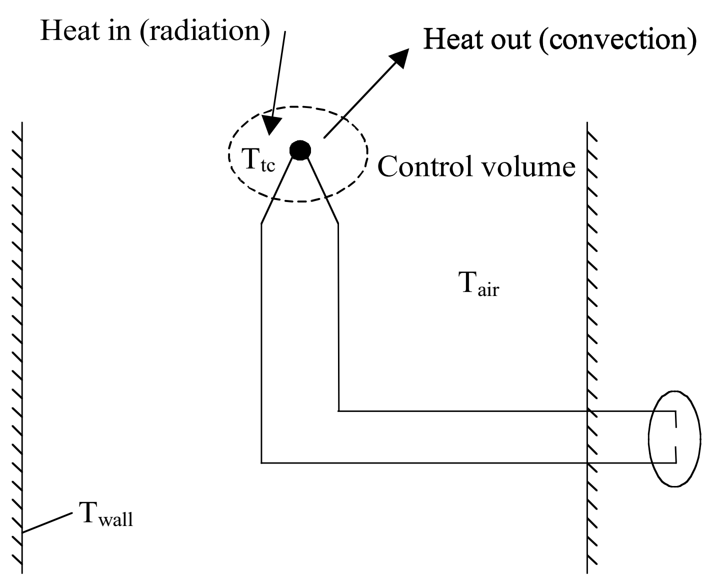


Figure 9.7: Effect of radiation heat transfer on measured temperature.

The heat balance is

$$hA\left(T_{tc} - T_{air}\right) = \sigma A\left(T_{wall}^{\ 4} - T_{tc}^{\ 4}\right) \quad (9.6)$$

where A is the area of the thermocouple. Substituting the numerical values gives

$$15\left(T_{tc} - 293\right) = 5.67 \times 10^{-8}\left(\ 373^4 - T_{tc}^{\ 4}\right)$$

from which we find T_{tc} = 51 °C = 324 K. The thermocouple thus sees a higher temperature than the air. We could reduce this error by shielding the thermocouple as shown in Figure 9.8.

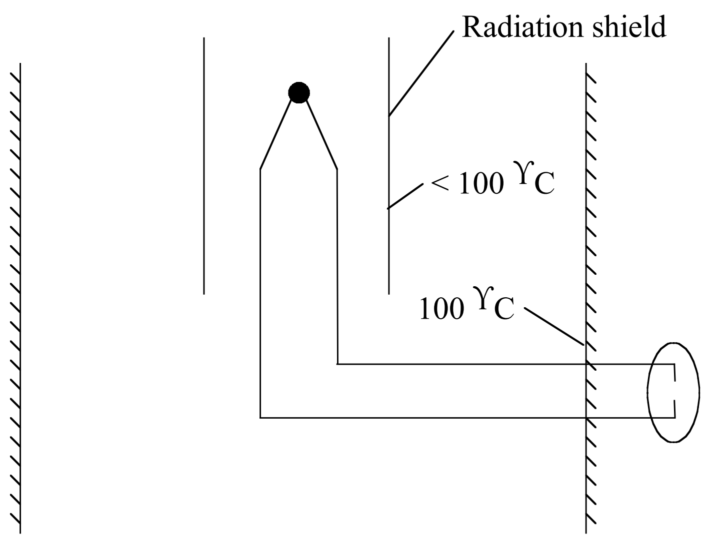


Figure 9.8: Shielding a thermocouple to reduce radiation heat transfer error

Muddy points

Which bodies does the radiation heat transfer occur between in the thermocouple?(MP HT.21)

Still muddy about thermocouples. (MP HT.22)

Why does increasing the local flow velocity decrease the temperature error for the thermocouple? (MP HT.23)

9.4 Radiation Heat Transfer Between Black Surfaces of Arbitrary Geometry

In general, for any two objects in space, a given object 1 radiates to object 2, and to other places as well, as shown in Figure 9.9.

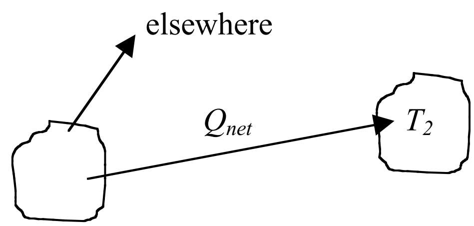


Figure 9.9: Radiation between two bodies

We want a general expression for energy interchange between two surfaces at different temperatures. This is given by the radiation *shape factor* or *view factor*, F_{i-j}. For the situation in Figure 9.10

F_{1-2} = fraction of energy leaving *1* which reaches *2*
F_{2-1} = fraction of energy leaving *2* which reaches *1*
F_{1-2} , F_{2-1} are functions of geometry only

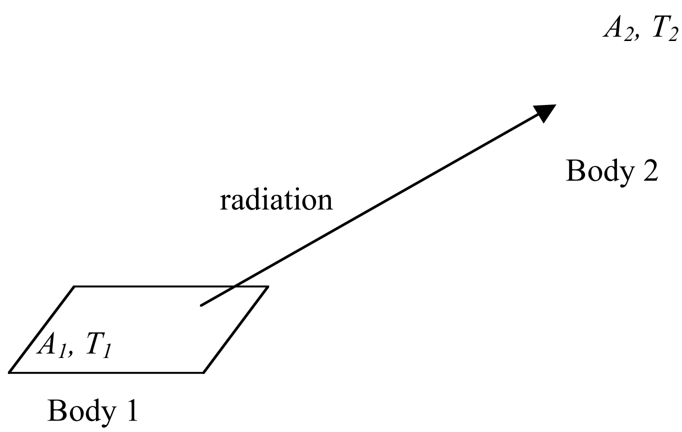


Figure 9.10: Radiation between two arbitrary surfaces

For body 1, we know that E_b is the emissive power of a black body, so the energy leaving body 1 is $E_{b1} A_1$. The energy leaving body 1 and arriving (and being absorbed) at body 2 is $E_{b1} A_1 F_{1-2}$. The energy leaving body 2 and being absorbed at body 1 is $E_{b2} A_2 F_{2-1}$. The net energy interchange from body 1 to body 2 is

$$E_{b1} A_1 F_{1-2} - E_{b2} A_2 F_{2-1} = \dot{Q}_{1-2}. \qquad (9.7)$$

Suppose both surfaces are at the same temperature so there is no net heat exchange. If so,

$E_{b1} A_1 F_{1-2} - E_{b2} A_2 F_{2-1} = 0$, but also $E_{b1} = E_{b2}$. Thus

$$A_1 F_{1-2} = A_2 F_{2-1}. \qquad (9.8)$$

Equation (9.7) is the *shape factor reciprocity relation*. The net heat exchange between the two surfaces is

$$\dot{Q}_{1-2} = A_1 F_{1-2}\left(E_{b1} - E_{b2}\right) \quad \left[\text{or } A_2 F_{2-1}\left(E_{b1} - E_{b2}\right)\right]$$

Example: Concentric cylinders or concentric spheres

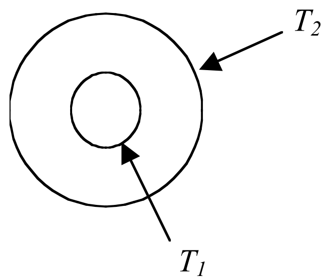


Figure 9.11: Radiation heat transfer for concentric cylinders or spheres

The net heat transfer from surface 1 to surface 2 is

$$\dot{Q}_{1-2} = A_1 F_{1-2}\left(E_{b1} - E_{b2}\right) .$$

We know that $F_{1-2} = 1$, i.e. that all of the energy emitted by 1 gets to 2. Thus

$$\dot{Q}_{1-2} = A_1\left(E_{b1} - E_{b2}\right)$$

This can be used to find the net heat transfer from 2 to 1.

$$\dot{Q}_{2-1} = A_2 F_{2-1}\left(E_{b2} - E_{b1}\right) = A_1 F_{1-2}\left(E_{b2} - E_{b1}\right) = A_1\left(E_{b2} - E_{b1}\right)$$

View factors for other configurations can be found analytically or numerically. Shape factors are given in textbooks and reports (they are tabulated somewhat like Laplace transforms), and examples of the analytical forms and numerical values of shape factors for some basic engineering configurations are given in Figures 9.12 through 9.15, taken from the book by Incropera and DeWitt.

Metals			Nonmetals		
Surface	*Temperature* (^oC)	ε	*Surface*	*Temperature* (^oC)	ε
Aluminum			Asbestos	40	0.93 - 0.97
Polished, 98% pure	200 - 600	0.04 - 0.06	Brick		
Commercial sheet	90	0.09	Red, Rough	40	0.93
Heavily oxidized	90 - 540	0.2 - 0.33	Silica	980	0.8 - 0.85
Brass			Fireclay	9980	0.75
Highly polished	260	0.03	Ordinary refractory	1090	0.59
Dull plate	40 - 260	0.22	Magnesite refractory	980	0.38
Oxidized	40 - 260	0.46 - 0.56	White refractory	1090	0.29
Copper			Carbon		
Highly polished electrolytic	90	0.02	Filament	1040 - 1430	0.53
Slightly polished, to dull	40	0.12 - 0.15	Lampsoot	40	0.95
Black oxidized	40	0.76	Concrete, rough	40	0.94
Gold: pure, polished	90 - 600	0.02 - 0.035	Glass		
Iron and Steel			Smooth	40	0.94
Mild steel, polished	150 - 480	0.14 - 0.32	Quartz glass (2mm)	260 - 540	0.96 - 0.66
Steel, polished	40 - 260	0.07 - 0.1	Pyrex	260 - 540	0.94 - 0.74
Sheet steel, rolled	40	0.66	Gypsum	40	0.8 - 0.9
Sheet steel, strong rough oxide	40	0.8	Ice	0	0.97 - 0.98
Cast iron, oxidized	40 - 260	0.57 - 0.66	Limestone	40 - 260	0.95 - 0.83
Iron, rusted	40	0.61 - 0.85	Marble	40	0.93 - 0.95
Wrought iron, smooth	40	0.35	Mica	40	0.75
Wrought iron, dull oxidized	20 - 360	0.94	Paints		
Stainless, polished	40	0.07 - 0.17	Black gloss	40	0.9
Stainless, after repeated heating	230 - 900	0.5 - 0.7	White paint	40	0.89 - 0.97
Lead			Lacquer	40	0.8 - 0.95
Polished	40 - 260	0.05 - 0.08	Various oil paints	40	0.92 - 0.96
Oxidized	40 - 200	0.63	Red lead	90	0.93
Mercury: pure, clean	40 - 90	0.1 - 0.12	Paper		
Platinum			White	40	0.95 - 0.98
Pure, polished plate	200 - 590	0.05 - 0.1	Other colors	40	0.92 - 0.94
Oxidized at 590^{o}C	260 - 590	0.07 - 0.11	Roofing	40	0.91
Drawn wire and strips	40 - 1370	0.04 - 0.19	Plaster, rough lime	40 - 260	0.92
Silver	200	0.01 - 0.04	Quartz	100 - 1000	0.89 - 0.58
Tin	40 - 90	0.05	Rubber	40	0.86 - 0.94
Tungsten			Snow	10 - 20	0.82
Filament	540 - 1090	0.11 - 0.16	Water, thickness $\geq$ 0.1mm	40	0.96
Filament	2760	0.39	Wood	40	0.8 - 0.9
			Oak, planed	20	0.9

Table 9.1:Total emittances for different surfaces [Adapted from: *A Heat Transfer Textbook*, J. Lienhard]

Figure 9.12: View Factors for Three - Dimensional Geometries

Figure 9.13: Fig. 13.4--View factor for aligned parallel rectangles

Figure 9.14: Fig 13.5--View factor for coaxial parallel disk

Figure 9.15: Fig 13.6--View factor for perpendicular rectangles with a common edge

[from: *Fundamentals of Heat Transfer,* F.P. Incropera and D.P. DeWitt, John Wiley and Sons]

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Muddiest points on heat transfer

HT.1 *How do we quantify the contribution of each mode of heat transfer in a given situation?*

Developing the insight necessary to address the important parts of a complex situation (such as turbine heat transfer) and downplay (neglect or treat approximately) the other aspects is part of having a solid working knowledge of the fundamentals. This is an important issue, because otherwise every problem will seem very complex. One way to sort out what is important is to make order of magnitude estimates (similar to what we did to answer when the one-dimensional heat transfer approximation was appropriate) to see whether all three modes have to be considered. Sometimes one can rule out one or two modes on the basis of the problem statement. For example if there were a vacuum between the two surfaces in the thermos bottle, we would not have to consider convection, but often the situation is more subtle.

HT.2 *How specific do we need to be about when the one-dimensional assumption is valid? Is it enough to say that dA/dx is small?*

The answer really is "be specific enough to enable one to have faith in the analysis or at least some idea of how good the answer is". This is a challenge that comes up a great deal. For now, if we say that A is an area defined per unit depth normal to the blackboard then saying dA/dx is small, which is a statement involving a non-dimensional parameter, is a reasonable criterion.

HT.3 *Why is the thermal conductivity of light gases such as helium (monoatomic) or hydrogen (diatomic) much higher than heavier gases such as argon (monoatomic) or nitrogen (diatomic)?*

To answer this, we need some basics of the kinetic theory of gases. A reference for this is Castellan, *Physical Chemistry*, Benjamin/Cummings Publishers, 1986. Two factor contribute, the collision cross section and the average molecular velocity. For the gases mentioned above the dominant factor appears to be the velocity. The kinetic energy per molecule at a given temperature is the same and so the lower the molecular weight the higher the average molecular velocity.

HT.4 *What do you mean by continuous?*

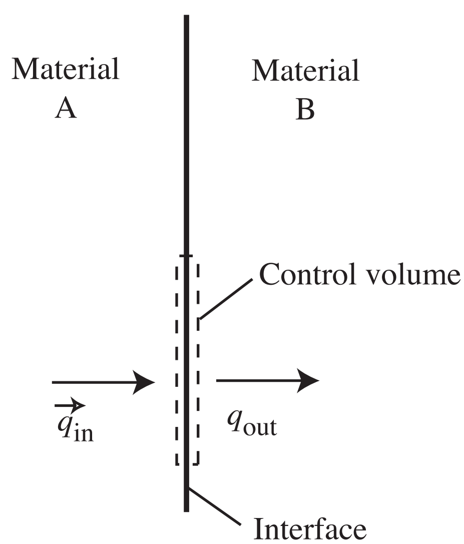
The meaning is similar to the definition you have seen in the math subjects. A way to state it is that the function at a given location has the same value as we approach the location independent of the direction we approach from. To say this in a more physical manner, the temperature as a function of x has the same value at $x = a$, say, whether we approach location a from the left or from the right. In terms of what the function looks like, it will have no "jumps" or discontinuous (step) changes in value.

HT.5 Why is temperature continuous in the composite wall problem? Why is it continuous at the interface between two materials?

We can argue this point by supposing T were *not* continuous, i.e., there was a different temperature on the two sides of an interface between two materials, with the interface truly a sharp plane. If so, there would be a finite temperature difference across an infinitesimal distance, leading to a very large (infinite in the limit) heat flux which would immediately erase the temperature difference. This argument could also be applied to any location inside a solid of uniform composition, guaranteeing that the temperature is continuous in each material.

HT.6 Why is the temperature gradient dT/dx not continuous?

As derived in class, across an interface the heat flux is continuous. From the first law, for a thin control volume that encloses the interface the *net* heat flow into the control volume is zero. As sketched below, the contribution from the heat flux at the upper and lower ends of the control volume is negligible so the heat flux in one side must be the same as the flux out of the other. The heat flux, however, is related to the temperature gradient by $\vec{q} = -k\nabla T$, or, for one-dimension, $q_x = -k\left(\frac{dT}{dx}\right)$. If the heat flux is continuous, and if the thermal conductivity, k, is not the same in the two materials, then dT/dx is not continuous.



HT.7 Why is ΔT the same for the two elements in a parallel thermal circuit? Doesn't the relative area of the bolt to the wood matter?

In terms of the bolt through the wood wall, the approximation made is that the bolt and the wood are both exposed to the same conditions at the two sides of the wall. The relative areas of the bolt and the wood indeed do matter. Suppose we consider a square meter area of wood without bolts. It has a certain heat resistance. If we now add bolts to the wall, the resistance of *each* bolt is $R_{bolt} = \dfrac{L_{bolt}}{k_{bolt} A_{bolt}}$. If there are *n* bolts, they will be in parallel, and the effective area will be *n* times the area for one bolt. The thermal resistance of the wood will be increased by a factor of $\left(A_{\substack{without \\ bolts}} / A_{\substack{with \\ n\ bolts}} \right)$, which is larger than unity.

In summary, the amount by which the heat transfer is increased depends on the fractional area with low thermal resistance compared to the fractional area with high thermal resistance.

HT.8 *<u>How do we know that δ' is not a fluid property?</u>*

The term $\left(\dfrac{\delta'}{k}\right)$ represents the resistance to heat transfer for a unit area. The resistance to heat transfer per unit area (1/heat transfer coefficient) can be computed for cases of laminar flow, or measured experimentally where we cannot compute it, and it is found that for the same state variables, it can have a range of values of several orders of magnitude depending on the parameters I described (Reynolds number, surface condition, surface shape…). Put another way, if the value of the resistance is affected by the surface condition (smooth, bumpy, corrugated, etc.) how can the resistance be just a property of the fluid?

HT.9 *<u>What is the "analogy" that we are discussing? Is it that the equations are similar?</u>*

While the equations are similar, the concepts are deeper than that. The analogy is drawn between the heat transfer process (transfer of heat represented by heat flux) and the momentum transfer process (transfer of momentum represented by shear stress)

HT.10 *<u>In what situations does the Reynolds analogy "not work"?</u>*

The Reynolds Analogy is just that. It is not a law of nature, but rather a plausible hypothesis that allows useful estimation of the heat transfer coefficients in many situations in which little or no explicit heat transfer information exists. In the form we have derived it, the Reynolds Analogy is appropriate for use in air, but it is not strictly applicable if there are pressure gradients, or if the Prandtl number ($Pr = \dfrac{\mu c_p}{k}$) is not unity. However, the conceptual framework provided by the analogy has been found useful enough that the analogy has been extended (in a more complex form, as briefly discussed in class) for application to these situations.

HT.11 In the expression $\frac{1}{h.A}$, what is A?

A is the area normal to the heat flow. For a turbine blade, for example, it would be the outer surface area of the blade.

HT.12 It seems that we have simplified convection a lot. Is finding the heat transfer coefficient, h, really difficult?

We have indeed "simplified convection a lot". We will look at heat transfer by convection in more depth in a few lectures, but to answer the question in a few sentences, finding the heat transfer coefficient is a difficult problem, because it necessitates determining the fluid dynamics; the latter is key to predictions of heat transfer. Even with present computational power, calculating the flow around aerospace devices with the accuracy needed to be confident about heat transfer coefficients is not by any means a "standard" calculation. For some circumstances, it is still beyond the state of the art. We will concentrate on describing (i) the basic *mechanisms* of convective heat transfer and (ii) ways of *estimating* the heat transfer coefficient from known fluid dynamic information.

HT.13 What does the "K" in the contact resistance formula stand for?

The definition of the resistance comes out of $\dot{Q} = \frac{\Delta T_{driving}}{R_{thermal}}$. The units of $\dot{Q}$ are Watts/meter^2, so the units of $R_{thermal}$ are [degrees x meters^2/Watts]. The *K* is thus the symbol for Kelvin, or degree centigrade.

HT.14 In the equation for the temperature in a cylinder (3.22), what is "r"?

The variable *r* denotes the radial coordinate, in other words the location of the point at which we want to know the temperature.

HT.15 For an electric heated strip embedded between two layers, what would the temperature distribution be if the two side temperatures were not equal?

If the two temperatures on the outer surfaces of the composite layer were T_1 and T_2, the heat flux from the two sides of the electrical heater would be different, but the sum of the two heat fluxes would still be equal to the heat generated per unit area and unit time. In fact heat could be coming into one side of the heater if one of the temperatures were high enough. (Sketch this out and prove it to yourself.) The temperature distribution would be linear from the heater temperature to the two surface temperatures. If this example is not clear, please come and see me.

HT.16 Why did you change the variable and write the derivative $\frac{d^2T}{dx^2}$ as $\frac{d^2(T - T_\infty)}{dx^2}$ in the equation for heat transfer in the fin?

The heat balance and derivation of the equation for temperature (5.3) is given in Section 5.0 of the notes. This is

$$\frac{d^2T}{dx^2} - \frac{Ph}{Ak}(T - T_\infty) = 0 \tag{5.3}$$

It is not *necessary* to change variables, and one could solve (5.3) as is (or in the form $\frac{d^2T}{dx^2} - \frac{Ph}{Ak}T = T_\infty$). However, since: (1) the reference temperature is T_∞, and what is of interest is the difference $T - T_\infty$, (2) making the substitution results in a simpler form of the equation to be solved, and (3) the derivative of $T - T_\infty$ is the same as the derivative of T, I found it convenient to put it the equation in the form of Eq. (5.5),

$$\frac{d^2(T - T_\infty)}{dx^2} - \frac{Ph}{Ak}(T - T_\infty) = 0. \tag{5.5}$$

HT.17 What types of devices use heat transfer fins?

A number of types of heat exchangers use fins. Examples of the use of fins you may have seen are cooling fins on motorcycle engine heads, cooling fins on electric power transformers, or cooling fins on air conditioners.

HT.18 Why did the Stegosaurus have cooling fins? Could the stegosaurus have "heating fins"?

My knowledge of this issue extends only to reading about it in the text by Lienhard (see reference in notes). The journal article referenced in that text is:
Farlow, J. O., Thompson, C. V., and Rosner, D.E., "Plates of the Dinosaur Stegosaurus: Forced Convection Heat Loss Fins?, *Science*, vol. 192, no. 4244, 1976, pp. 1123-1125 and cover.

HT.19 In equation $\dot{Q}_{in} = \rho.V.c.\frac{dT}{dt}$ (6.3), what is c?

c is the specific heat for a unit mass. We don't have to differentiate between the two specific heats for a solid because the volume changes are very small, unlike a gas.

HT.20 In the lumped parameter transient heat transfer problem, does a high density "slow down" heat transfer?

It doesn't. The high density slows down the rate at which the object changes temperature; high density means more "heat capacity".

HT.21 Which bodies does the radiation heat transfer occur between in the thermocouple?

The radiation heat transfer was between the walls (or more generally the boundaries of the duct) and the thermocouple. The boundaries may not be at the same temperature of the flowing fluid, for example in a turbine. If the boundaries are not at the same temperature as the fluid, and they radiate to the thermocouple, there can be an error in the temperature that the thermocouple reads. The discussion was about different possible sources of this type of error.

HT.22 Still muddy about thermocouples.

I didn't mean to strew confusion about these. As on page 59 of the notes, if we take a pair of dissimilar wires (say copper and constantan (an alloy of tin and several other metals) or platinum and rhodium) which are joined at both ends and subject the two junctions to a temperature difference, it is found that a voltage difference will be created. If we know the temperature of one junction (say by use of an ice bath) and we know the relation between voltage and temperature difference (these have been measured in detail) we can find the temperature of the other junction from measurement of the voltage. The assumption is that the temperature of the junction is the temperature that is of interest; the possible error in this thinking is the subject of muddy point 1.

HT.23 Why does increasing the local flow velocity decrease the temperature error for the thermocouple?

The heat transferred by convection is $\dot{Q}_{convection} = h\left(T_{thermocouple} - T_{fluid}\right)$. If we neglect conduction from the wire junction (in other words assume the thermocouple wires are thin and long), the heat balance is between convection heat transfer and radiation heat transfer. For a fixed $\dot{Q}_{convection}$ if the heat transfer increases the temperature difference between the thermocouple and the fluid decreases. We have seen, however, that the heat transfer may be estimated using the Reynolds analogy (Section 3.1). For fixed skin friction coefficient the higher the velocity the higher the heat transfer coefficient.