16.323 Principles of Optimal Control
Spring 2008

16.323 Final Exam

This is a closed-book exam, but you are allowed 3 pages of notes (both sides). You have 3 hours. There are five **5** questions with the relative values clearly marked.

1. (total 15pts) John Howe, who hasn't taken 16.323 and has no idea about the Riccati equation, is taking the final exam for an optimization class. The instructor loves to challenge the students by asking interesting, though possibly unexpected, questions, gave the following LQR problem in their exam, although it hasn't been covered in class:

$$\text{minimize } J \equiv \int_0^\infty (x^2 + \rho u^2) dt \tag{1}$$

$$\text{subject to} \quad \dot{x} = ax + bu, \quad x(0) = x_0. \tag{2}$$

The only hint the instructor gave is that the optimal solution should have the form: $u(t) = -kx(t)$.

Avoiding panic, John managed to solve the problem using what he learned in class as follows:

(a) (5pts) Express the optimal cost $J^\star$ in terms of k and other given parameters.

(b) (5pts) Find k that minimizes the above expression of $J^\star$.

(c) (5pts) John realizes that the same result can be derived from the Riccati equation, while talking to one of his committee members, Prof. Jon How, after the exam. Can you also verify this?

2. (total 20pts) A discrete system described by the difference equation

$$x_{k+1} = x_k + u_k$$

is to be controlled to minimize the performance measure

$$J = \sum_{k=1}^{2} (2|x_k - 0.1k^2| + |u_{k-1}|)$$

The state and control values must satisfy the constraints

$$0.0 \leq x_k \leq 0.4, \qquad k = 0, 1, 2 \tag{3}$$
$$-0.2 \leq u_k \leq 0.2, \qquad k = 0, 1 \tag{4}$$

(a) (12pts) Use the dynamic programming algorithm to determine the optimal control law $u^\star(x_k, k)$.

   • Quantize the state into the values $x_k = 0, 0.1, 0.2, 0.3, 0.4$ $(k = 0, 1, 2)$ and the control into the values $u_k = -0.2, -0.1, 0, 0.1, 0.2$ $(k = 0, 1)$.

(b) (8pts) Determine the optimal sequence $\{u_0^\star, u_1^\star\}$ if the initial value is $x_0 = 0.2$.

3. (total 15pts) Find the control input $u(t)$ sequence that minimizes the cost functional

$$J = -y(t_f)$$

subject to the state constraints

$$\dot{y}(t) = y(t)u(t) - y(t) - u^2(t)$$

for an initial condition $y(0) = 1$ and final time $t_f = 5$. Give both the control, the state response, and the costate.

Hint: A solution to the differential equation $\dot{y} = ay^2 - y$ can be written as

$$y(t) = \frac{1/a}{1 + \beta e^t}$$

with β being an arbitrary constant.

4. (total 30pts) **LQG control for an unstable system.** Consider the unstable second order system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_2 + u + w$$

with the continuous measurements

$$y = x_1 + x_2 + v$$

where w and v are zero-mean white noise processes with spectral densities R_w and R_v respectively.

(a) (10pts) For the performance index

$$J = \int_0^\infty (x_1^2 + x_2^2 + R_{uu} u^2) \, dt$$

Find the closed-loop pole locations for the expensive control problem. Show that if $R_{uu} = 1$, then the steady-state LQR gains are:

$$K = [\, 1 \quad 3 \,]$$

and that the closed-loop poles are both at $s = -1$.

(b) (10pts) For $R_w = R_v = 1$ show that the steady-state LQE gains are:

$$L = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and that the closed-loop poles are both at $s = -1$.

(c) (5pts) Find the transfer function of the resulting LQG compensator $G_c(s)$. As best as possible identify the main compensator elements in terms of classical components such as lead, lag, PID. In particular, indicate why this compensator appears to be doing and why this makes sense to you.

(d) (5pts) Fix the compensator dynamics and *sketch* a root locus for this system versus the overall plant gain (so that $G(s) \to \alpha G(s)$, where nominally $\alpha = 1$).

   • Clearly mark the optimal closed-loop pole locations on this locus.

5. (total 20pts) Consider the minimum-time optimal control problem for the following harmonic oscillator:
$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + u, \qquad |u| \leq 1 \tag{5}$$
that leads both states to the origin: $\mathbf{x}(t_f) = (0,0)$.

(a) (6pts) Form the Hamiltonian for this system and find the optimal control law.

(b) (6pts) Use your result in part (a) to support the following conclusions:

- The time-optimal control must be piecewise-constant and switches between ± 1.
- The time-optimal control can remain constant for no more than π units of time.
- There is no upperbound to the number of switches of the time-optimal control.
- The function $p_2(t)$ cannot be zero for a finite period of time, so there is no possibility of singular control.

(c) (4pts) Given the results of (b), now let us consider the response of the system under the control action $u = \pm 1$. Using the fact that the state equation can be manipulated to the form:
$$(x_1 - u)\dot{x}_1 + x_2\dot{x}_2 = 0, \tag{6}$$
show that each state can be expressed as
$$x_1 - u = c\cos\theta, \qquad x_2 = c\sin\theta \tag{7}$$
for fixed u (Hint: Eq. 6 can be regarded as the time derivative of some $A(\mathbf{x}, u)^2 + B(\mathbf{x}, u)^2 = R^2$ expression.)

(d) (4pts) Use the result in (c) to show that the response curves are circles in the x_1-x_2 plane, so that if $u = 1$, they are circles about $(1,0)$, and if $u = -1$, they are circles about $(-1,0)$. Given this information, sketch what you think the time optimal response will be in the (x_1, x_2) plane given an initial point of $(1,1)$.