16.323 Principles of Optimal Control
Spring 2008

April 11, 2007

16.323 Final Exam

This is a closed-book exam, but you are allowed 3 pages of notes (both sides). You have 3 hours. There are six **6** questions with the relative values clearly marked.

Some handy formulas:

$$\mathbf{u}^\star(t) = \arg\left\{\min_{\mathbf{u}(t)\in\mathcal{U}} H(\mathbf{x},\mathbf{u},\mathbf{p},t)\right\}$$

$$A^T P_{ss} + P_{ss}A + R_{\mathrm{xx}} - P_{ss}B_u R_{\mathrm{uu}}^{-1} B_u^T P_{ss} = 0$$

$$K_{ss} = R_{\mathrm{uu}}^{-1} B_u^T P_{ss}$$

$$\lambda_i\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \frac{a + d \pm \sqrt{a^2 - 2ad + d^2 + 4bc}}{2}$$

$$\begin{aligned}
&(i) && P_N = H \\
&(ii) && F_k = -\left[R_k + B_k^T P_{k+1} B_k\right]^{-1} B_k^T P_{k+1} A_k \\
&(iii) && P_k = Q_k + F_k^T R_k F_k + \{A_k + B_k F_k\}^T P_{k+1} \{A_k + B_k F_k\}
\end{aligned}$$

1. (15%) Consider a model of a first order unstable plant with the state equation

$$\dot{x}(t) = ax(t) - au(t) \qquad\qquad a > 0$$

and a **new** cost functional

$$J = \int_0^\infty e^{2bt}\left[x^2(t) + \rho u^2(t)\right] dt \qquad\qquad \rho > 0\,,\, b \geq 0$$

It can be shown that, with a change of variables in the system to $e^{bt}x(t) \rightarrow \tilde{x}(t)$ and $e^{bt}u(t) \rightarrow \tilde{u}(t)$, then the only real modification required to solve for the static LQR controller with this cost functional is to use $A + bI$ in the algebraic Riccati equation instead of A.

(a) Given this information, determine the LQR gain K as a function of ρ, a, b.
(b) Determine the location of the closed-loop poles for $0 < \rho < \infty$
(c) Use these results to explain what the primary effect of b is on this control design.

2. (15%) Given the optimal control problem for a scalar nonlinear system:

$$\begin{aligned} \dot{x} &= xu \qquad x(0) = 1 \\ J(u) &= x(1)^2 + \int_0^1 (x(t)u(t))^2 dt \end{aligned}$$

find the optimal feedback strategy by solving the associated HJB equation. Hint: show that the HJB differential equation admits a solution of the form $J^\star = p(t)x(t)^2$.

3. (15%) For the system given by the dynamics:

$$\begin{aligned} \dot{x}_1 &= x_2 + u \\ \dot{x}_2 &= u \end{aligned}$$

with $|u(t)| \leq 1$, find the time optimal controller that drives the system to the origin $x_1 = x_2 = 0$. In your solution, clearly state:

- The switching law,
- Show the switching lines,
- Sketch the system response for $x_1(0) = x_2(0) = 1$.

4. (15%) Consider the following discrete system:

$$x_{k+1} = x_k - 0.4x_k^2 + u_k$$

Assume that the state space is quantized to be $(0, 0.5, 1)$, and the control is quantized to be $(-0.4, -0.2, 0, 0.2, 0.4)$. The cost to be minimized is

$$J = 4|x_2| + \sum_{k=0}^{1} |u_k|$$

Use dynamic programming to find the optimal control sequence and complete the following tables. If in the process you find a state that is not at the quantized value, assign it the nearest quantized value.

x_0	$J^\star_{0,2}(x_0)$	$u^\star_0$
0.0		
0.5		
1.0		

x_1	$J^\star_{1,2}(x_1)$	$u^\star_1$
0.0		
0.5		
1.0		

Use these results to find the optimal control sequence if the initial state is $x_0 = 1$.

5. (20%) Optimal control of linear systems:

   (a) Given the linear system

   $$\dot{x} = -x + u$$

   with $x(0) = 1$ and $x(1) = 1$, find the controller that optimizes the cost functional

   $$J = \frac{1}{2}\int_0^1 (3x^2 + u^2)dt$$

   Please solve this problem by forming the Hamiltonian, solve the differential equations for the state/costate, and then use these to provide the control law, which you should write as an explicit function of time.

   (b) Consider the following system:

   $$\begin{aligned} \dot{x}_1 &= -x_1 + u \\ \dot{x}_2 &= -2x_2 \\ \text{with } J &= \int_0^\infty (x_2^2(t) + u^2(t))dt \end{aligned}$$

   - Comment on the stabilizability and detectability of this system.
   - Find the optimal steady state regulator feedback law for the system. Why does this answer make sense?
   - Given $x(0) = [\ 1 \quad 1\ ]$ what is the minimum value of the initial cost?

6. (20%) LQG control for an unstable system: Consider the unstable second order system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_2 + u + w \end{aligned}$$

with the continuous measurements

$$y = x_1 + v$$

where w and v are zero-mean white noise processes with spectral densities $R_{\rm ww}$ and $R_{\rm vv}$ respectively and the performance index is

$$J = \int_0^\infty (R_{\rm xx} x_1^2 + R_{\rm uu} u^2)\, dt$$

You analyzed this system in Homework #3 and showed that for $R_{\rm xx}/R_{\rm uu} = 1$ the steady-state LQR gains are $K = [\ 1 \quad \sqrt{3}+1\ ]$ and the closed-loop poles are at $s = -(\sqrt{3} \pm j)/2$.

(a) *Sketch* by hand the locus of the estimation error poles versus the ratio $R_{\rm ww}/R_{\rm vv}$ for the steady-state LQE case. Show the pole locations for the noisy sensor problem.

(b) For $R_{\rm ww}/R_{\rm vv} = 1$ show analytically that the steady-state LQE gains are

$$L = \begin{bmatrix} \sqrt{3}+1 \\ \sqrt{3}+2 \end{bmatrix}$$

and that the closed-loop poles are at $s = -(\sqrt{3} \pm j)/2$.

(c) Find the transfer function of the corresponding steady state LQG compensator.

(d) As best as possible, provide a classical explanation of this compensator and explain why it is a good choice for this system.