16.323 Principles of Optimal Control
Spring 2008

16.323, #15

March 8th, 2007

16.323 Midterm #1

This is a closed-book exam, but you are allowed 1 page of notes (both sides).

You have 1.5 hours.

There are three **3** questions of **equal value**.

Hint: To maximize your score, initially give a brief explanation of your approach before getting too bogged down in the equations.

1. For the following cost function, $F = x^2 + y^2 - 6xy - 4x - 5y$

   (a) Minimize the cost subject to the constraints,

   $$f_1 : -2x + y + 1 \geq 0$$

   $$f_2 : x + y - 4 \leq 0$$

   (b) How is the optimal cost affected if the constraint f_1 is changed to,

   $$f_1' = -2x + y + 1.1 \geq 0$$

   Estimate this difference and explain your answer.

2. The first order discrete system,

   $$x_{k+1} = x_k + u_k$$

   is to be transferred to the origin in two stages ($x_2 = 0$). The performance measure to be minimized is,

   $$J = \sum_{k=0}^{1} (|x_k| + 5|u_k|)$$

   The possible state and control values are:

   $$x_k \in \{3, 2, 1, 0, -1, -2, -3\}$$

   $$u_k \in \{2, 1, 0, -1, -2\}$$

   (a) Use dynamic programming to determine the optimal control law and the associated cost for each possible value of x_0.

   (b) Use the results from (a) to determine the optimal control sequence $\{u_0^*, u_1^*\}$ for the initial state $x_0 = -2$.

3. Consider a disturbance rejection problem that minimizes:

$$J = \frac{1}{2}\mathbf{x}(t_f)^T H \mathbf{x}(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \mathbf{x}^T(t) R_{\mathrm{xx}}(t)\mathbf{x}(t) + \mathbf{u}(t)^T R_{\mathrm{uu}}(t)\mathbf{u}(t)\ dt \tag{1}$$

subject to

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t) + \mathbf{w}(t). \tag{2}$$

To handle the disturbance term, the optimal control should consist of both a feedback term and a feedforward term (assume $\mathbf{w}(t)$ is known).

$$\mathbf{u}^\star(t) = -K(t)\mathbf{x}(t) + \mathbf{u}_{fw}(t), \tag{3}$$

Using the Hamilton-Jacobi-Bellman equation, show that a possible optimal value function is of the form

$$J^\star(\mathbf{x}(t), t) = \frac{1}{2}\mathbf{x}^T(t)P(t)\mathbf{x}(t) + b^T(t)\mathbf{x}(t) + \frac{1}{2}c(t), \tag{4}$$

where

$$K(t) = R_{uu}^{-1}(t)B^T(t)P(t), \quad \mathbf{u}_{fw} = -R_{uu}^{-1}(t)B^T(t)b(t) \tag{5}$$

In the process demonstrate that the conditions that must be satisfied are:

$$\begin{aligned}
-\dot{P}(t) &= A^T(t)P(t) + P(t)A(t) + R_{\mathrm{xx}}(t) - P(t)B(t)R_{\mathrm{uu}}^{-1}(t)B^T(t)P(t) \\
\dot{b}(t) &= -\left[A(t) - B(t)R_{\mathrm{uu}}^{-1}(t)B^T(t)P(t)\right]^T b(t) - P(t)\mathbf{w}(t) \\
\dot{c}(t) &= b^T(t)B(t)R_{\mathrm{uu}}^{-1}(t)B^T(t)b(t) - 2b^T(t)\mathbf{w}(t).
\end{aligned}$$

with boundary conditions: $P(t_f) = H$, $b(t_f) = 0$, $c(t_f) = 0$.