Vector algebra using tensorial notation

Definition of the Kronecker Delta $\Rightarrow$ $\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$

and the Levi-Civita tensor $\Rightarrow$ $\varepsilon_{ijk} = \begin{cases} 0 & \text{for } i=j \text{ or } i=k \text{ or } j=k \\ \text{-1 for an odd index permutation} \\ \text{1 for an even index permutation} \end{cases}$

$$\varepsilon_{ijk}\varepsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}$$ in particular, note that if $i=l$ then $\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$

RULE: Whenever an index is repeated, a summation is assumed over that index

$$\mathrm{A}_i\mathrm{B}_i = \sum_{i=1}^{3} \mathrm{A}_i\mathrm{B}_i \text{ in 3D}$$

Dot product: $\vec{\mathrm{A}} \cdot \vec{\mathrm{B}} = \mathrm{A}_i\mathrm{B}_i$ Cross product: $\left(\vec{\mathrm{A}} \times \vec{\mathrm{B}}\right)_i = \varepsilon_{ijk}\mathrm{A}_j\mathrm{B}_k$ Gradient: $\nabla_i = \frac{\partial}{\partial x_i}$

That's all we need to know, here are a few examples:

1) Prove that $\vec{\mathrm{A}} \times \vec{\mathrm{B}} \times \vec{\mathrm{C}} = \vec{\mathrm{B}}(\vec{\mathrm{A}} \cdot \vec{\mathrm{C}}) - \vec{\mathrm{C}}(\vec{\mathrm{A}} \cdot \vec{\mathrm{B}})$

$$\left(\vec{\mathrm{A}} \times \vec{\mathrm{B}} \times \vec{\mathrm{C}}\right)_i = \varepsilon_{ijk}\mathrm{A}_j\left(\vec{\mathrm{B}} \times \vec{\mathrm{C}}\right)_k = \varepsilon_{ijk}\mathrm{A}_j\varepsilon_{klm}\mathrm{B}_l\mathrm{C}_m = \varepsilon_{ijk}\varepsilon_{klm}\mathrm{A}_j\mathrm{B}_l\mathrm{C}_m = \varepsilon_{kij}\varepsilon_{klm}\mathrm{A}_j\mathrm{B}_l\mathrm{C}_m$$
$$= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\mathrm{A}_j\mathrm{B}_l\mathrm{C}_m = \mathrm{A}_j\mathrm{B}_i\mathrm{C}_j - \mathrm{A}_j\mathrm{B}_j\mathrm{C}_i \equiv \mathrm{B}_i\left(\vec{\mathrm{A}} \cdot \vec{\mathrm{C}}\right) - \mathrm{C}_i\left(\vec{\mathrm{A}} \cdot \vec{\mathrm{B}}\right)$$

> The k index was permuted to the left 2 times, this gives a (+1)

2) Prove that $\nabla \cdot \nabla \times \vec{\mathrm{A}} = 0$

$$\nabla \cdot \nabla \times \vec{\mathrm{A}} = \frac{\partial}{\partial x_i}\left(\nabla \times \vec{\mathrm{A}}\right)_i = \frac{\partial}{\partial x_i}\varepsilon_{ijk}\frac{\partial}{\partial x_j}\mathrm{A}_k = \varepsilon_{ijk}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\mathrm{A}_k = -\varepsilon_{jik}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\mathrm{A}_k$$
$$= -\varepsilon_{jik}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}\mathrm{A}_k = -\varepsilon_{ijk}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\mathrm{A}_k = 0$$

> The i index was permuted once with j, thus giving a (–1)

> The derivation order was inverted, no harm done since x_j and x_i are independent variables.

> No index permutation, we just decided to change notation such that all i's transform into j's and vice versa.

> These two expressions are identical except for their signs. This contradiction can only be avoided if the expression is equal to zero.

3) Prove that $\nabla\cdot\left(\vec{A}\times\vec{B}\right)=\left(\nabla\times\vec{A}\right)\cdot\vec{B}-\left(\nabla\times\vec{B}\right)\cdot\vec{A}$

$$\nabla\cdot\left(\vec{A}\times\vec{B}\right)=\frac{\partial}{\partial x_i}\left(\vec{A}\times\vec{B}\right)_i=\frac{\partial}{\partial x_i}\varepsilon_{ijk}A_jB_k=\varepsilon_{ijk}\frac{\partial}{\partial x_i}A_jB_k=\varepsilon_{ijk}\left(A_j\frac{\partial}{\partial x_i}B_k+B_k\frac{\partial}{\partial x_i}A_j\right)$$

$$=\varepsilon_{ijk}A_j\frac{\partial}{\partial x_i}B_k+\varepsilon_{ijk}B_k\frac{\partial}{\partial x_i}A_j=\varepsilon_{ikj}A_k\frac{\partial}{\partial x_i}B_j+\varepsilon_{kij}B_k\frac{\partial}{\partial x_i}A_j$$

Again, no index permutation, we changed all *j*'s for *k*'s and vice versa.

Two permutations of *k* to the left yield a (+1)

$$=-\varepsilon_{kij}A_k\frac{\partial}{\partial x_i}B_j+\varepsilon_{kij}B_k\frac{\partial}{\partial x_i}A_j=-A_k\varepsilon_{kij}\frac{\partial}{\partial x_i}B_j+B_k\varepsilon_{kij}\frac{\partial}{\partial x_i}A_j$$

A single permutation of k to the left gives a (-1)

$$=-\vec{A}\cdot\left(\nabla\times\vec{B}\right)+\vec{B}\cdot\left(\nabla\times\vec{A}\right)$$

4) Prove that $\vec{B}\times\left(\nabla\times\vec{B}\right)=\nabla\left(\frac{B^2}{2}\right)-\left(\vec{B}\cdot\nabla\right)\vec{B}$

$$\left[\vec{B}\times\left(\nabla\times\vec{B}\right)\right]_i=\varepsilon_{ijk}B_j\varepsilon_{klm}\frac{\partial}{\partial x_l}B_m=\varepsilon_{kij}\varepsilon_{klm}B_j\frac{\partial}{\partial x_l}B_m=\left(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}\right)B_j\frac{\partial}{\partial x_l}B_m$$

$$=B_j\frac{\partial}{\partial x_i}B_j-B_j\frac{\partial}{\partial x_j}B_i=\frac{1}{2}\frac{\partial}{\partial x_i}B_jB_j-B_j\frac{\partial}{\partial x_j}B_i=\frac{\partial}{\partial x_i}\left(\frac{B^2}{2}\right)-\left(\vec{B}\cdot\nabla\right)B_i$$

$$\frac{\partial}{\partial x_i}B_jB_j=2B_j\frac{\partial}{\partial x_i}B_j$$

In this way, we can deal with arbitrarily complicated vector representations by following simple rules.

As an exercise, try other identities on your own.

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