Session 3-4: Mission Analysis for Electric Propulsion

Previously, we described the benefits of electric propulsion over chemical rockets, namely, propellant savings albeit at increased mass from the supply, delivery and processing of electric power. Given this tradeoff, it is expected that there will exist a value of specific impulse that will maximize payload mass for some particular mission parameters. In this Lecture we analyze a few cases amenable to analytical (or straightforward numerical) solutions.

The fundamental idea behind mission analysis is to write an objective function and then optimize it with respect to some variables under specific constraints. We begin by writing the total (wet) mass of the spacecraft,

$$m_0 = m_{ps} + m_p + m_s + m_{pay} \tag{1}$$

The mass of the propulsion/power system is,

$$m_{ps} = \alpha P = \frac{\alpha F c}{2\eta} \tag{2}$$

If the mission time t_m is fixed and we assume constant mass flow rate,

$$m_p = \dot{m} t_m = \frac{F}{c} t_m \tag{3}$$

Taking the structural and payload mass as constant (or at least independent of specific impulse), we apply the optimality condition (maximum payload),

$$\left.\frac{dm_{pay}}{dc}\right|_{F,P,t} = 0 \qquad \text{or} \qquad \frac{F t_m}{c_{opt}^2} - \frac{\alpha F}{2\eta} = 0 \tag{4}$$

Solving for the optimal specific impulse we obtain,

$$c_{opt} = \sqrt{\frac{2\eta t_m}{\alpha}} \tag{5}$$

Missions requiring thrusting for long periods of time, for example to counteract drag when orbiting in LEO, would benefit from high specific impulse.

In finding the optimal specific impulse Eq. (5) we only specified the mission duration and assumed constant mass flow rate and thrust independent of I_{sp}. If in addition there is now a constraint in Δv then we should use $m_p = m_0\left(1 - e^{-\Delta v/c}\right)$ for the propellant mass. Also, the propulsion/power system mass can be written as,

$$m_{ps} = \alpha P = \frac{\alpha \dot{m} c^2}{2\eta} = \frac{\alpha m_0 c^2}{2\eta t_m}\left[1 - \exp\left(-\frac{\Delta v}{c}\right)\right] \tag{6}$$

Here we notice the same group of parameters of Eq. (5). We define this group as a characteristic velocity,

$$v_{ch} = \sqrt{\frac{2\eta t_m}{\alpha}} \quad \text{(Stuhlinger velocity)} \tag{7}$$

with the understanding that this may not be the optimal value of the specific impulse given the Δv constraint. The meaning of this velocity, from the definition of α is that, if the propulsion/power mass were to be accelerated by converting all of the electrical energy generated during time t_m, it would then reach the velocity v_{ch}. This is therefore the upper limit of Δv that can be achieved with an electric propulsion system.

The balance of Eq. (1) can be written as,

$$H = \frac{m_{pay} + m_s}{m_0} = \exp\left(-\frac{\Delta v/v_{ch}}{c/v_{ch}}\right) - \left(\frac{c}{v_{ch}}\right)^2 \left[1 - \exp\left(-\frac{\Delta v/v_{ch}}{c/v_{ch}}\right)\right] \tag{8}$$

where we have grouped the problem invariants into an objective function H (in this case non-dimensional, accounting for the payload and structural mass) to be maximized in terms of c/v_{ch} for a given $\Delta v/v_{ch}$.

Fig. 1 shows the shape of H as a function of c/v_{ch} with $\Delta v/v_{ch}$ as a parameter. The existence of an optimum specific impulse in each case is apparent in the figure. This optimum c is seen to be near v_{ch}. If $\Delta v/c$ is taken to be relatively small, expansion of the exponentials in Eq. (8) allows an approximate analytical expression for the optimum c:

$$\frac{c_{opt}}{v_{ch}} \approx 1 - \frac{1}{2}\frac{\Delta v}{v_{ch}} - \frac{1}{24}\left(\frac{\Delta v}{v_{ch}}\right)^2 + \ldots \tag{9}$$

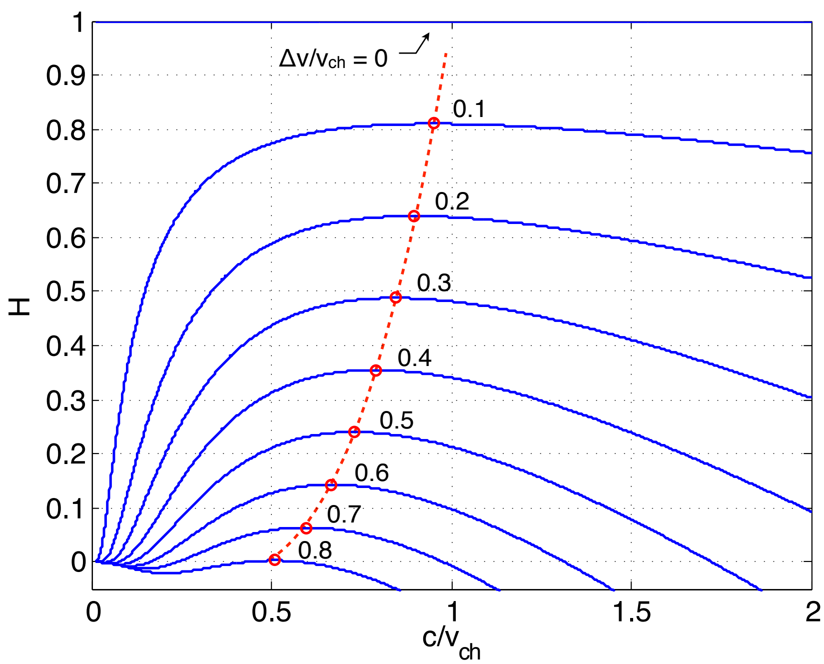

Figure 1: Objective function H as a function of c/v_{ch} with $\Delta v/v_{ch}$ as a parameter

Fig. 1 also shows that, as anticipated, the maximum Δv for which a positive payload can be carried (with negligible m_s) is of the order of $0.8v_{ch}$. Even at this high Δv, Eq. (9) holds fairly well.

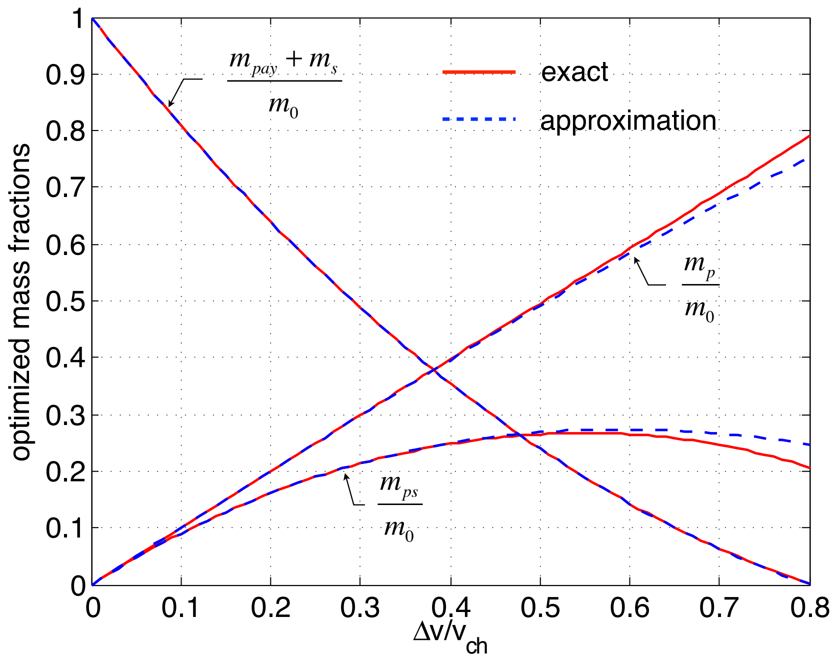

Figure 2: Optimized mass fractions as a function of $\Delta v/v_{ch}$

The effects of (constant) efficiency, propulsion/power specific mass and mission time are all lumped into the parameter v_{ch}. Eq. (9) then shows that a high specific impulse is indicated when the propulsion/power system is light and/or the mission has a long duration. Fig. 2 shows that, for a fixed Δv, these same attributes tend to give a high payload fraction and small (and comparable) propulsion/power and propellant fractions. Of course the same breakdown trends can be realized by reducing Δv for a fixed v_{ch}. This regime is appropriately known as the *trucking* regime. At the opposite end (short mission, heavy propulsion/power) we have a low v_{ch}, hence low optimum specific impulse, and from Fig. 2, small payload and large propellant fractions. Correspondingly, this is known as the *sports-car* regime.

We have, so far, regarded the efficiency η as a constant, independent of the choice of specific impulse. This is not, in general, a good assumption for electric thrusters where the physics of the gas acceleration process can change significantly as the power loading (hence the jet velocity) is increased. For each thruster family and for each propellant and design, one can typically establish a connection between Δv and c alone. Thus, as we will see later in detail, η increases with c in ion, Hall, MPD and electrospray thrusters, whereas it typically decreases for arcjets (beyond a certain c). In general, then, one needs to return to Eq. (8) with $\eta = \eta(c)$ in order to discover the best choice of specific impulse in each case. It is instructive to consider the particular case of ion and electrospray thrusters, both because relatively simple and accurate laws can be obtained in those cases.

Losses for these thrusters can be fairly well characterized by a constant voltage drop per accelerated ion or charged droplet. If this is called $\Delta\phi$, the energy spent per particle of mass m_i and charge q is,

$$\frac{1}{2}m_i c^2 + q\Delta\phi \tag{10}$$

of which only $m_i c^2/2$ is useful. The efficiency is then,

$$\eta = \frac{\eta_0}{1 + (v_L/c)^2} \tag{11}$$

Eq. (11) includes a factor $\eta_0 < 1$ to account for power processing and other losses and where v_L is a *loss velocity*, equal to the velocity to which charged particles would be accelerated by the voltage drop $\Delta\phi$,

$$v_L = \sqrt{2\frac{q}{m_i}\Delta\phi} \tag{12}$$

This expression indicates that losses are reduced if the specific charge q/m_i is low. This shows the importance of high atomic mass in ion engines and electrosprays working in the pure ionic regime, since $\Delta\phi$ is not very sensitive to propellant choice, and v_L can be reduced if m_i is large. Eq. (11) also shows the rapid loss of efficiency when c is reduced below v_L.

Using Eq. (11), we can rewrite Eq. (8) as,

$$H = e^{-\Delta v/c} - \frac{c^2 + v_L^2}{v_{ch}^2}\left(1 - e^{-\Delta v/c}\right) \tag{13}$$

where the definition of v_{ch} in Eq. (7) is now made using η_0 instead of η. Once again, only approximate expressions in $\Delta v/v_{ch}$ are feasible for the optimum specific impulse and mass fractions. Normalizing all velocities by v_{ch}:

$$x = \frac{c}{v_{ch}}, \quad \nu = \frac{\Delta v}{v_{ch}} \quad \text{and} \quad \delta = \frac{v_L}{v_{ch}} \tag{14}$$

we obtain,

$$x_{opt} = \sqrt{1+\delta^2} - \frac{\nu}{2} - \frac{\nu^2}{24\sqrt{1+\delta^2}} + ... \tag{15}$$

$$H_{max} = 1 - 2\nu\sqrt{1+\delta^2} + \nu^2 - \frac{\nu^3}{12\sqrt{1+\delta^2}} + ... \tag{16}$$

$$\left.\frac{m_p}{m_0}\right|_{opt} = \frac{\nu}{\sqrt{1+\delta^2}} - \frac{1}{24}\left(\frac{\nu}{\sqrt{1+\delta^2}}\right)^3 + ... \tag{17}$$

The main effect of the losses characterized by δ can be seen to be:

(a) An increase of the optimum specific impulse, seeking to take advantage of the higher efficiency thus obtained.

(b) A reduction of the maximum payload,

(c) A reduction of the fuel fraction.

Both these last effects indicate a higher propulsion/power mass fraction, due to the need to raise rated power to compensate for the efficiency loss. It is worth noting also that the losses are felt least in the *trucking* mode (high v_{ch}, i.e. light engine or long duration operation).

As was mentioned, there is no a priori reason to operate an electric thruster at a constant thrust or specific impulse, even if the power is indeed fixed. We examine here a simple case to illustrate this point, namely, one with a constant efficiency as in the classical *Stuhlinger* optimization, but allowing F, $\dot{m}$ and c to vary in time if this is advantageous. Of course these variations are linked by the constancy of the power,

$$P = \frac{\dot{m}(t)c^2(t)}{2\eta} = \frac{F(t)c(t)}{2\eta} \tag{18}$$

Consider the rate of change of the inverse mass with time:

$$\frac{d(1/m)}{dt} = -\frac{1}{m^2}\frac{dm}{dt} = \frac{\dot{m}}{m^2} = \frac{\dot{m}^2}{m^2\dot{m}} = \frac{F^2}{m^2\dot{m}c^2} = \frac{a^2}{2\eta P} \tag{19}$$

where $a = F/m$ is the acceleration due to thrust. Integrating,

$$\frac{1}{m_f} - \frac{1}{m_0} = \frac{1}{2\eta P}\int_0^{t_m} a^2 dt \tag{20}$$

On the other hand, the mission Δv is,

$$\Delta v = \int_0^{t_m} a dt \tag{21}$$

and is a prescribed quantity. We wish to select the function $a(t)$ which will give a maximum m_f while preserving this value of Δv. The problem reduces to finding the shape of $a(t)$, whose square integrates to a minimum while its own value has a fixed integral. The solution (which can be found by various mathematical techniques, but is intuitively clear) is that a should be a constant.

Using this condition, Eq. (20) and (21) integrate immediately. Eliminating a between these, we obtain,

$$\frac{m_f}{m_0} = \left(1 + \frac{m_0 \Delta v^2}{2\eta t_m P}\right)^{-1} \tag{22}$$

The level of power is yet to be selected. It will determine the average specific impulse, and it is to be expected that an optimum will also exist. The final mass consists of the payload and propulsion/power system, $m_f = m_{pay} + m_{ps}$. Using the definition of v_{ch} and $P = m_{ps}/\alpha$, our objective function is now,

$$H = \frac{m_{pay}}{m_0} = \frac{m_{ps}}{m_0}\left[\frac{1}{(m_{ps}/m_0) + (\Delta v/v_{ch})^2} - 1\right] \tag{23}$$

and select the value of m_{ps}/m_0 that will maximize H. This is found to be,

$$\left.\frac{m_{ps}}{m_0}\right|_{opt} = \frac{\Delta v}{v_{ch}}\left(1-\frac{\Delta v}{v_{ch}}\right) \tag{24}$$

which, when used back in Eq. (23) gives,

$$H_{max} = \left.\frac{m_{pay}}{m_0}\right|_{opt} = \left(1-\frac{\Delta v}{v_{ch}}\right)^2 \tag{25}$$

and then,

$$\left.\frac{m_p}{m_0}\right|_{opt} = \frac{\Delta v}{v_{ch}} \tag{26}$$

These are, within the assumptions, exact expressions. They could be compared to the results in Fig. 2 which were found to apply when c, and not a, was assumed constant. The difference is noticeable only for the highest values of $\Delta v/v_{ch}$ and is negligible for smaller values.

It is of some interest to inquire at this point how the jet velocity c should vary with time in order to keep the acceleration constant. We have,

$$a = \frac{\dot{m}c}{m} = \frac{c}{m}\left(\frac{2\eta P}{c^2}\right) \qquad \text{then} \qquad c = \left(\frac{2\eta P}{a}\right)\frac{1}{m} \tag{27}$$

and since $1/m$ varies linearly with time (because a is constant), so will c. At the final time, when $m = m_f$,

$$c_f = \left(\frac{2\eta P}{a}\right)\frac{1}{m_f} = \left[\frac{2\eta(m_{ps}/\alpha)}{\Delta v/t_m}\right]\frac{1}{m_f} = \left(\frac{v_{ch}^2}{\Delta v}\right)\frac{m_{ps}/m_0}{1-m_p/m_0} = \left(\frac{v_{ch}^2}{\Delta v}\right)\frac{\Delta v}{v_{ch}} = v_{ch} \tag{28}$$

From Eqs. (19) and (27), the rate of change of c becomes,

$$\frac{dc}{dt} = \left(\frac{2\eta P}{a}\right)\frac{d(1/m)}{dt} = \left(\frac{2\eta P}{a}\right)\frac{a^2}{2\eta P} = a \tag{29}$$

so that, altogether, for some intermediate time t,

$$c(t) = v_{ch} - a(t_m - t) \tag{30}$$

Thus c varies between $v_{ch} - \Delta v$ at $t = 0$ and v_{ch} at $t = t_m$. The approximate result $c_{opt} \approx v_{ch} - \Delta v/2$ found when c was constrained to remain constant is therefore quite reasonable. Notice that Eq. (30), in the ideal case with no external forces acting on the vehicle, implies a constant absolute velocity of the exhaust,

$$c_{abs} = c - v = c - (at + v_0) = v_{ch} - \Delta v - v_0 \tag{31}$$

This means that in the optimal case, exhaust particles move at constant velocity with respect to an inertial observer. Even though some energy was spent in realizing this absolute motion, in time, no more energy is effectively added to the exhaust, thus minimizing energy losses.