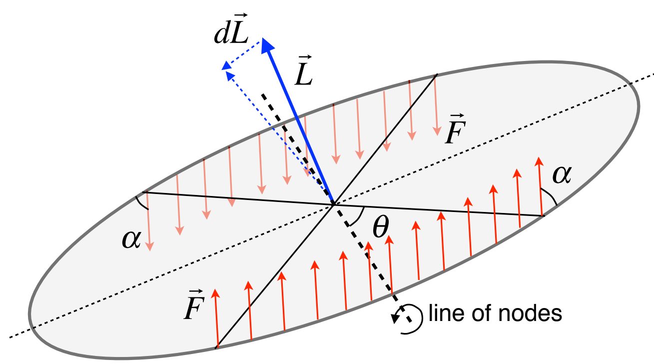
Session 7: **Sub-Optimal Climb and Plane Change**

In this lecture we present another application of a low thrust maneuver, but this time including some optimality notions. From an initially circular orbit of radius r, the objective is to climb while changing the inclination of the orbit. This would be required, for instance, when starting from a circular LEO after launch from KSC (N28.5°) and then transferring to GEO, on the equatorial plane. The generic configuration of such maneuver is illustrated in the figure below.



Plane change using impulsive maneuvers requires thrusting exactly over the line of nodes. The thrust vector $\vec{F}$ should be normal to the original plane to change the inclination, i. Thrusting at some angle from the normal will change the orbit to an elliptical one by raising the apoaxis. Using this strategy for low thrust propulsion is highly inefficient, since it would require a substantial amount of time to obtain a noticeable change in the orbital altitude and inclination. Instead, thrust is distributed around the line of nodes as shown in the figure. In this case, firing occurs inside arcs defined by the angle θ and there will be some particular thrust angle distribution $\alpha = \alpha(\theta)$ that would optimize the trajectory for a particular net change in orbital radius Δr and inclination Δi.

To analyze this problem, we start by calculating the rate of change of the magnitude of the orbital angular momentum,

$$\frac{dL}{dt} = L\frac{di}{dt} = (F\sin\alpha)r\cos\theta \tag{1}$$

Since the angular momentum for a circular orbit is given by,

$$L = mh = m\sqrt{\mu r} \tag{2}$$

and the angular velocity of the orbit is,

$$\frac{d\theta}{dt} = \sqrt{\frac{\mu}{r^3}} \tag{3}$$

then we can write an equation for the rate of change of the orbital inclination with respect to θ (true anomaly),

$$\frac{di}{d\theta} = \frac{F}{m}\frac{r^2}{\mu}\sin\alpha\cos\theta \tag{4}$$

The inclination and time increment (one period) after one orbit will then be,

$$(\Delta i)_1 = \frac{F}{m}\frac{r^2}{\mu}\int_0^{2\pi} \sin\alpha\cos\theta\; d\theta \qquad \text{and} \qquad (\Delta t)_1 = 2\pi\sqrt{\frac{r^3}{\mu}} \tag{5}$$

So, the rate of change of inclination over one orbit will be,

$$\left\langle \frac{di}{dt} \right\rangle = \frac{1}{2\pi}\frac{F}{m}\sqrt{\frac{r}{\mu}}\int_0^{2\pi} \sin\alpha\cos\theta\; d\theta \tag{6}$$

To perform a similar averaging for the rate of change of the radius, we start from a power balance,

$$m\frac{d}{dt}\left(-\frac{\mu}{2r}\right) = (F\cos\alpha)\sqrt{\frac{\mu}{r}} \qquad \rightarrow \qquad \frac{dr}{dt} = 2\frac{F}{m}\sqrt{\frac{r^3}{\mu}}\cos\alpha \tag{7}$$

The average over one orbit is then,

$$\left\langle \frac{dr}{dt} \right\rangle = \frac{1}{\pi}\frac{F}{m}\sqrt{\frac{r^3}{\mu}}\int_0^{2\pi} \cos\alpha\; d\theta \tag{8}$$

These equations could be solved numerically for a particular profile $\alpha = \alpha(\theta)$, or incorporated into an optimization process to figure out what profile minimizes some cost function.

Instead of dealing with this general problem, Edelbaum (1961, 1973) just kept $|\alpha|$ constant during each orbit, then optimized $|\alpha|(r)$. In this case,

$$\alpha(\theta) = \begin{cases} +\alpha & \text{for} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ -\alpha & \text{for} \quad \frac{\pi}{2} < \theta < \frac{3\pi}{2} \end{cases} \tag{9}$$

From Eq. (4), but now (using $\alpha \equiv$ constant),

$$\langle \sin\alpha\cos\theta \rangle_\theta = \frac{1}{\pi}\sin\alpha\int_{-\pi/2}^{\pi/2} \cos\theta\; d\theta = \frac{2}{\pi}\sin\alpha$$

therefore,

$$\left\langle \frac{di}{d\theta} \right\rangle = \frac{2}{\pi}\frac{F}{m}\frac{r^2}{\mu}\sin\alpha \tag{10}$$

Similarly, we still have from Eqs. (5) and (7),

$$\frac{dr}{d\theta} = 2\frac{F}{m}\frac{r^3}{\mu}\cos\alpha \tag{11}$$

which needs no averaging. Dividing Eq. (10) by Eq. (11) and dropping the averaging sign,

$$\frac{di}{dr} = \frac{\tan\alpha}{\pi r} \tag{12}$$

We also have,

$$\frac{d\Delta v}{dr} = \frac{F/m}{dr/dt}$$

which combined with Eq. (7) results in,

$$\frac{d\Delta v}{dr} = \frac{1}{2}\frac{\sqrt{\mu/r^3}}{\cos\alpha} \tag{13}$$

To optimize $\alpha(r)$ we look to minimize $d\Delta v/dr$ for a given di/dr. We introduce the following cost function, or Hamiltonian,

$$H = \frac{1}{2}\frac{\sqrt{\mu/r^3}}{\cos\alpha(r)} + \lambda\left(\frac{di}{dr} - \frac{\tan\alpha(r)}{\pi r}\right) \tag{14}$$

The control variable is $\alpha(r)$, the state variable is i and the independent variable (replacing time) is r. The optimality and "transversality" conditions are,

$$\begin{aligned}\frac{\partial H}{\partial \alpha} &= 0 \\ \frac{d\lambda}{dr} &= \frac{\partial H}{\partial i}\end{aligned} \qquad \text{and} \qquad [\lambda \delta i]_{r_1}^{r_2} = 0 \tag{15}$$

with the second being satisfied automatically, because i is prescribed at both ends. Since H does not depend explicitly on i,

$$\frac{d\lambda}{dr} = 0 \qquad \rightarrow \qquad \lambda \equiv \text{constant} \tag{16}$$

And using $\frac{1}{\cos\alpha} = \sqrt{1+\tan^2\alpha}$, the first of the conditions in Eq. (15) is,

$$\frac{1}{2}\frac{\sqrt{\mu/r^3}\tan\alpha}{\sqrt{1+\tan^2\alpha}} - \frac{\lambda}{\pi r} = 0 \qquad \rightarrow \qquad \sin\alpha = \frac{2\lambda}{\pi}\sqrt{\frac{r}{\mu}} \tag{17}$$

Pending determination of λ, Eq. (17) indicates that the thrust tilt amplitude $\alpha(r)$ increases over the mission, so that most of the plane-change activity is deferred to the last part of the climb, when the orbital velocity is lower.

To find λ, use the constraint on the total Δi, from Eq. (12),

$$\Delta i = \int_{r_1}^{r_2} \frac{\tan\alpha}{\pi r} dr \tag{18}$$

From Eq. (17) we solve for r, so that,

$$r = \mu \left(\frac{\pi}{2\lambda}\right)^2 \sin^2 \alpha \qquad \rightarrow \qquad \frac{dr}{r} = 2\frac{d(\sin\alpha)}{\sin\alpha} = 2\frac{d\alpha}{\tan\alpha} \tag{19}$$

Hence,

$$\Delta i \;=\; \int_{\alpha_1}^{\alpha_2} \frac{2}{\pi} d\alpha \qquad \rightarrow \qquad \Delta i \;=\; \frac{2}{\pi}\left(\alpha_2 - \alpha_1\right) \tag{20}$$

A separate relationship between α_1 and α_2 comes from Eq. (17),

$$\frac{\sin\alpha_1}{\sin\alpha_2} \;=\; \sqrt{\frac{r_1}{r_2}} \;=\; \frac{v_2}{v_1} \tag{21}$$

where v_1 and v_2 denote the initial and final orbital velocities. Combining Eqs. (20) and (21),

$$\frac{\sin\left(\alpha_1 + \frac{\pi}{2}\Delta i\right)}{\sin\alpha_1} = \frac{v_1}{v_2} \qquad \rightarrow \qquad \cos\left(\frac{\pi}{2}\Delta i\right) + \cot\alpha_1 \sin\left(\frac{\pi}{2}\Delta i\right) = \frac{v_1}{v_2}$$

we obtain,

$$\cot\alpha_1 = \frac{\dfrac{v_1}{v_2} - \cos\left(\dfrac{\pi}{2}\Delta i\right)}{\sin\left(\dfrac{\pi}{2}\Delta i\right)} \qquad \text{or} \qquad \sin\alpha_1 = \frac{\sin\left(\dfrac{\pi}{2}\Delta i\right)}{\sqrt{1 + \left(\dfrac{v_1}{v_2}\right)^2 - 2\dfrac{v_1}{v_2}\cos\left(\dfrac{\pi}{2}\Delta i\right)}} \tag{22}$$

The most important quantity is the optimized Δv. From Eq. (13),

$$\Delta v \;=\; \frac{1}{2}\int_{r_1}^{r_2} \sqrt{\frac{\mu}{r^3}}\frac{dr}{\cos\alpha} \;=\; \frac{1}{2}\int_{r_1}^{r_2} \sqrt{\frac{\mu}{r}}\frac{dr/r}{\cos\alpha}$$

and combining with Eq. (19),

$$\Delta v \;=\; \frac{1}{2}\int_{\alpha_1}^{\alpha_2} \left(\frac{2\lambda}{\pi\sin\alpha}\right)\left(\frac{1}{\cos\alpha}\right)\frac{2d\alpha}{\tan\alpha} \;=\; \frac{2\lambda}{\pi}\int_{\alpha_1}^{\alpha_2} \frac{d\alpha}{\sin^2\alpha} \;=\; \frac{2\lambda}{\pi}\left(\cot\alpha_1 - \cot\alpha_2\right) \tag{23}$$

From Eq. (17) we have $2\lambda/\pi = v_1 \sin\alpha_1$, and using,

$$\cot\alpha_2 = \frac{\cos\left(\dfrac{\pi}{2}\Delta i\right) - \dfrac{v_2}{v_1}}{\sin\left(\dfrac{\pi}{2}\Delta i\right)}$$

we calculate the optimized Δv,

$$\Delta v \;=\; \frac{v_1 \sin\left(\dfrac{\pi}{2}\Delta i\right)}{\sqrt{1 + \left(\dfrac{v_1}{v_2}\right)^2 - 2\dfrac{v_1}{v_2}\cos\left(\dfrac{\pi}{2}\Delta i\right)}} \frac{\left[\dfrac{v_1}{v_2} - \cos\left(\dfrac{\pi}{2}\Delta i\right)\right] - \left[\cos\left(\dfrac{\pi}{2}\Delta i\right) - \dfrac{v_2}{v_1}\right]}{\sin\left(\dfrac{\pi}{2}\Delta i\right)}$$

and simplifying,

$$\Delta v = \sqrt{v_1^2 + v_2^2 - 2v_1 v_2 \cos\left(\frac{\pi}{2}\Delta i\right)} \tag{24}$$

Geometrically, Δv appears as the vector difference of the final and initial velocities, except the angle between $\vec{v}_1$ and $\vec{v}_2$ is not the actual Δi, but $\frac{\pi}{2}\Delta i$. The extra factor reflects the inefficiency associated with thrusting through the full $\pi/2$ in each out-of-plane direction.

Example

Consider a LEO (400 Km) to GEO, with:

- $\Delta i = 28.5°$
- $v_1 = 7673$ m/s
- $v_2 = 3072$ m/s

From Eq. (24),

$$\Delta v = \sqrt{7673^2 + 3072^2 - 2(7673)(3072)\cos\left(\frac{\pi}{2}28.5°\right)} = 5903 \text{ m/s}$$

This is noticeably worse than the true optimum $\Delta v = 5768$ m/s calculated for the case when $\alpha = \alpha(\theta)$ is also modulated as $\tan\alpha = \cos\theta$.

The initial and final tilt angles are,

$$\sin\alpha_1 = \frac{\sin\left(\frac{\pi}{2}28.5°\right) \times 3072}{5903} = 0.3665 \qquad \alpha_1 = 21.5°$$

$$\alpha_2 = \alpha_1 + \frac{\pi}{2}\Delta i \qquad \alpha_2 = 66.3°$$

These are smaller than the peak values $\alpha_{1_{\max}} = 30.5°$ and $\alpha_{2_{\max}} = 72.2°$ in this optimal case, but, of course, they are applied for the whole half-orbit.

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