

16.55 Ionized Gases

Problem Set #3

Problem 1

- (a) In many situations, the distribution function of some particles is known to be approximately isotropic, namely, $f(t, \vec{x}, \vec{w}) = f(t, \vec{x}, w)$, where w is the magnitude of the velocity. Alternatively, in that case one can put $f = f(t, \vec{x}, K)$, where $K = mw^2/2$ is the kinetic energy of a particle. Show that the Vlasov equation can then be written as,

$$\frac{\partial f}{\partial t} + \vec{w} \cdot \nabla f + (\vec{F} \cdot \vec{w}) \frac{\partial f}{\partial K} = 0$$

where \vec{F} is the force on the particle.

- (b) Assume in addition that the force derives from a force potential, $\vec{F} = -\nabla V$, and show that in that case the Vlasov equation becomes,

$$\frac{\partial f}{\partial t} + \vec{w} \cdot \left(\nabla f - \frac{\partial f}{\partial K} \nabla V \right) = 0$$

- (c) As a further assumption, suppose the problem is in steady state. Show that the distribution then depends only upon the total energy,

$$f = f(E) \quad ; \quad E = K(w) + V(\vec{x})$$

- (d) The form of the function $f(E)$ depends on the boundary conditions. As an example, consider a spherically symmetric potential bump or potential well, and suppose there is a uniform region of zero potential (“at infinity”) where the distribution is Maxwellian,

$$f = n_{\infty} \left(\frac{m}{2\pi kT_{\infty}} \right)^{3/2} \exp \left(-\frac{mw^2}{2kT_{\infty}} \right)$$

Show that the distribution at a location where the potential V is not zero is either,

$$f = n_{\infty} \left(\frac{m}{2\pi kT_{\infty}} \right)^{3/2} \exp \left(-\frac{mw^2}{2kT_{\infty}} \right) \exp \left(-\frac{V(\vec{x})}{kT_{\infty}} \right)$$

or $f = 0$, the latter being true for velocities that would make the total energy negative (which requires $V < 0$). Notice that T_{∞} is determined by the conditions at infinity, but at locations with $V < 0$, T_{∞} is no longer the local temperature, due to the absence of some of the velocity ranges. Calculate the local density $n(\vec{x})$, and show that it is,

$$\begin{aligned} n &= n_{\infty} e^{-V/kT_{\infty}} & V > 0 \\ n &= n_{\infty} e^{-V/kT_{\infty}} \left(\operatorname{erfc} \sqrt{\frac{-V}{kT_{\infty}}} + \frac{2}{\sqrt{\pi}} \sqrt{\frac{-V}{kT_{\infty}}} e^{V/kT_{\infty}} \right) & V < 0 \end{aligned}$$

Calculate the limit of the second of these expressions when $-V/kT \gg 1$.

- (e) As a second example, suppose the potential bump (or well) has cylindrical symmetry instead. The velocity component parallel to the axis is now conserved, and so the sum of the potential energy and the perpendicular kinetic energy is separately conserved, and cannot become negative. Show that in this case, if $V < 0$, the local density n is everywhere equal to n_∞ .

NOTE: In this case the distribution does not remain isotropic, since only cylindrical symmetry exists. However, $f = f(E)$ still applies, because E is a constant of the motion.

Problem 2

For approximate calculations, an often used kinetic model for a single stationary gas is to replace the collisional integral of Boltzmann's equation by a simpler relaxation term that also has the property of becoming zero for the equilibrium distribution:

$$\frac{\partial f}{\partial t} + \vec{w} \cdot \nabla f + \frac{\vec{F}}{m} \cdot \nabla_w f = \nu(f_M - f)$$

where the “collision frequency” is some function of T and n , and,

$$f_M = n \left(\frac{m}{2\pi kT} \right)^{3/2} e^{-\frac{mw^2}{2kT}}$$

is the Maxwellian distribution. Since T and n are themselves moments of f , this is still an integro-differential equation, but it is less complex than the complete Boltzmann equation. This is sometimes referred to as Krook's model, or the BGK model.

- (a) Consider a gas at constant pressure, but with a temperature gradient dT/dx in the x -direction. In the absence of external forces, and in steady state, the equation for f reduces to,

$$w_x \frac{\partial f}{\partial x} = \nu(f_M - f)$$

Assume a small perturbation expansion for f , of the form $f = f_M + \phi$, where $\phi \ll f$. To first order in ϕ , show that,

$$\phi \approx -\frac{w_x}{\nu} \frac{\partial f_M}{\partial x}$$

and using the fact that f_M depends on x only through T (notice that $n = p/kT$), show that,

$$f = f_M \left[1 + \frac{w_x}{\nu T} \left(\frac{5}{2} - \frac{mw^2}{2kT} \right) \frac{\partial T}{\partial x} - \frac{w_x}{\nu p} \frac{\partial p}{\partial x} \right]$$

- (b) Calculate the heat flux,

$$q_x = \left\langle \frac{1}{2} m w^2 w_x \right\rangle$$

and the corresponding thermal conductivity K .

Note:
$$\iiint w_x^2 w^{2i} f_M d^3w = \frac{1}{3} \iiint w^{2i+2} f_M d^3w = \frac{4\pi}{3} \int_0^\infty w^{2i+4} f_M dw$$

- (c) For a simple generalization, consider now ν to be a specified function of the particle velocity w , which for light particles like electrons, is also the relative velocity g . Assume a power-law interaction $F = \alpha/r^s$ and calculate the particle diffusive flux $\Gamma_x = \langle n w_x \rangle$ due to gradients of T and p . In particular, give the results for Maxwellian collisions ($s = 5$) and for hard spheres ($s \rightarrow \infty$). The effect of ∇T is called “thermal diffusion”, and its existence was once questioned; eventually it was put to use for uranium isotope separation.

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