Lecture 10

Fourier Analysis of Finite Difference Methods

In this lecture, we determine the stability of PDE discretizations using Fourier analysis. First, we begin with Fourier analysis of PDE's, and then extend these ideas to finite difference methods.

10.1 Fourier Analysis of PDE's

We will develop Fourier analysis in one dimension. The basic ideas extend easily to multiple dimensions. We will consider the convection-diffusion equation,

$$\frac{\partial U}{\partial t} + u\frac{\partial U}{\partial x} = \mu\frac{\partial^2 U}{\partial x^2}.$$

We will assume that the velocity, u, and the viscosity, μ are constant.

The solution is assumed to be periodic over a length L. Thus,

$$U(x + mL, t) = U(x, t)$$

where m is any integer. As we saw in Lecture 9, as the mesh is refined, the eigenvalues of the systems with general boundary conditions tend to approach the eigenvalues of the periodic case. Thus, we expect this periodicity assumption to still lead to insight into more general boundary conditions especially as the mesh is refined.

A Fourier series with periodicity over length L is given by,

$$U(x,t) = \sum_{m=-\infty}^{+\infty} \hat{U}_m(t)e^{ik_m x} \qquad \text{where} \qquad k_m = \frac{2\pi m}{L}. \tag{10.1}$$

k_m is generally called the wavenumber, though m is the number of waves occurring over the length L. We note that $\hat{U}_m(t)$ is the amplitude of the m-th wavenumber and it is generally complex (since we have used complex exponentials). Substituting the Fourier series into the convection-diffusion equation gives,

$$\frac{\partial}{\partial t}\left[\sum_{m=-\infty}^{+\infty} \hat{U}_m(t)e^{ik_m x}\right] + u\frac{\partial}{\partial x}\left[\sum_{m=-\infty}^{+\infty} \hat{U}_m(t)e^{ik_m x}\right] = \mu\frac{\partial^2}{\partial x^2}\left[\sum_{m=-\infty}^{+\infty} \hat{U}_m(t)e^{ik_m x}\right].$$

$$\sum_{m=-\infty}^{+\infty} \frac{d\hat{U}_m}{dt} e^{ik_m x} + u \sum_{m=-\infty}^{+\infty} ik_m \hat{U}_m e^{ik_m x} = \mu \sum_{m=-\infty}^{+\infty} (ik_m)^2 \hat{U}_m e^{ik_m x}.$$

Noting the $i^2 = -1$ and collecting terms gives,

$$\sum_{m=-\infty}^{+\infty} \left[\frac{d\hat{U}_m}{dt} + \left(iuk_m + \mu k_m^2 \right) \hat{U}_m \right] e^{ik_m x} = 0. \tag{10.2}$$

The next step is to utilize the orthogonality of the different Fourier modes over the length L, specifically,

$$\int_0^L e^{-ik_n x} e^{ik_m x} = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases} \tag{10.3}$$

By multiplying Equation (10.2) by $e^{-ik_n x}$ and integrating from 0 to L, the orthogonality condition gives,

$$\frac{d\hat{U}_n}{dt} + \left(iuk_n + \mu k_n^2 \right) \hat{U}_n = 0, \qquad \text{for any integer value of } n. \tag{10.4}$$

Thus, the evolution of the amplitude for an individual wavenumber is independent of the other wavenumbers. The solution to Equation (10.4),

$$\hat{U}_n(t) = \hat{U}_n(0) e^{-iuk_n t} e^{-\mu k_n^2 t}.$$

The convection term, which results in the complex time dependent behavior, $e^{-iuk_n t}$, only oscillates and does not change the magnitude of $\hat{U}_n$. The diffusion term causes the magnitude to decrease as long as $\mu > 0$. But, if the diffusion coefficient were negative, then the magnitude would increase unbounded with time. Thus, in the case of the convection-diffusion PDE, as long as $\mu \geq 0$, this solution is stable.

10.2 Semi-Discrete Fourier Analysis

Now we move on to Fourier analysis of a semi-discrete equation. That is, we discretize in space but not time. To be specific and simple, let consider pure convection discretized with central differences. This is exactly the case considered in Example 9.2 so we already know the results. But here, we derive the eigenvalues of the central difference discretization using Fourer analysis. The semi-discrete equation is,

$$\frac{dU_j}{dt} + u\,\delta_{2x} U_j = 0, \qquad \Rightarrow \qquad \frac{dU_j}{dt} + \frac{u}{2\Delta x} \left(U_{j+1} - U_{j-1} \right) = 0. \tag{10.5}$$

Note that we have switched our indices from the usual i's to j's to avoid confusion in this discussion because the Fourier series we are about to introduce will give rise to the imaginary number, i.

For the analysis of PDE's, a Fourier series of infinite dimension was used (i.e. m ranged from $\pm\infty$ in Equation (10.1)). In that case, the infinite number of terms in the Fourier series corresponds to the fact that there are infinitely many values of U over the periodic domain (because the space from $0 \leq x \leq L$ has an infinite number of points). In the semi-discrete case, only a finite number of values exist over the periodic domain. Specifically, if there are N equally-spaced nodes in the domain, then there will be $N-1$ unique values (because the values at the beginning and end of the domain must be the same due to periodicity). Summarizing, there can only be $N-1$ terms in the Fourier series used to describe a finite difference solution on a periodic domain with N points.

The most common set of $N-1$ modes used to analyze finite difference schemes is,

$$U_j(t) = \sum_{m=-N/2+1}^{N/2-1} \hat{U}_m(t) e^{ik_m j\Delta x} \qquad \text{for even } N, \tag{10.6}$$

and,

$$U_j(t) = \sum_{m=-(N-3)/2}^{(N-1)/2} \hat{U}_m(t) e^{ik_m j\Delta x} \qquad \text{for odd } N \tag{10.7}$$

Note that for large N, the limits on either Fourier series approach $\pm N/2$.

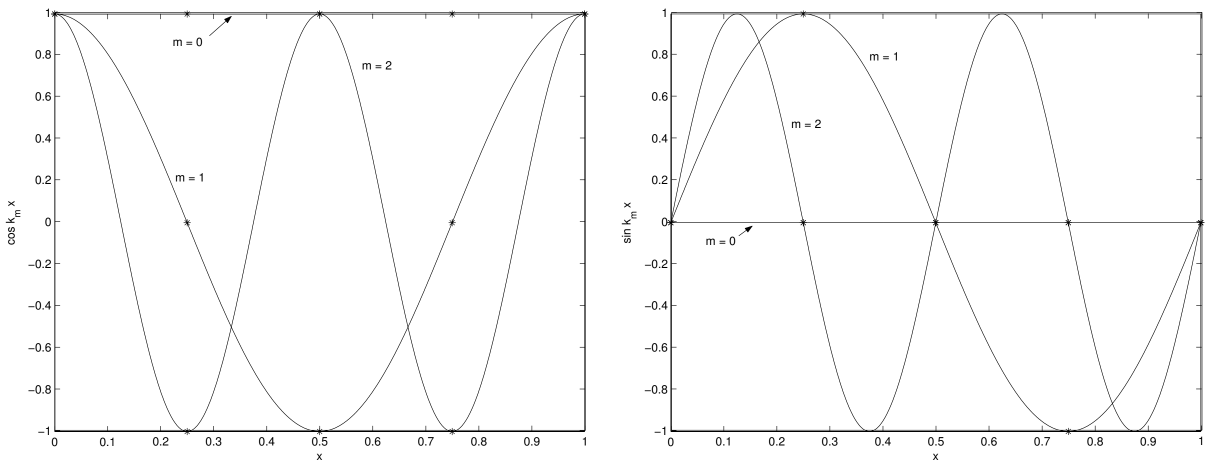
So what happens to the modes with higher wavenumbers? The answer is that the higher wavenumbers are aliased with the lower wavenumber when fewed only at a finite number of nodes. To demonstrate this, consider the case with $N=5$ nodes. In this case, the values of m in the Fourier series are from -1 to 2. The real and imaginary parts of the modes are plotted in Figure 10.1 for m =0, 1, and 2. Although the mode shape is shown for all values of x, the only values of concern are those at the nodes. We can already see the potential for aliasing by looking at the imaginary part of the $m=2$ mode. In this case, the nodal values of this mode are all zero. Thus, at the nodes, the imaginary part of $m=2$ mode is identical to the imaginary part of the $m=0$ mode (i.e. the are both zero at the nodes). The $m=2$ mode is not completely aliased with the $m=0$ mode, however, because the real part of the modes are different. Specifically, the real part of the $m=0$ mode is constant (equal to one at all nodes), while the real part of the $m=2$ mode oscillates between ± 1. This ± 1 oscillation between nodes is often called an odd-even or a sawtooth mode. To demonstrate what happens with higher wave numbers, Figure 10.2 shows the $m=3$ mode, overlayed with the $m=-1$ mode. As can be seen from this plot, these two modes are indistinguishable from each other at the nodes.

If we proceed along the analogous lines to the Fourier analysis of PDE's in the previous section, we would substitute the Fourier series into Equation (10.5) and utilize a similar orthogonality relationship to arrive at the conclusion that each mode of the Fourier series behaves independently. Thus, from now on, we will simply substitute an individual mode into the discrete equations. That is, we assume that

$$U_j(t) = \hat{U}_m(t) e^{ik_m j\Delta x},$$

for a valid m. Substitution of this individual mode into Equation (10.5) gives,

$$\frac{d\hat{U}_m}{dt} e^{ik_m j\Delta x} + \frac{u}{2\Delta x}\left[e^{ik_m(j+1)\Delta x} - e^{ik_m(j-1)\Delta x}\right]\hat{U}_m = 0.$$



(a) Real $e^{ik_m x} = \cos k_m x$ (b) Imag $e^{ik_m x} = \sin k_m x$

Figure 10.1: Plots of $\cos k_m x$ and $\sin k_m x$ for $L = 1$ including the nodal values for a five node grid ($\Delta x = 0.2$).

Factoring out the term $e^{ik_m j\Delta x}$ gives,

$$\frac{d\hat{U}_m}{dt} + \frac{u}{2\Delta x}\left[e^{ik_m\Delta x} - e^{-ik_m\Delta x}\right]\hat{U}_m = 0.$$

$$\frac{d\hat{U}_m}{dt} + i\frac{u}{\Delta x}\sin(k_m\Delta x)\,\hat{U}_m = 0.$$

Thus, for each mode m we may write,

$$\frac{d\hat{U}_m}{dt} = \lambda_m\hat{U}_m,$$

where

$$\lambda_m = -i\frac{u}{\Delta x}\sin(k_m\Delta x).$$

Comparing these λ_m to the eigenvalues found in Example 9.2, we can show that they are identical.

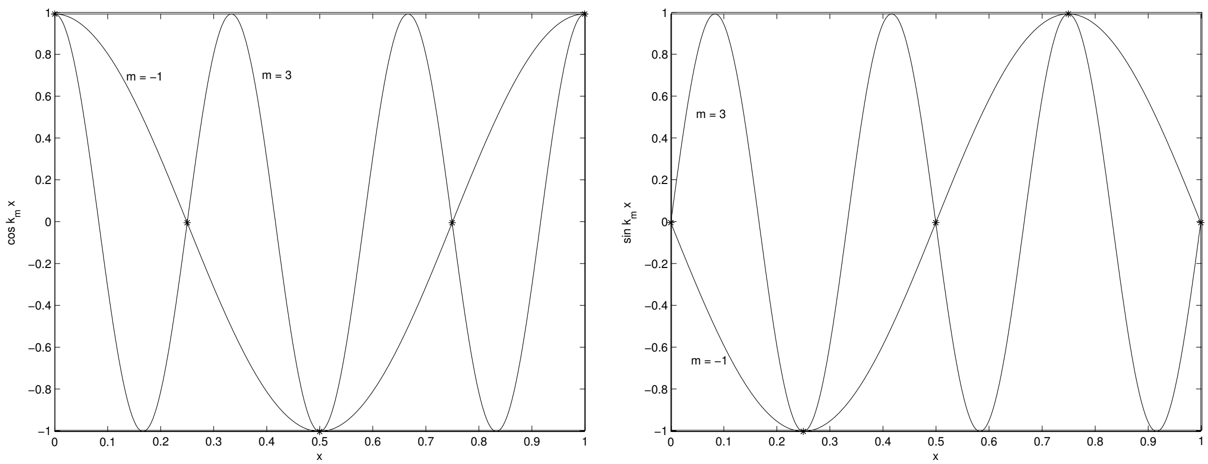
A parameter which occurs throughout Fourier analysis of spatial discretizations is,

$$\beta_x \equiv k_m\Delta x. \tag{10.8}$$

Since $k_m = 2\pi m/L$, then

$$\beta_m = 2\pi\frac{m\Delta x}{L}.$$

Since β_m varies with m, we can determine the range of β_m values corresponding to the range of m values. In particular, for large N, the limiting values of m approach $\pm N/2$, thus the



(a) Real $e^{ik_m x} = \cos k_m x$ (b) Imag $e^{ik_m x} = \sin k_m x$

Figure 10.2: Demonstration of aliasing of an $m = 3$ mode to $m = -1$ for a five node grid.

limiting values of β_m will be,

$$\lim_{\substack{m\to\pm N/2 \\ N\to\infty}} \beta_m = \pm 2\pi \frac{N\Delta x}{2L} = \pm\pi \frac{N\Delta x}{L} = \pm\pi \frac{N}{N-1} = \pm\pi.$$

Thus, assuming large N, the eigenvalue calculation for any spatial differencing method reduces to:

1. Substitute $U_j(t) = \hat{U}_m(t) e^{ij\beta_m}$ into the semi-discrete equation.
2. Determine the $\lambda_m(\beta_m)$ such that $\frac{d}{dt}\hat{U}_m = \lambda_m \hat{U}_m$.
3. Determine the limitations on the timestep such that $\lambda_m(\beta_m)\Delta t$ will be inside the stability region for a chosen time integration method for all $-\pi \le \beta_m \le \pi$.

Example 10.1 (Analysis of a first-order upwind discretization of convection) *In this example, we perform a Fourier analysis of a first-order upwind discretization of convection. Assuming the velocity is positive, then an upwind discretization of convection is,*

$$\frac{dU_j}{dt} + u\,\delta_x^- U_j = 0, \qquad \Rightarrow \qquad \frac{dU_j}{dt} + \frac{u}{\Delta x}\left(U_j - U_{j-1}\right) = 0.$$

Following the steps outlined above, we substitute the Fourier mode,

$$\frac{d\hat{U}_m}{dt} e^{ij\beta_m} + \frac{u}{\Delta x}\left[e^{ij\beta_m} - e^{i(j-1)\beta_m}\right]\hat{U}_m = 0.$$

Factoring out the term $e^{ik_m j\Delta x}$ gives,

$$\frac{d\hat{U}_m}{dt} + \frac{u}{\Delta x}\left(1 - e^{-i\beta_m}\right)\hat{U}_m = 0.$$

Thus,

$$\lambda_m \Delta t = -\frac{u\Delta t}{\Delta x}\left(1 - e^{-i\beta_m}\right) = -\frac{u\Delta t}{\Delta x}\left(1 - \cos\beta_m + i\sin\beta_m\right).$$

The upwind approximation gives both a real and imaginary part to the eigenvalue. In fact, the imaginary part is identical to the central difference part. The real part is negative, thus the upwind approximation adds stability (in the sense that a negative real eigenvalue corresponds to amplitudes that decrease in time). A plot of the eigenvalues is shown in Figure 10.3 for $u\Delta t/\Delta x = 1$*. The Matlab script for generating these eigenvalue plots is given below:*

```
bm = linspace(-pi,pi,21);
CFL = 1;
lamdt = -CFL*(1 - exp(-i*bm));

plot(lamdt,'*');
xlabel('Real \lambda\Delta t');
ylabel('Imag \lambda\Delta t');
grid on;
axis('equal');
```

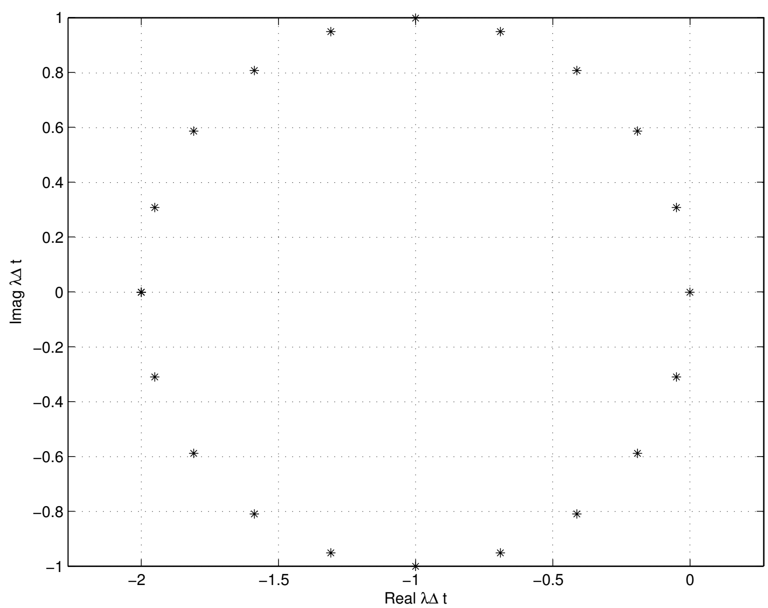


Figure 10.3: $\lambda_m \Delta t$ for a first-order upwind discretization of convection for CFL $= u\Delta t/\Delta x = 1$.