Lecture 15

Introduction to the Monte Carlo Method

In this lecture, we begin our exploration of probabilistic methods, i.e. numerical methods which are used to quantify the impact of uncertainty. In particular, we will focus on the Monte Carlo method as it is the most common probabilistic method and is the foundation for many others.

To make our discussion concrete, we will consider a simplified model for the heat transfer through a cooled turbine blade as shown in Figure 15.1. A one-dimensional model of the heat transfer along the dashed line is,

$$\dot{q} = h_{gas}\left(T_{gas} - T_{TBC}\right), \tag{15.1}$$

$$\dot{q} = \frac{k_{TBC}}{L_{TBC}}\left(T_{TBC} - T_{mh}\right), \tag{15.2}$$

$$\dot{q} = \frac{k_m}{L_m}\left(T_{mh} - T_{mc}\right), \tag{15.3}$$

$$\dot{q} = h_{cool}\left(T_{mc} - T_{cool}\right). \tag{15.4}$$

Then, given the values of h_{gas}, T_{gas}, k_{TBC}, L_{TBC}, k_m, L_m, h_{cool}, and T_{cool}, we can solve these

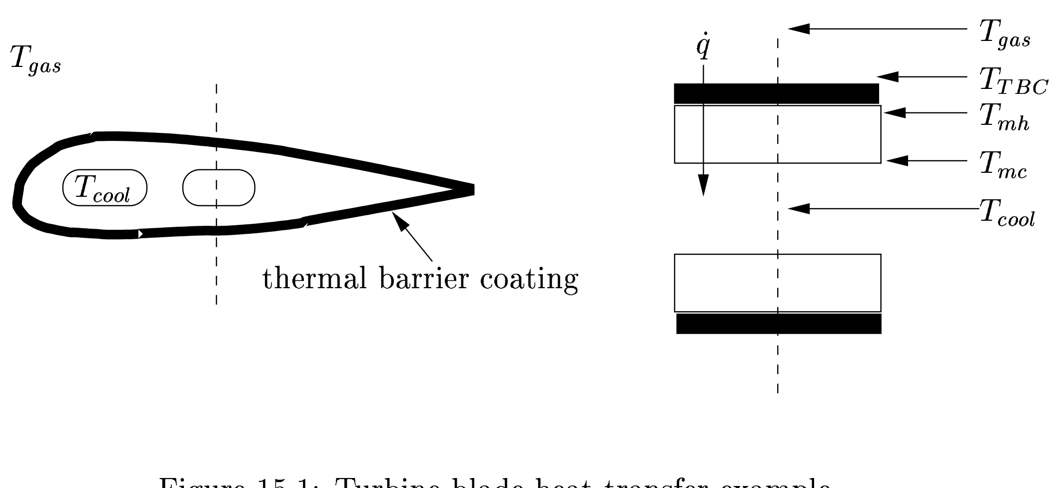


Figure 15.1: Turbine blade heat transfer example

four equations to determine, T_{TBC}, T_{mh}, T_{mc}, and $\dot{q}$. In the design of cooled turbine blades, a key parameter is the hot-side metal temperature, T_{mh}, because as this temperature increases, the durability (usable life) of a blade decreases. Thus, the goal of the heat transfer design is to maintain the metal temperatures at an acceptablly low value while minimizing cost.

Example 15.1 (Nominal Analysis of Turbine Blade Heat Transfer) *A typical, deterministic analysis of the turbine blade heat transfer problem would assume that all of the input parameters are at their nominal, i.e. design-intent, values. Suppose the design-intent values were,*

$$\begin{aligned} h_{gas} &= 3000\,W/(m^2K), & h_{cool} &= 1000\,W/(m^2K), \\ T_{gas} &= 1500\,K, & T_{cool} &= 600\,K, \\ k_{TBC} &= 1\,W/(mK), & k_m &= 20\,W/(mK), \\ L_{TBC} &= 0.0005\,m, & L_m &= 0.003\,m. \end{aligned}$$

The results of this design-intent analysis give,

$$T_{TBC} = 1348.7\,K, \quad T_{mh} = 1121.8\,K, \quad T_{mc} = 1053.8\,K, \quad \dot{q} = 453780\,W/m^2.$$

Due to manufacturing variability, the parameters of the actual manufactured blades are not exactly the design intent values but rather are distributed. For example, due to the difficulty of applying the thermal barrier coating on the outside of the blades, the thickness of the thermal barrier coating is variable. The role of probabilistic methods is to quantify the impact of this type of variability on properties of interest (e.g. the hot-side metal temperature). The results of the probabilistic analysis can take many forms depending on the specific application. In the example of the turbine blade where the hot-side metal temperature is critical, the following information might be desired from a probabilistic analysis:

- The distribution of T_{mh} that would be observed in the population of manufactured blades.
- The probability that T_{mh} is above some critical value (indicating the blade's life will be unacceptable low).
- Instead of determining the entire distribution of T_{mh}, sometimes knowing the mean value, $\mu_{T_{mh}}$, is sufficient.
- To have some indication of the variability of T_{mh} without requiring accurate estimation of the entire distribution, the standard deviation, $\sigma_{T_{mh}}$, can be used.

The Monte Carlo method is based on the idea of taking a small, randomly-drawn sample from a population and estimating the desired outputs from this sample. For the outputs described above, this would involve:

- Replacing the distribution of T_{mh} that would be observed over the entire population of manufactured blades with the distribution (i.e. histogram) of T_{mh} observed in the random sample.

- Replacing the probability that T_{mh} is above a critical value for the entire population of manufactured blades with the fraction of blades in the random sample that have T_{mh} greater than the critical value.
- Replacing the mean value of T_{mh} for the entire population with the mean value of the random sample.
- Replacing the standard deviation of T_{mh} for the entire population with the standard deviation of the random sample.

Since this exactly what is done in the field of statistics, the analysis of the Monte Carlo method is a direct application of statistics.

In summary, the Monte Carlo method involves essentially three steps:

1. Generate a random sample of the input parameters according to the (assumed) distributions of the inputs.
2. Analyze (deterministically) each set of inputs in the sample.
3. Estimate the desired probabilistic outputs, and the uncertainty in these outputs, using the random sample.

15.1 Monte Carlo Method for Uniform Distributions

To demonstrate the Monte Carlo method in more detail, let's consider the specific case where the thermal barrier coating in the previous turbine blade example is known to be uniformly distributed from $0.00025\,m < L_{TBC} < 0.00075\,m$ as shown from the probability distribution function (PDF) of L_{TBC} in Figure 15.2.

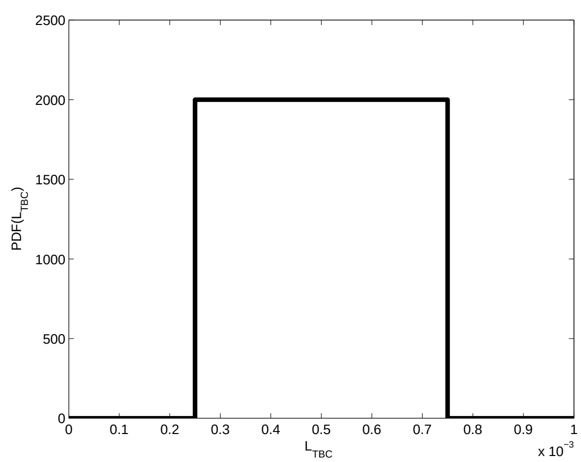


Figure 15.2: Probability distribution function (PDF) of L_{TBC} uniformly-distributed from $0.00025\,m < L_{TBC} < 0.00075\,m$.

The first step is to generate a random sample of L_{TBC}. The basic approach relies on the ability to generate random numbers which are uniformly distributed from 0 to 1. This type of functionality exists within many different scientific programming environments or languages. In Matlab, the command **rand** provides this capability. Then, using the uniform distribution from 0 to 1, a uniform distribution of L_{TBC} over the desired range can be created,

$$\mathbf{L}_{TBC} = 0.00025 + 0.0005\mathbf{u},$$

where $\mathbf{u}$ is a random variable uniformly distributed from 0 to 1. This approach is used to create the samples shown as histograms in Figure 15.3 for samples of size $N = 100$, 1000, and 10000. For the smaller sample size (specifically $N = 100$), the fact that the sample was drawn from a uniform distribution is not readily apparent. However, as the number of samples increases, the uniform distribution becomes more evident. Clearly, the sample size will have a direct impact on the accuracy of the probabilistic estimates in the Monte Carlo method.

The following is a Matlab script that implements the Monte Carlo method for this uniform distribution of L_{TBC}. The distributions of T_{mh} shown in Figure 15.4 correspond to the L_{TBC} distributions shown in Figure 15.3 and were generated with this script.

```
clear all;

% Nominal values of input parameters

hgas  = 3000;    % TBC-gas heat transfer coef. (W/(m^2 K))
Tgas  = 1500;    % Mixed gas temperature (K)
ktbc  = 1;       % TBC   thermal conduct. (W/mK)
km    = 20;      % Metal thermal conduct. (W/mK)
Lm    = 0.003;   % Metal thickness (m)
hcool = 1000;    % Coolant-metal heat transfer coef. (W/(m^2 K))
Tcool =  600;    % Coolant temperature (K)

% Number of Monte Carlo trials
N = 100;
Ltbc = zeros(N,1);
Tmh  = zeros(N,1);

for n = 1:N,

  % generate Ltbc values using a uniform distribution

  Ltbc(n) = 0.00025 + 0.0005*rand;

  % Solve heat transfer problem

  [Ttbc, Tmh(n), Tmc, q] = blade1D(hgas, Tgas, ...
   ktbc, Ltbc(n), ...
   km, Lm, ...
```

```
   hcool, Tcool);

end

figure(1);
hist(Ltbc,20);
xlabel('L_{tbc} (m)');
figure(2);
hist(Tmh,20);
xlabel('T_{mh} (K)');
```

15.2 Monte Carlo Method for Non-Uniform Distributions

Input variability can be distributed in many ways beyond the simple uniform distribution considered above. In this section, we discuss a common approach used to implement the Monte Carlo method for non-uniform distributions. However, we note that for many of the most common distribution types, random number generators widely available. For example, in Matlab, the function **randn** returns random numbers that are normally distributed with a mean of zero and a standard deviation of one. In Matlab's Statistics Toolbox, many other distribution types are available (see the documentation for the **random** function for details).

In these notes, we will discuss the inversion method for generating random numbers with non-uniform distributions. While other methods exist for generating random numbers, they are often based on the inversion method. The basic principle of the inversion method is to utilize the inverse of the cumulative distribution function (CDF) to transform a uniform distribution to a desired distribution. Recall that the CDF is defined as the integral of the PDF,

$$F(x) = \int_{-\infty}^{x} f(\xi)\, d\xi,$$

and that the CDF is related to probability by,

$$F(x) \equiv P\left\{\mathbf{x} \leq x\right\}.$$

That is, the probability of the random variable $\mathbf{x} \leq x$ is the CDF evaluated at x. As shown in Figure 15.5, $F(x)$ ranges from 0 to 1. The inversion method for generating random numbers of an arbitrary distribution consists of the following two steps:

1. Generate a random number, u, from a uniform distribution between 0 and 1.

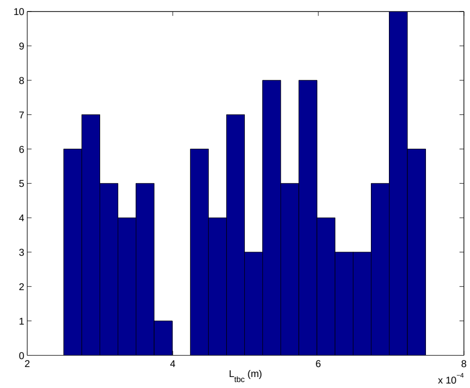
2. Given u, find the value of x at which $u = F(x)$. In otherwords, invert $F(x)$ such that, $x = F^{-1}(u)$.

15.2.1 Triangular Distributions

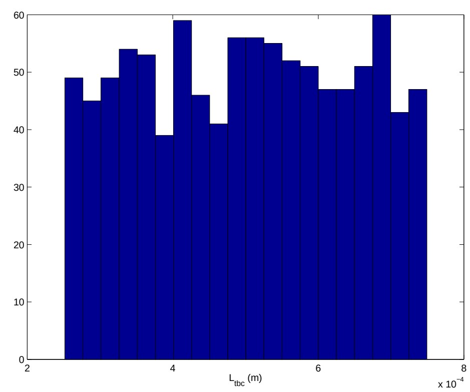
This was discussed in class. Hand-written notes were distributed.

15.2.2 Empirical Distributions

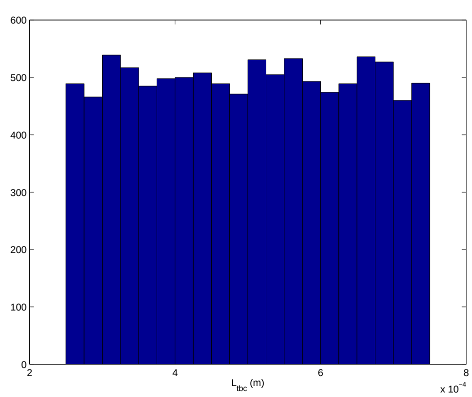
This was discussed in class.



(a) $N = 100$

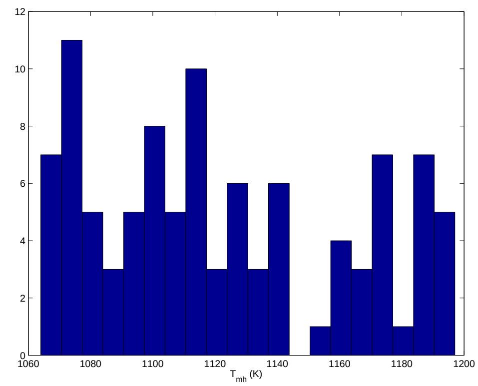


(b) $N = 1000$

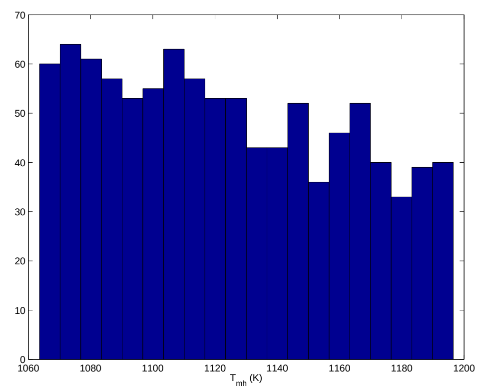


(c) $N = 10000$

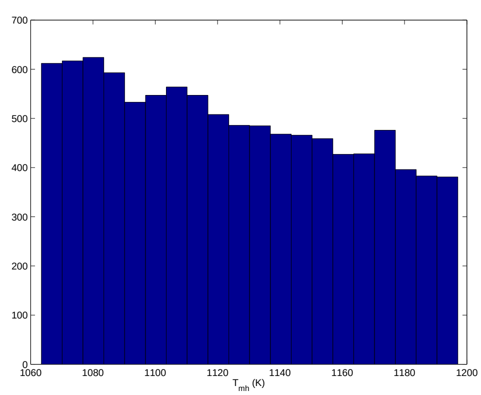
Figure 15.3: Distribution of a random sample from a uniformly-distributed L_{TBC} for a sample size of $N = 100$, 1000, and 10000.



(a) $N = 100$



(b) $N = 1000$



(c) $N = 10000$

Figure 15.4: Distribution of a T_{mh} from a uniformly-distributed L_{TBC} for a sample size of $N = 100$, 1000, and 10000.

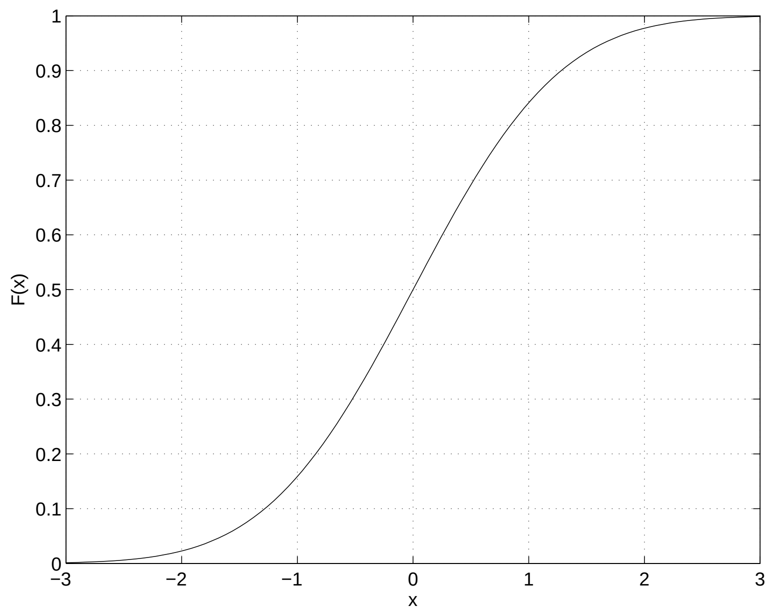


Figure 15.5: Cumulative distribution function (CDF) of a normally distributed variables with zero mean and unit variance.