Lecture 4

Systems of ODE's and Eigenvalue Stability

Until know, we have only addressed the integration of a single ODE. In this lecture, we consider numerical methods for systems of ODE's.

4.1 Nonlinear Systems

For a system of ODE's, we have the same canonical form as for a scalar (see Equation 1.5),

$$u_t = f(u, t), \tag{4.1}$$

except that u and f are vectors of the same length, d:

$$u = [u_1, u_2, u_3, \cdots, u_d]^T \qquad f = [f_1, f_2, f_3, \cdots, f_d]^T$$

Example 4.1 *Nonlinear Pendulum*

One manner in which a system of ODE's occurs is for higher-order ODE's. A classic example of this are second-order oscillators such as a pendulum. The nonlinear dynamics of a pendulum of length L satisfy the following second-order system of equations:

$$\theta_{tt} + \frac{g}{L}\sin\theta = 0. \tag{4.2}$$

To transform this into a system of first-order equations, we define the angular rate, ω,

$$\theta_t = \omega.$$

Then, Equation 4.2 becomes,

$$\omega_t + \frac{g}{L}\sin\theta = 0.$$

For this example,

$$u = \begin{pmatrix} \omega \\ \theta \end{pmatrix} \qquad f = \begin{pmatrix} -\frac{g}{L}\sin\theta \\ \omega \end{pmatrix}$$

A forward Euler method was used to simulate the motion of a pendulum (with $L = 1$ m, $g = 9.8$ m/sec^2) released from rest at an angle of $45°$ at a timestep of $\Delta t = 0.02$ seconds. The results are shown in Figure 4.1. While the oscillatory motion is evident, the amplitude is growing which is not expected physically. This would indicate some kind of numerical stability problem. Note, however, that if a smaller Δt were used, the amplification would still be present but not as significant.

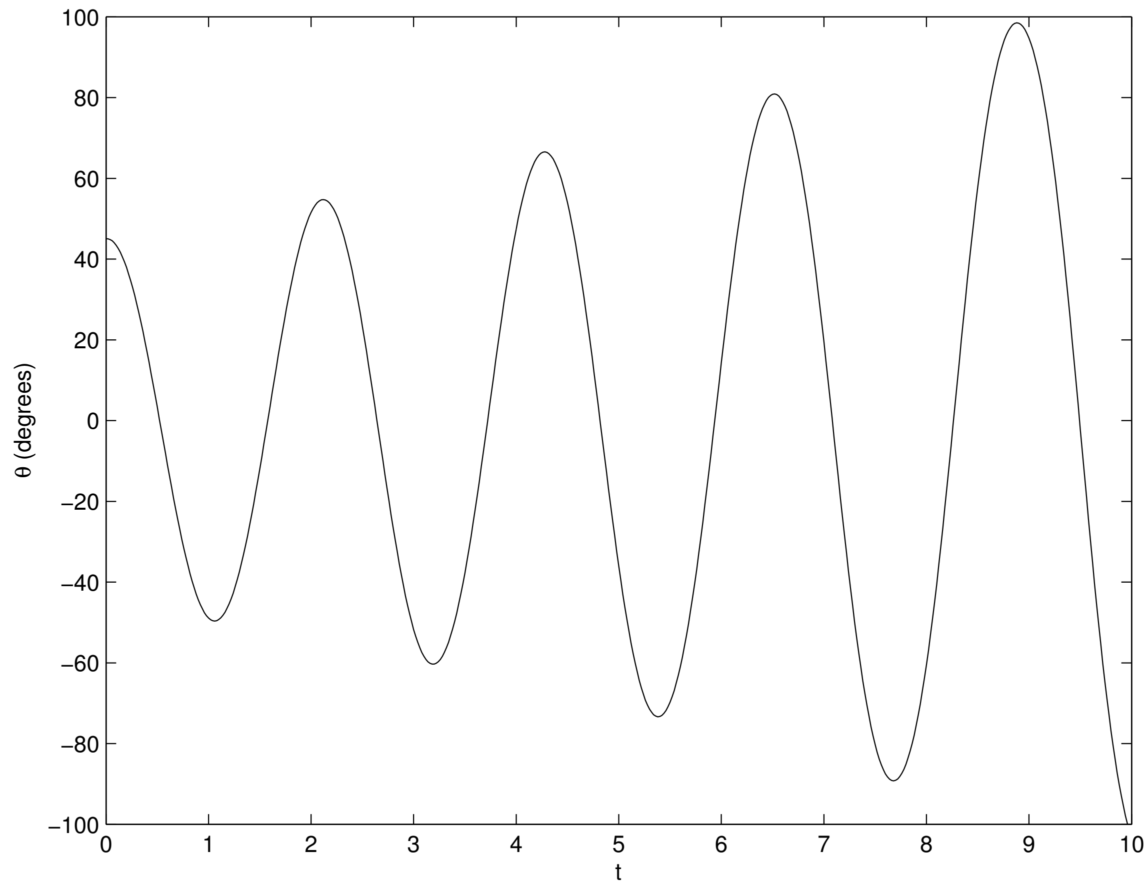

Figure 4.1: Forward Euler solution for nonlinear pendulum with $L = 1$ m, $g = 9.8$ m/sec^2, and $\Delta t = 0.02$ seconds.

The same problem was also simulated using the midpoint method. These results are shown in Figure 4.2. For this method and Δt choice, the oscillation amplitude is constant and indicates that the midpoint method is a better choice for this problem than the forward Euler method.

4.2 Linear Constant Coefficient Systems

The analysis of numerical methods applied to linear, constant coefficient systems can provide significant insight into the behavior of numerical methods for nonlinear problems. Consider

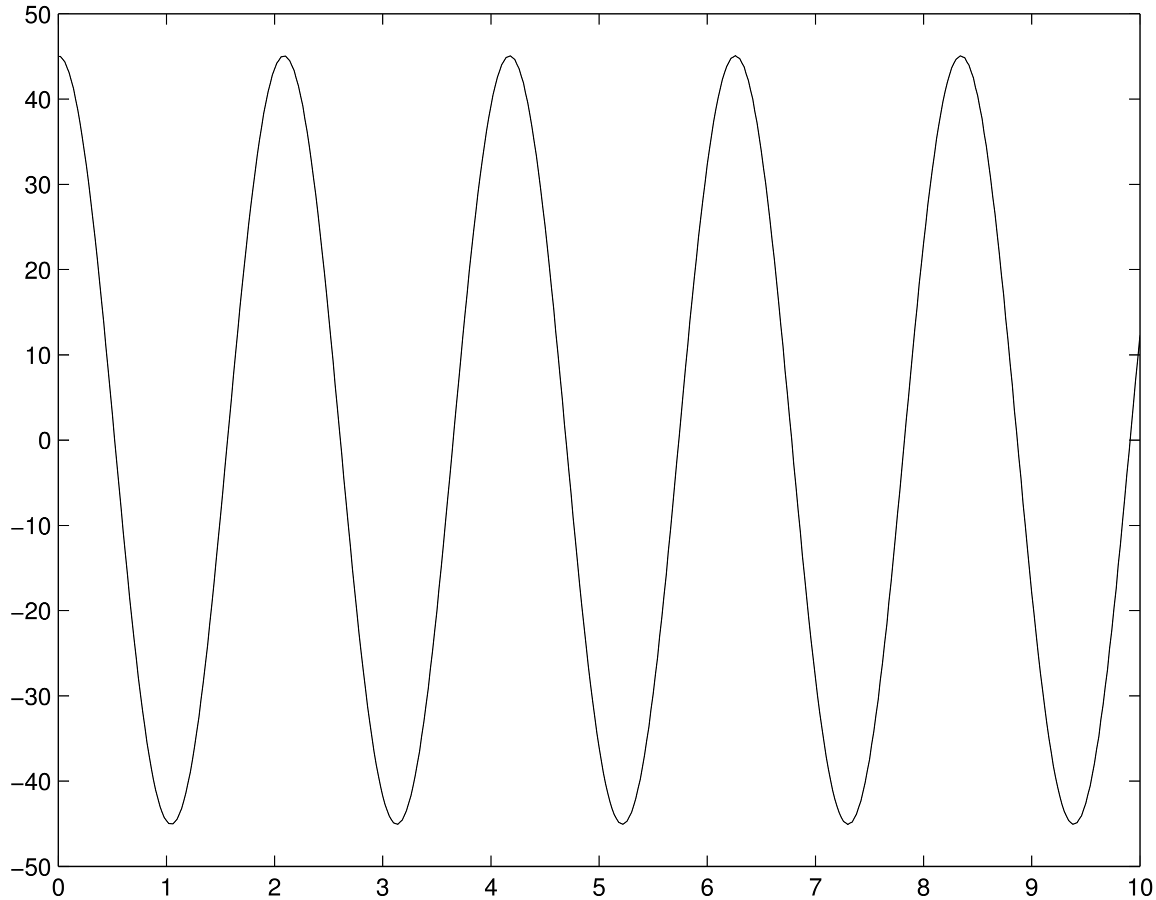

Figure 4.2: Midpoint solution for nonlinear pendulum with $L = 1$ m, $g = 9.8$ m/sec^2, and $\Delta t = 0.02$ seconds.

the following problem,

$$u_t = Au, \tag{4.3}$$

where A is a $d \times d$ matrix. Assuming that a complete set of eigenvectors exists, the matrix A can be decomposed as,

$$A = R\Lambda R^{-1}, \qquad \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_d), \qquad R = \left(r_1 \middle| r_2 \middle| r_3 \middle| r_4 \middle| r_5 \right) \tag{4.4}$$

The solution to Equation 4.3 can be derived as follows,

$$\begin{aligned} u_t &= Au \\ u_t &= R\Lambda R^{-1}u \\ R^{-1}u_t &= \Lambda R^{-1}u \end{aligned}$$

Then, defining $w = R^{-1}u$,

$$w_t = \Lambda w.$$

This system of equations is actually uncoupled from each other, so that each of the eigenmodes has its own independent evolution equation,

$$(w_j)_t = \lambda_j w_j, \qquad \text{for each } i = 1 \text{ to } d$$

Since each of the eigenmodes has a solution $w_j(t) = w_j(0)\exp(\lambda_j t)$, then the solution for $u(t)$ can be written as,

$$u(t) = \sum_{i=1}^{d} w_j(0) r_j e^{\lambda_j t}. \tag{4.5}$$

Note that the eigenvalues are in general complex, $\lambda_j = \lambda_{j_r} + i\lambda_{j_i}$. The imaginary part of the eigenvalues determines the frequency of oscillations, and the real part of the eigenvalues determines the growth or decay rate. Specifically,

$$e^{\lambda t} = e^{(\lambda_r + i\lambda_i)t} = (\cos\lambda_i t + i\sin\lambda_i t)\, e^{\lambda_r t}.$$

Thus, when $\lambda_r > 0$, the solution will grow unbounded as $t \to \infty$.

4.3 Eigenvalue Stability for a Linear ODE

As we have seen, while numerical methods can be convergent, they can still exhibit instabilities as n increases for finite Δt. For example, when applying the midpoint method to either the ice particle problem in Example 1.5 or the simpler model problem in Example 2.2, instabilities were seen in both cases as n increased. Similarly, for the nonlinear pendulum problem in Example 4.1, the forward Euler method had a growing amplitude again indicating an instability. The key to understanding these results is to analyze the stability for finite Δt. This analysis is different than the stability analysis we performed in Section 3.4 since that analysis was for the limit of $\Delta t \to 0$.

Suppose we are interested in solving the linear ODE,

$$u_t = \lambda u.$$

Consider the Forward Euler method applied to this problem,

$$v^{n+1} = v^n + \lambda \Delta t v^n. \tag{4.6}$$

Similar to the zero stability analysis, we will assume that the solution has the following form,

$$v^n = g^n v^0, \tag{4.7}$$

where g is the amplification factor (and the superscript n acting on g is again raising to a power). As in the zero stability analysis, we wish to determine under what conditions $|g| > 1$ since this would mean that v^n will grow unbounded as $n \to \infty$. Substituting Equation 4.7 into Equation 4.6 gives,

$$g^{n+1} = (1 + \lambda\Delta t) g^n.$$

Thus, the only non-zero root of this equation gives,

$$g = 1 + \lambda \Delta t,$$

which is the amplification factor for the forward Euler method. Now, we must determine what values of $\lambda \Delta t$ lead to instability (or stability). A simple way to do this for multi-step methods is to solve for the stability boundary for which $|g| = 1$. To do this, let $g = e^{i\theta}$ (since $|e^{i\theta}| = 1$) where $\theta = [0, 2\pi]$. Making this substitution into the amplification factor,

$$e^{i\theta} = 1 + \lambda \Delta t \quad \Rightarrow \quad \lambda \Delta t = e^{i\theta} - 1.$$

Thus, the stability boundary for the forward Euler method lies on a circle of radius one centered at -1 along the real axis and is shown in Figure 4.3.

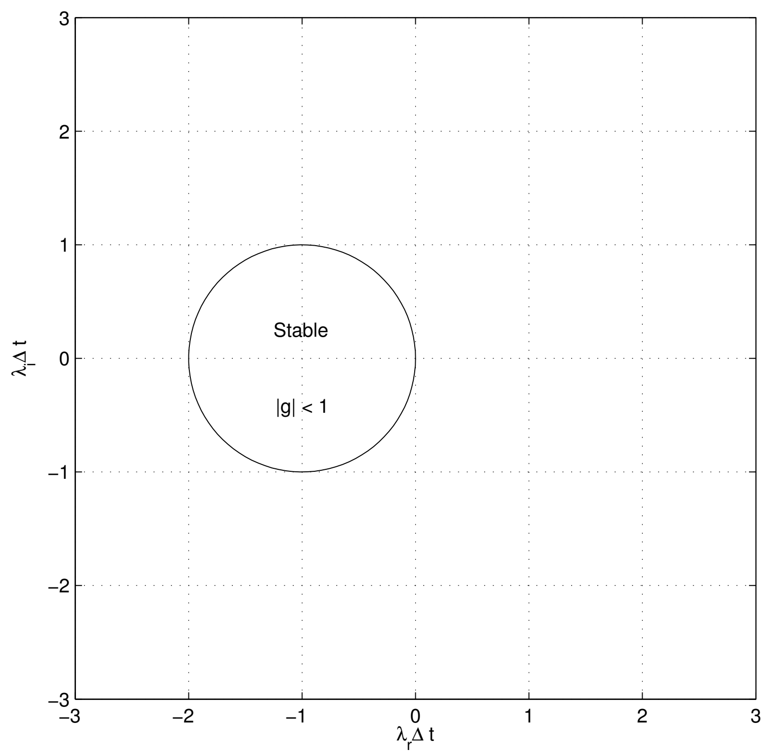

Figure 4.3: Forward Euler stability region

For a given problem, i.e. with a given λ, the timestep must be chosen so that the algorithm remains stable for $n \to \infty$. Let's consider some examples.

Example 4.2 *Let's return to the previous example, $u_t = -u^2$ with $u(0) = 1$. To determine the timestep restrictions, we must estimate the eigenvalue for this problem. As described in Lecture 1, linearizing this problem about a known state gives the eigenvalue as $\lambda = \partial f / \partial u = -2u$. Since the solution will decay from the initial condition (since $u_t < 0$ because $-u^2 < 0$), the largest magnitude of the eigenvalue occurs at the initial condition when $u(0) = 1$ and thus, $\lambda = -2$. Since this eigenvalue is a negative real number, the maximum Δt will occur* *at the maximum extent of the stability region along the negative real axis. Since this occurs when $\lambda\Delta t = -2$, this implies the $\Delta t < 1$. To test the validity of this analysis, the forward Euler method was run for a $\Delta t = 0.9$ and 1.1. The results are shown in Figure 4.4 which are stable for $\Delta t = 0.9$ but are unstable for $\Delta t = 1.1$.*

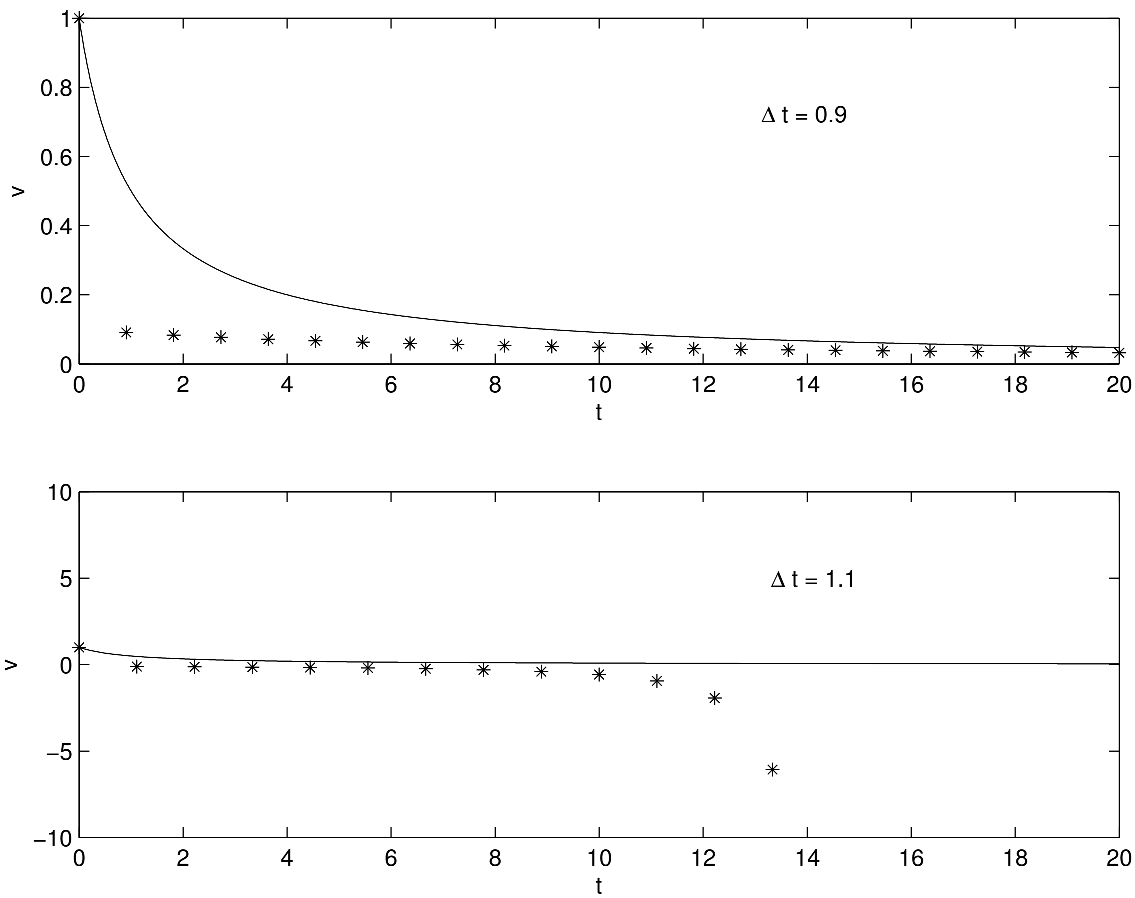

Figure 4.4: Forward Euler solution for $u_t = -u^2$ with $u(0) = 1$ with $\Delta t = 0.9$ and 1.1.

Example 4.3 *Next, let's consider the application of the forward Euler method to the pendulum problem. For this case, the linearization produces a matrix,*

$$\frac{\partial f}{\partial u} = \begin{pmatrix} 0 & -\frac{g}{L}\cos\theta \\ 1 & 0 \end{pmatrix}$$

The eigenvalues can be found from the roots of the determinant of $\partial f/\partial u - \lambda I$:

$$\begin{aligned} \det\left(\frac{\partial f}{\partial u} - \lambda I\right) &= \det\begin{pmatrix} -\lambda & -\frac{g}{L}\cos\theta \\ 1 & -\lambda \end{pmatrix} \\ &= \lambda^2 + \frac{g}{L}\cos\theta = 0 \\ &\Rightarrow \lambda = \pm i\sqrt{\frac{g}{L}\cos\theta} \end{aligned}$$

Thus, we see that the eigenvalues will always be imaginary for this problem. As a result, since the forward Euler stability region does not contain any part of the imaginary axis (except the origin), no finite timestep exists which will be stable. This explains why the amplitude increases for the pendulum simulations in Figure 4.1.

In-class Discussion 4.1 (Midpoint method eigenvalue stability region)