Lecture 9

Matrix Stability Analysis of Finite Difference Methods

In this lecture, we a matrix approach to analyzing the eigenvalue stability of PDE discretizations. This method builds upon our understanding of eigenvalue stability for systems of ODE's.

As we saw in Section 8.1, finite difference (or finite volume) approximations can potentially be written in a semi-discrete form as,

$$\frac{dU}{dt} = AU + b. \tag{9.1}$$

While there are some PDE discretization methods that cannot be written in that form, the majority can be. So, we will take the semi-discrete Equation (9.1) as our starting point. Note: the term semi-discrete is used to signify that the PDE has only been discretized in space.

Let $U(t)$ be the exact solution to the semi-discrete equation. Then, consider perturbation $e(t)$ to the exact solution such that the perturbed solution, $V(t)$, is:

$$V(t) = U(t) + e(t).$$

The questions that we wish to resolve are: (1) can the perturbation $e(t)$ grow in time for the semi-discrete problem, and (2) what the stability limits are on the timestep for a chosen time integration method.

First, we substitute $V(t)$ into Equation (9.1),

$$\begin{aligned}
\frac{dV}{dt} &= AV + b \\
\frac{d(U+e)}{dt} &= A(U+e) + b \\
\frac{de}{dt} &= Ae.
\end{aligned}$$

Thus, the perturbation must satisfy the homogeneous equation, $e_t = Ae$. Having studied the behavior of linear system of equations in Section 4.2, we know that $e(t)$ will grow unbounded as $t \to \infty$ if the real parts of the eigenvalues of A are positive.

The problem is that determining the eigenvalues of A can be non-trivial. In fact, for a general problem finding the eigenvalues of A can be about as hard as solving the specific problem. So, while the matrix stability method is quite general, it can also require a lot of time to perform. Still, the matrix stability method is an indispensible part of the numerical analysis toolkit.

As we saw in the eigenvalue analysis of ODE integration methods, the integration method must be stable for all eigenvalues of the given problem. One manner that we can determine whether the integrator is stable is by plotting the eigenvalues scaled by the timestep in the complex $\lambda \Delta t$ plane and overlaying the stability region for the desired ODE integrator. In fact, we have already plotted the eigenvalues for one-dimensional diffusion using a central difference discretization in Example 5.1. Then, Δt can be adjusted to attempt to bring all eigenvalues into the stability region for the desired ODE integrator.

Example 9.1 (Matrix Stability of FTCS for 1-d convection) *In Example 8.1, we used a forward time, central space (FTCS) discretization for 1-d convection,*

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + u_i^n \delta_{2x} U_i^n = 0. \tag{9.2}$$

Since this method is explicit, the matrix A does not need to be constructed directly, rather Equation (9.2) can be used to find the new values of U at each point i. The Matlab script given in Example 8.1 does exactly that. However, if we are interested in calculating the eigenvalues to analyze the eigenvalue stability, then the A matrix is required. The following script does exactly that (i.e. calculates A, determines the eigenvalues of A, and then plots the eigenvalues scaled by Δt overlayed with the forward Euler stability region). The script can set either the inflow/outflow boundary conditions described in Example 8.1, or can set periodic boundary conditions. We will look at the eigenvalues of both cases.

```
% This Matlab script calculates the eigenvalues of
% the one-dimensional convection equation discretized by
% finite differences.  The discretization uses central
% differences in space and forward Euler in time.
%
% Periodic bcs are set if periodic_flag == 1.
%
% Otherwise, an inflow (dirichlet) bc is set and at
% the outflow a one-sided (backwards) difference is used.
%

clear all;

periodic_flag = 1;

% Set-up grid
xL = -4;
xR =  4;
```

```
Nx = 21; % number of points
x = linspace(xL,xR,Nx);

% Calculate cell size in control volumes (assumed equal)
dx = x(2) - x(1);

% Set velocity
u = 1;

% Set timestep
CFL = 1;
dt = CFL*dx/abs(u);

% Set bc state at left (assumes u>0)
UL = exp(-xL^2);

% Allocate matrix to hold stiffness matrix (A).
%
A = zeros(Nx-1,Nx-1);

% Construct A except for first and last row
for i = 2:Nx-2,
  A(i,i-1) =  u/(2*dx);
  A(i,i+1) = -u/(2*dx);
end

if (periodic_flag == 1), % Periodic bcs

  A(1   ,2   ) = -u/(2*dx);
  A(1   ,Nx-1) =  u/(2*dx);
  A(Nx-1,1   ) = -u/(2*dx);
  A(Nx-1,Nx-2) =  u/(2*dx);

else % non-periodic bc's

  % At the first interior node, the i-1 value is known (UL).
  % So, only the i+1 location needs to be set in A.
  A(1,2) = -u/(2*dx);

  % Outflow boundary uses backward difference
  A(Nx-1,Nx-2) =  u/dx;
  A(Nx-1,Nx-1) = -u/dx;

end
```

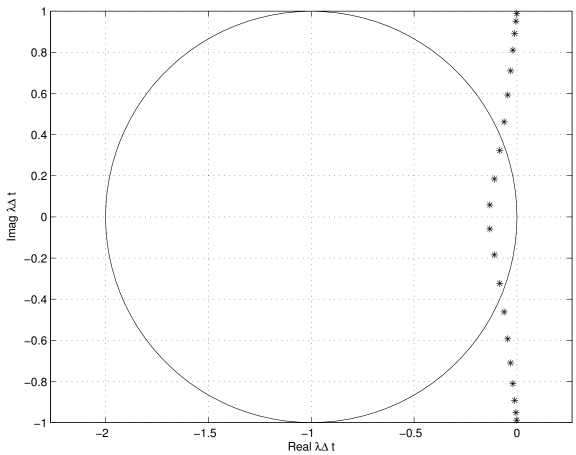
```
% Calculate eigenvalues of A
lambda = eig(A);

% Plot lambda*dt
plot(lambda*dt,'*');
xlabel('Real \lambda\Delta t');
ylabel('Imag \lambda\Delta t');

% Overlay Forward Euler stability region
th = linspace(0,2*pi,101);
hold on;
plot(-1 + sin(th),cos(th));
hold off;
axis('equal');
grid on;
```

Figure 9.1 shows a plot of $\lambda\Delta t$ for a CFL set to one. Recall that for this one-dimensional problem, the CFL number was defined as,

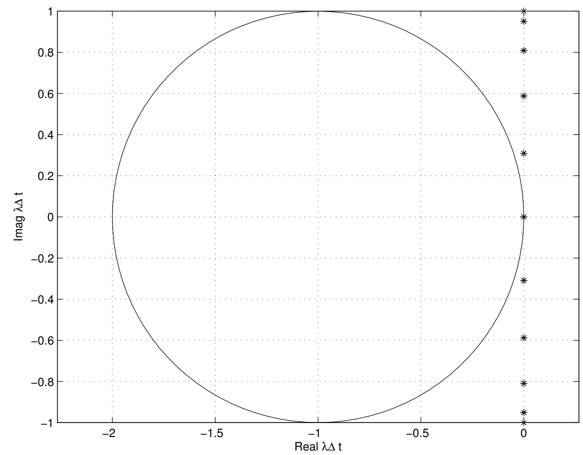
$$CFL = \frac{|u|\Delta t}{\Delta x}.$$

In the inflow/outflow boundary condition case (shown in Figure 9.1 the eigenvalues lay slightly inside the negative real half-plane. As they move away from the origin, they approach the imaginary axis at $\pm i$. The periodic boundary conditions give purely imaginary eigenvalues but these also approach $\pm i$ as the move away from the origin. Note that the periodic boundary conditions actually give a zero eigenvalue so that the matrix A is actually singular (Why is this?). Regardless what we see is that for a $CFL = 1$, some $\lambda\Delta t$ exist which are outside of the forward Euler stability region. We could try to lower the timestep to bring all of the $\lambda\Delta t$ into the stability region, however that will prove to be practically impossible since the extreme eigenvalues approach $\pm i$ (i.e. they are purely imaginary). Thus, no finite value of Δt exists for which these eigenvalues can be brought inside the circular stability region of the forward Euler method (i.e. the FTCS is unstable for convection).

In-class Discussion 9.1 (Behavior of FTCS eigenvalues with decreased Δx) *We will discuss Figure 9.2.*



(a) Dirichlet inflow and upwinded outflow



(b) Periodic boundary conditions

Figure 9.1: $\lambda\Delta t$ distribution for one-dimensional convection example using two different boundary conditions. Note: Δt set such that CFL $= 1$.

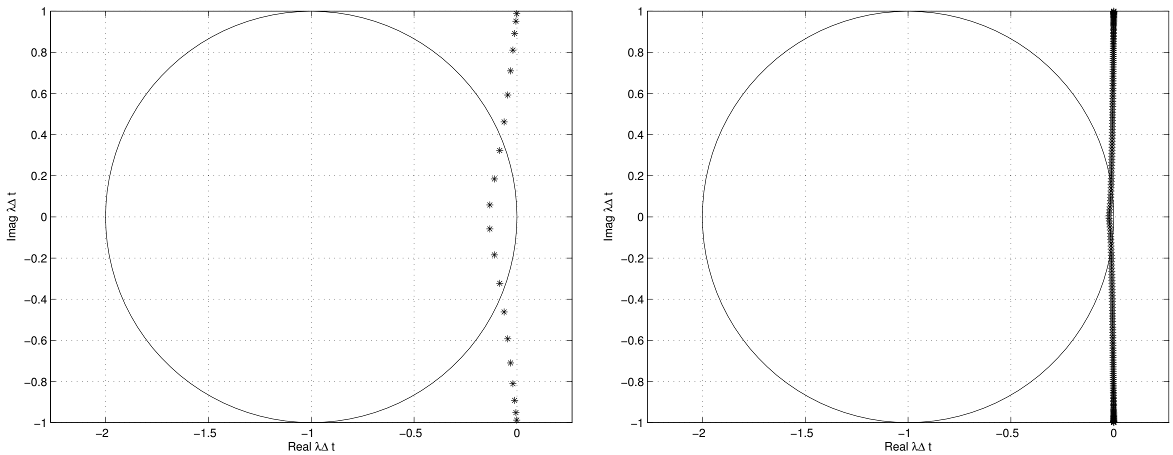
In-class Discussion 9.2 (Behavior of FTCS eigenvalues with diffusion) *We will discuss Figure 9.3.*

Though the eigenvalues of A typically require numerical techniques for the general problem, a special case of practical interest occurs when the matrix is 'periodic'. That is, the column entries shift a column every row. Thus, the matrix has the form,

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_N \\ a_N & a_1 & a_2 & \dots & a_{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \dots & a_1 \end{pmatrix}$$

This type of matrix is known as a circulant matrix. Circulant matrices have eigenvalues given by,

$$\lambda_n = \sum_{j=1}^{N} a_j e^{i\,2\pi(j-1)\frac{n}{N}} \qquad \text{for} \quad n = 0, 1, \dots, N-1 \tag{9.3}$$



(a) $\Delta x = 0.4$ (b) $\Delta x = 0.04$

Figure 9.2: Effect of Δx on $\lambda \Delta t$ distribution for one-dimensional convection example using Dirichlet inflow and upwinded outflow conditions. Note: Δt set such that CFL = 1.

Example 9.2 *As we saw in Example 9.1, when periodic boundary conditions are assumed, the central space discretization of one-dimensional convection gives purely imaginary eigenvalues, and when scaled by a timestep for which the CFL number is one, the eigenvalues stretch along the axis until $\pm i$. Since for a convection problem with constant velocity and periodic boundary conditions gives a circulant matrix, we can use Equation (9.3) to determine the eigenvalues analytically. We begin by finding the coefficients, a_j. For a central space discretization, we find,*

$$a_2 = -\frac{u}{2\Delta x}, \quad a_N = \frac{u}{2\Delta x}, \text{ and for all other } j, \quad a_j = 0.$$

Then, substituting these a_j into Equation (9.3) gives,

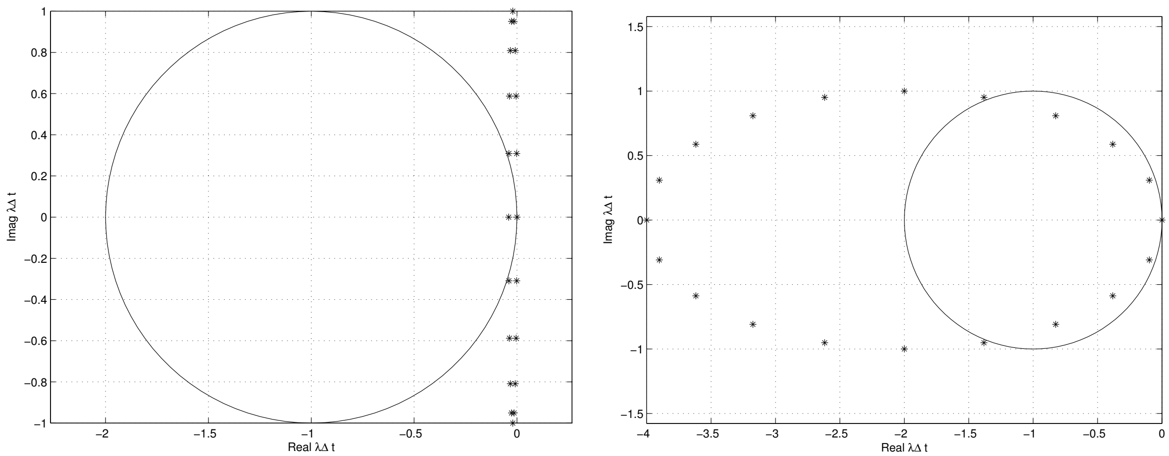
$$\begin{aligned}
\lambda_n &= -\frac{u}{2\Delta x} e^{i\,2\pi \frac{n}{N}} + \frac{u}{2\Delta x} e^{i\,2\pi(N-1)\frac{n}{N}}, \\
&= -\frac{u}{2\Delta x} e^{i\,2\pi \frac{n}{N}} + \frac{u}{2\Delta x} e^{i\,2\pi n} e^{-i\,2\pi \frac{n}{N}}.
\end{aligned}$$

Since $e^{i\,2\pi n} = 1$ (because n is an integer), then,

$$\begin{aligned}
\lambda_n &= -\frac{u}{2\Delta x} e^{i\,2\pi \frac{n}{N}} + \frac{u}{2\Delta x} e^{-i\,2\pi \frac{n}{N}}, \\
&= -\frac{u}{2\Delta x}\left(e^{i\,2\pi \frac{n}{N}} - e^{-i\,2\pi \frac{n}{N}}\right), \\
&= -i\frac{u}{\Delta x}\sin\left(2\pi \frac{n}{N}\right).
\end{aligned}$$

Multiplying by the timestep,

$$\lambda_n \Delta t = -i\frac{u\Delta t}{\Delta x}\sin\left(2\pi \frac{n}{N}\right).$$



(a) $u\Delta x/\mu = 100$ (b) $u\Delta x/\mu = 1$

Figure 9.3: Effect of viscosity on $\lambda\Delta t$ distribution for one-dimensional convection-diffusion example using periodic boundary conditions.Note: Δt set such that $\text{CFL} = 1$.

As observed in Example 9.1, the eigenvalues are purely imaginary and will extend to $\pm i$ *when* $CFL = |u|\Delta t/\Delta x = 1$.