**SMA 5212 - Numerical methods for Partial Differential Equations.**

Massachusetts Institute of Technology — Singapore - MIT Alliance

**3nd Problem Set - Variational Methods**

Solution

# Design of a thermal fin

**Problem Statement**

The problem of designing a thermal fin to effectively remove heat from a surface has been considered. The two-dimensional fin, shown in Figure 1 is characterized by a five–component parameter vector, or "input," $\underline{\mu} = (\mu^1, \mu^2, \ldots, \mu^5)$, where $\mu^i = k^i,\ i = 1, \ldots, 4$, and $\mu^5 = \text{Bi}$; $\underline{\mu}$ may take on any value in a specified design set $\mathcal{D} \subset \mathbb{R}^5$.

Here $k^i$ is the thermal conductivity of the $i^{th}$ subfin (normalized relative to the post conductivity $k^0 \equiv 1$); and Bi is the Biot number, a nondimensional heat transfer coefficient reflecting convective transport to the air at the fin surfaces. The post is of width unity and height four; the subfins are of fixed thickness $t = 0.25$ and length $L = 2.5$.

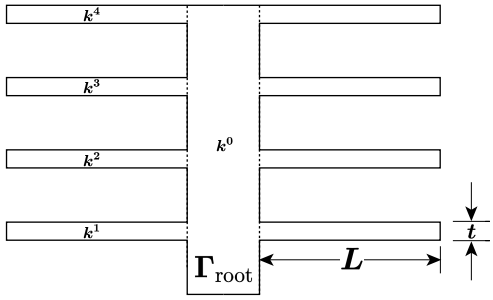


Figure 1: Thermal Fin

The steady–state temperature distribution within the fin, $u(\underline{\mu})$, is governed by the elliptic partial differential equation

$$-k^i \nabla^2 u^i = 0 \text{ in } \Omega^i,\ i = 0, \ldots, 4; \qquad (1)$$

$u^i$ refers to the restriction of $u$ to $\Omega^i$. We must also ensure continuity of temperature and heat flux at the conductivity–discontinuity interfaces $\Gamma^i_{\text{int}} \equiv \partial\Omega^0 \cap \partial\Omega^i,\ i = 1, \ldots, 4$, where $\partial\Omega^i$ denotes the boundary of $\Omega^i$:

$$\left.\begin{array}{rcl} u^0 & = & u^i \\ -(\nabla u^0 \cdot \hat{\mathbf{n}}^i) & = & -k^i(\nabla u^i \cdot \hat{\mathbf{n}}^i) \end{array}\right\} \text{ on } \Gamma^i_{\text{int}},\ i = 1, \ldots, 4; \qquad (2)$$

here $\hat{\mathbf{n}}^i$ is the outward normal on $\partial\Omega^i$. Finally, we introduce a Neumann flux boundary condition on the fin root

$$-(\nabla u^0 \cdot \hat{\mathbf{n}}^0) = -1 \text{ on } \Gamma_{\text{root}}, \qquad (3)$$

which models the heat source; and a Robin boundary condition

$$-k^i(\nabla u^i \cdot \hat{\mathbf{n}}^i) = \text{Bi}\ u^i \text{ on } \Gamma^i_{\text{ext}},\ i = 0, \ldots, 4, \qquad (4)$$

which models the convective heat losses. Here $\Gamma^i_{\text{ext}}$ is that part of the boundary of $\Omega^i$ exposed to the flowing fluid; note that $\cup_{i=0}^4 \Gamma^i_{\text{ext}} = \Gamma \backslash \Gamma_{\text{root}}$.

The output considered was $T_{\text{root}}(\underline{\mu})$, the average steady-state temperature of the fin root normalized by the prescribed heat flux into the fin root.

$$T_{\text{root}}(\underline{\mu}) \equiv \ell^O(v) = \int_{\Gamma_{\text{root}}} v. \qquad (5)$$

**Part 1 - Finite Element Approximation**

$\alpha$) In the statement of the problem the strong form of the equations was described. To obtain the temperature distribution $u(\underline{\mu})$ inside the thermal fin, we need to solve (1), with the boundary conditions given in (2),(3),(4). We want to verify that the solution of this problem will satisfy the weak formulation which is the starting point for the finite element method. In this question we assume that $u(\underline{\mu})$ is sufficiently smooth, so that integration by parts and relatedly Gauss and Green's theorems can be applied.

The weak form of the problem is to find $u(\underline{\mu}) \in X \equiv H^1(\Omega)$ which satisfies

$$a(u(\underline{\mu}), v; \underline{\mu}) = \ell(v), \ \forall v \in X, \tag{6}$$

where

$$\begin{aligned} a(w, v; \underline{\mu}) &= \sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla w \cdot \nabla v \, dA + \mathrm{Bi} \int_{\Gamma \setminus \Gamma_{\mathrm{root}}} wv \, dS, \\ \ell(v) &= \int_{\Gamma_{\mathrm{root}}} v \, dS. \end{aligned}$$

We want to show that

$$\underbrace{\sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla u^i(\underline{\mu}) \cdot \nabla v \, dA}_{I_1} = -\mathrm{Bi} \int_{\Gamma \setminus \Gamma_{\mathrm{root}}} u^i(\underline{\mu}) \, v \, dS + \int_{\Gamma_{\mathrm{root}}} v \, dS, \ \forall v \in X. \tag{7}$$

We start from $I_1$ and apply Gauss theorem

$$\begin{aligned} I_1 &= \sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla u^i(\underline{\mu}) \cdot \nabla v \, dA \\ &= \sum_{i=0}^{4} k^i \left( \int_{\Omega^i} \nabla \cdot (v \nabla u^i(\underline{\mu})) \, dA - \int_{\Omega^i} v \, \nabla^2 u^i(\underline{\mu}) \, dA \right). \end{aligned}$$

The second term in the equation above vanishes, because from (1), $-k^i \, \nabla^2 u^i = 0$ in $\Omega^i, \ i = 0, \ldots, 4$. For the first term we apply Green's theorem and

$$I_1 = \sum_{i=0}^{4} k^i \int_{\partial \Omega^i} v (\nabla u^i(\underline{\mu}) \cdot \hat{\mathbf{n}}^i) \, dS.$$

Since for each domain $\Omega^i, i = 1, \ldots, 4$ the boundary can be decomposed as $\partial \Omega^i = \Gamma^i_{\mathrm{int}} \cup \Gamma^i_{\mathrm{ext}}, i = 1, \ldots, 4$, and for domain $\Omega^0$, $\partial \Omega^0 = \Gamma^0_{\mathrm{int}} \cup \Gamma^0_{\mathrm{ext}} \cup \Gamma_{\mathrm{root}}$

$$I_1 = \sum_{i=0}^{4} k^i \int_{\Gamma^i_{\mathrm{ext}}} v (\nabla u^i(\underline{\mu}) \cdot \hat{\mathbf{n}}^i) \, dS + \underbrace{\sum_{i=0}^{4} k^i \int_{\Gamma^i_{\mathrm{int}}} v (\nabla u^i(\underline{\mu}) \cdot \hat{\mathbf{n}}^i) \, dS}_{I_2} + \int_{\Gamma_{\mathrm{root}}} v (\nabla u^0(\underline{\mu}) \cdot \hat{\mathbf{n}}^i) \, dS;$$

for the last integral $k^0 = 1$ was used. We next show that the term $I_2$ is equal to zero.

$$\begin{aligned} I_2 &= \sum_{i=1}^{4} k^i \int_{\Gamma^i_{\mathrm{int}}} v (\nabla u^i(\underline{\mu}) \cdot \hat{\mathbf{n}}^i) \, dS + \underbrace{\int_{\Gamma^0_{\mathrm{int}}} v (\nabla u^0(\underline{\mu}) \cdot \hat{\mathbf{n}}^0) \, dS}_{I_3} \\ &= \sum_{i=1}^{4} \int_{\Gamma^i_{\mathrm{int}}} v \left[ k^i (\nabla u^i(\underline{\mu}) \cdot \hat{\mathbf{n}}^i) - (\nabla u^0(\underline{\mu}) \cdot \hat{\mathbf{n}}^i) \right] \, dS = 0 \end{aligned}$$

and vanishes from the continuity of heat flux boundary condition (2). Note that for $I_3$ we have used the following expression

$$I_3 = \int_{\Gamma^0_{\text{int}}} v(\nabla u^0(\underline{\mu}) \cdot \hat{\mathbf{n}}^0)\, dS = \sum_{i=1}^{4} \int_{\Gamma^i_{\text{int}}} v(\nabla u^0(\underline{\mu}) \cdot \hat{\mathbf{n}}^0)\, dS = -\sum_{i=1}^{4} \int_{\Gamma^i_{\text{int}}} v(\nabla u^0(\underline{\mu}) \cdot \hat{\mathbf{n}}^i)\, dS;$$

since $\Gamma^0_{\text{int}} = \cup_{i=1}^{4} \Gamma^i_{\text{int}}$, and $\hat{\mathbf{n}}^0 = -\hat{\mathbf{n}}^i$ on $\Gamma^i_{\text{int}},\ i = 1, \ldots, 4$. Therefore,

$$I_1 \quad = \quad \sum_{i=0}^{4} k^i \int_{\Gamma^i_{\text{ext}}} v(\nabla u^i(\underline{\mu}) \cdot \hat{\mathbf{n}}^i)\, dS + \int_{\Gamma_{\text{root}}} v(\nabla u^0(\underline{\mu}) \cdot \hat{\mathbf{n}}^i)\, dS.$$

Now we apply the Neumann and Robin boundary conditions (3) and (4), to obtain

$$I_1 = -\sum_{i=0}^{4} \text{Bi} \int_{\Gamma^i_{\text{ext}}} u^i(\underline{\mu})\, v\, dS + \int_{\Gamma_{\text{root}}} v\, dS$$

since $\cup_{i=0}^{4} \Gamma^i_{\text{ext}} = \Gamma \backslash \Gamma_{\text{root}}$, $I_1$ becomes

$$I_1 = -\text{Bi} \int_{\Gamma \backslash \Gamma_{\text{root}}} u(\underline{\mu})\, v\, dS + \int_{\Gamma_{\text{root}}} v\, dS$$

which is exactly expression (7), that we wanted to prove.

$\beta$) We want to show that

$$u(\underline{\mu}) = \arg \min_{w \in X} J(w),$$

with

$$J(w) = \frac{1}{2} \sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla w \cdot \nabla w\, dA + \frac{\text{Bi}}{2} \int_{\Gamma \backslash \Gamma_{\text{root}}} w^2\, dS - \int_{\Gamma_{\text{root}}} w\, dS \tag{8}$$

Let $w = u(\underline{\mu}) + v$, since $X \equiv H^1(\Omega)$ (more precisely, $X = H^1(\Omega) \cap C^0(\Omega)$) is a linear space, then if $v$ and $u(\underline{\mu}) \in X$, $w$ will also be a member of $X$. Starting from (7),

$$\begin{aligned}
J(u(\underline{\mu}) + v) \quad &= \quad \frac{1}{2} \sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla(u(\underline{\mu}) + v) \cdot \nabla(u(\underline{\mu}) + v)\, dA + \frac{\text{Bi}}{2} \int_{\Gamma \backslash \Gamma_{\text{root}}} (u(\underline{\mu}) + v)^2\, dS - \int_{\Gamma_{\text{root}}} (u(\underline{\mu}) + v)\, dS \\
&= \quad \frac{1}{2} \sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla u(\underline{\mu}) \cdot \nabla u(\underline{\mu})\, dA + \frac{\text{Bi}}{2} \int_{\Gamma \backslash \Gamma_{\text{root}}} (u(\underline{\mu}))^2\, dS - \int_{\Gamma_{\text{root}}} u(\underline{\mu})\, dS \\
&\quad + \sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla u(\underline{\mu}) \cdot \nabla v\, dA + \text{Bi} \int_{\Gamma \backslash \Gamma_{\text{root}}} u(\underline{\mu})\, v\, dS - \int_{\Gamma_{\text{root}}} v\, dS \\
&\quad + \sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla v \cdot \nabla v\, dA + \text{Bi} \int_{\Gamma \backslash \Gamma_{\text{root}}} v\, dS \\
&= \quad J(u(\underline{\mu})) + \delta J_v(u(\underline{\mu})) + a(v, v; \underline{\mu}).
\end{aligned}$$

The first variation

$$\delta J_v(u(\underline{\mu})) = \sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla u(\underline{\mu}) \cdot \nabla v\, dA + \text{Bi} \int_{\Gamma \backslash \Gamma_{\text{root}}} u(\underline{\mu})\, v\, dS - \int_{\Gamma_{\text{root}}} v\, dS \equiv 0,\ \forall v \in X$$

as was shown in question $\alpha$. Therefore

$$J(u(\underline{\mu}) + v) = J(u(\underline{\mu})) + a(v, v; \underline{\mu}),\ \forall v \in X. \tag{9}$$

The last part of the proof is to prove that $a(v, v; \underline{\mu})$ is a symmetric positive definite bilinear form.

- Symmetry

$$\begin{aligned} a(w,v;\underline{\mu}) &= \sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla w \cdot \nabla v \, dA + \mathrm{Bi} \int_{\Gamma\setminus\Gamma_{\mathrm{root}}} wv \, dS \\ &= \sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla v \cdot \nabla w \, dA + \mathrm{Bi} \int_{\Gamma\setminus\Gamma_{\mathrm{root}}} vw \, dS \\ &= a(v,w;\underline{\mu}), \forall w, v \in X. \end{aligned}$$

- Positive Definiteness. We will prove that $a(w,w;\underline{\mu}) > 0,\ \forall w \in X,\ w \neq 0$. To start we consider only $a_1(w,v;\underline{\mu}) = \sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla w \cdot \nabla v \, dA$, the first part of $a(w,v;\underline{\mu})$. This is the pure Neumann problem, and as was discussed in the lecture notes has a non-trivial nullspace; any constant function $c$, makes $a_1(c,v;\underline{\mu}) = 0$. For all other functions $w \in X$, $a_1(w,v;\underline{\mu}) > 0$, therefore $\forall w \in X,\ w \neq c$, $a(w,v;\underline{\mu})$ will be positive, since the second term in $a(u,v;\underline{\mu})$ is non-negative. Finally if $w = c$,

$$\begin{aligned} a(c,c;\underline{\mu}) &= \underbrace{\sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla c \cdot \nabla c \, dA}_{=0} + \mathrm{Bi} \int_{\Gamma\setminus\Gamma_{\mathrm{root}}} cc \, dS \\ &= \mathrm{Bi}\, c^2 s; \end{aligned}$$

  with $s$ the perimeter of the Robin boundary. Since $s$ and Bi are positive,

$$a(c,c;\underline{\mu}) = \mathrm{Bi}\, c^2 s \geq 0 \rightarrow c^2 \geq 0$$

  so $a(c,c;\underline{\mu}) = 0 \leftrightarrow c = 0$. Therefore, we have proved that only $c = 0$ will make $a(c,c;\underline{\mu}) = 0$. Summarizing $a(w,w;\underline{\mu}) > 0,\ \forall w \in X,\ w \neq 0$, and we have proved positive definiteness. A corollary of this proof, is that $a(w,v;\underline{\mu})$ induces a norm, which is the energy norm $|||v||| \equiv a(v,v;\underline{\mu})^{1/2}$.

So from (9), we can conclude that

$$J(u(\underline{\mu}) + v) \geq J(u(\underline{\mu})),\ \forall v \in X, v \neq 0; \tag{10}$$

the minimization principle has been verified.

$\gamma$) To obtain the discrete approximation to the continuous problem, we will use the linear finite element space

$$X_h = \{v \in H^1(\Omega) |\ v|_{T_h} \in \mathbb{P}^1(T_h),\ \forall T_h \in \mathcal{T}_h\}.$$

By applying standard Galerkin projection to $X_h$, the discrete problem becomes, find $u_h(\underline{\mu}) \in X_h$ such that

$$a(u_h(\underline{\mu}), v; \underline{\mu}) = \ell(v),\ \forall v \in X_h; \tag{11}$$

the output of interest can be calculated similarly

$$T_{\mathrm{root}\,h}(\underline{\mu}) = \ell^O(u_h(\underline{\mu})). \tag{12}$$

To derive the elemental matrices we follow the same procedure presented in the lecture notes.

- Term $\int_{T_h^k} \nabla u \cdot \nabla v \, dA$.

  We will derive the expression for linear elements. We consider an element $T_h^k$ with three local nodes $\mathbf{x}_1^k = \{x_1^k, y_1^k\},\ \mathbf{x}_2^k = \{x_2^k, y_2^k\},\ \mathbf{x}_3^k = \{x_3^k, y_3^k\}$. Also let $\mathcal{H}_1^k,\ \mathcal{H}_2^k,\ \mathcal{H}_3^k$. be the restriction of the nodal functions that have support over the element $T_h^k$. These functions, from the definition of the finite element space are linear and satisfy $\mathcal{H}_\alpha^k(x_\beta^k) = \delta_{\alpha\,\beta},\ \alpha, \beta = 1, \ldots, 3$. Since $\mathcal{H}_\alpha^k$ are linear the following

expression can be used $\mathcal{H}^k_\alpha = c_\alpha + c_{x\,\alpha}x + c_{y\,\alpha}y,\ \alpha = 1,\ldots,3$. To obtain those coefficients we need to solve the following systems

$$\begin{pmatrix} 1 & x^k_1 & y^k_1 \\ 1 & x^k_2 & y^k_2 \\ 1 & x^k_3 & y^k_3 \end{pmatrix} \begin{pmatrix} c_\alpha \\ c_{x\,\alpha} \\ c_{y\,\alpha} \end{pmatrix} = \overbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}^{\alpha=1} \text{ or } \overbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}^{\alpha=2} \text{ or } \overbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}^{\alpha=3} \tag{13}$$

The elemental matrix can readily be calculated since,

$$A^{k\,1}_{\alpha\,\beta} = \int_{T^k_h} \frac{\partial \mathcal{H}^k_\alpha}{\partial x}\frac{\partial \mathcal{H}^k_\beta}{\partial x} + \frac{\partial \mathcal{H}^k_\alpha}{\partial y}\frac{\partial \mathcal{H}^k_\beta}{\partial y}\, dA = (Area)^k (c_{x\,\alpha}\, c_{x\,\beta} + c_{y\,\alpha}\, c_{y\,\beta}),\ \alpha,\ \beta = 1,\ldots,3.$$

Since the derivatives of the nodal functions are constant over each element $T^k_h$. An explicit form in terms of the coordinates follows

$$\begin{aligned}
\underline{B}^k_1 &= \begin{pmatrix} (y^k_2 - y^k_3)(y^k_2 - y^k_3) & (y^k_2 - y^k_3)(y^k_3 - y^k_1) & (y^k_2 - y^k_3)(y^k_1 - y^k_2) \\ (y^k_3 - y^k_1)(y^k_2 - y^k_3) & (y^k_3 - y^k_1)(y^k_3 - y^k_1) & (y^k_3 - y^k_1)(y^k_1 - y^k_2) \\ (y^k_1 - y^k_2)(y^k_2 - y^k_3) & (y^k_1 - y^k_2)(y^k_3 - y^k_1) & (y^k_1 - y^k_2)(y^k_1 - y^k_2) \end{pmatrix} \\
\underline{B}^k_2 &= \begin{pmatrix} (x^k_2 - x^k_3)(x^k_2 - x^k_3) & (x^k_2 - x^k_3)(x^k_3 - x^k_1) & (x^k_2 - x^k_3)(x^k_1 - x^k_2) \\ (x^k_3 - x^k_1)(x^k_2 - x^k_3) & (x^k_3 - x^k_1)(x^k_3 - x^k_1) & (x^k_3 - x^k_1)(x^k_1 - x^k_2) \\ (x^k_1 - x^k_2)(x^k_2 - x^k_3) & (x^k_1 - x^k_2)(x^k_3 - x^k_1) & (x^k_1 - x^k_2)(x^k_1 - x^k_2) \end{pmatrix} \\
\underline{A}^{k\,1} &= \frac{1}{2|-x^k_2 * y^k_1 + x^k_3 * y^k_1 + x^k_1 * y^k_2 - x^k_3 * y^k_2 - x^k_1 * y^k_3 + x^k_2 * y^k_3|}\left(\underline{B}^k_1 + \underline{B}^k_2\right)
\end{aligned} \tag{14}$$

Note that for the area of the element $T^k_h$ the formula

$$(\text{Area})^k = \frac{1}{2}\left|\begin{vmatrix} 1 & x^k_1 & y^k_1 \\ 1 & x^k_2 & y^k_2 \\ 1 & x^k_3 & y^k_3 \end{vmatrix}\right|$$

was used.

- Term $\displaystyle\int_{\Gamma\backslash\Gamma_{\text{root}}} u\, v\, dS$.

  This is an integral over the Robin boundary $\Gamma\backslash\Gamma_{\text{root}}$, and can be identified as the 1-d mass matrix. To evaluate the elemental matrix, we only need to look at a boundary segment $S^k_h$, with two local nodes $\mathbf{x}^k_1 = \{x^k_1, y^k_1\},\ \mathbf{x}^k_2 = \{x^k_2, y^k_2\}$. Note that $k$ is now an index over all the boundary segments, which are considered as 1-d elements. To evaluate this integral, we will map $S^k_h$ to a reference element $\hat{S} = (-1, 1)$; let $\eta$ be the spatial variable in the reference domain. The elemental matrix can then be calculated from

$$A^{k\,2}_{\alpha\,\beta} = \int_{\hat{S}} \mathcal{H}_\alpha\, \mathcal{H}_\beta\, \left(\frac{h^k}{2} d\eta\right),\ \alpha,\ \beta = 1, 2.$$

  $H_\alpha,\ \alpha = 1, 2$ are the Langrangian interpolants,

$$H_1 = \frac{1-\eta}{2},\ H_2 = \frac{1+\eta}{2},$$

  and the term $\frac{h^k}{2}$ is the Jacobian of the transformation (for more details look in the lecture notes.) Here, $h^k$ is the length of segment $S^k_h$ and can be calculated from

$$h^k = \sqrt{(x^k_1 - x^k_2)^2 + (y^k_1 - y^k_2)^2}.$$

  Using the above, and calculating the integrals the elemental matrix becomes

$$\underline{A}^{k\,2} = \frac{h^k}{3}\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}. \tag{15}$$

- Term $\int_{\Gamma_{\rm root}} v\, dS$.
  This term appears both in the calculation of the load vector $\underline{F}_h^k$, and the output vector $\underline{L}_h^k$. This is an integral over the fin root, $\Gamma_{\rm root}$, and the analysis is presented in the lecture notes. The elemental load vector can be calculated from

$$F_{h\,\alpha}^k = L_{h\,\alpha}^k = \frac{h^k}{2}\int_{-1}^{1} \mathcal{H}_\alpha\, d\eta,\ \alpha = 1, 2.$$

  By doing the integration,

$$F_h^k = L_h^k = \frac{h^k}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{16}$$

To form $\underline{A}_h(\underline{\mu})$, we then use the following algorithm:

```
A_h := 0
for all domains Ω^i, i = 0, ..., 4 do
  for all elements k ∈ Ω^i do
    Calculate the elemental matrix A^{k 1}
    for α = 1, ..., 3 do
      n_1 = θ^i(k, α)
      for β = 1, ..., 3 do
        n_2 = θ^i(k, β)
        A_{h n_1 n_2} = A_{h n_1 n_2} + k^i A^{k 1}_{α β}
      end for
    end for
  end for
end for
```

```
for all segments m on Γ\Γ_root do
  Calculate the elemental matrix A^{m 2}
  for α = 1, ..., 2 do
    n_1 = κ^1(m, α)
    for β = 1, ..., 2 do
      n_2 = κ^1(m, β)
      A_{h n_1 n_2} = A_{h n_1 n_2} + Bi A^{m 2}_{α β}
    end for
  end for
end for
```

For the load vector $\underline{F}_h$ and the output vector $\underline{L}_h$ the following algorithm is used:

```
F_h := 0
for all segments m on Γ_root do
  Calculate the load vector F_h^m
  for α = 1, ..., 2 do
    n_1 = κ^2(m, α)
    F_{h n_1} = F_{h n_1} + F^m_{h α}
  end for
end for
L_h := F_h
```

$\delta$) The implementation of the above algorithm for the data structures provided with the problem statement is given in Appendix 1 (file `FE.m`). For the configuration $\underline{\mu}_0 = \{0.4, 0.6, 0.8, 1.2, 0.1\}$, the solution $u_h(\underline{\mu}_0)$ has been computed using the triangulation $\mathcal{T}_{h_{\rm medium}}$. A plot of the temperature distribution is shown in Figure 2. The output was calculated

$$T_{\rm root\,h}(\underline{\mu}_0) = 1.7342.$$

$\epsilon$) We start by proving that

$$T_{\rm root}(\underline{\mu}) - T_{\rm root\ h}(\underline{\mu}) = a(e(\underline{\mu}), e(\underline{\mu})). \tag{17}$$

Since $\ell^O(v) = \ell(v),\ \forall v \in Y$,

$$\begin{aligned} T_{\rm root}(\underline{\mu}) &= \ell^O(u(\underline{\mu})) = \ell(u(\underline{\mu})), \\ T_{\rm root\ h}(\underline{\mu}) &= \ell^O(u_h(\underline{\mu})) = \ell(u(\underline{\mu})). \end{aligned}$$

Subtracting the above equations and using the linearity of $\ell(v)$

$$T_{\text{root}}(\underline{\mu}) - T_{\text{root }h}(\underline{\mu}) = \ell(u(\underline{\mu}) - u_h(\underline{\mu})) = \ell(e(\underline{\mu})),$$

where $e(\underline{\mu}) \equiv u(\underline{\mu}) - u_h(\underline{\mu})$ is the error introduced by the finite element approximation to the exact solution. We will also need Galerkin orthogonality

$$\left.\begin{array}{l} a(u(\underline{\mu}), v; \underline{\mu}) = \ell(v), \ \forall v \in X \\ a(u_h(\underline{\mu}), v; \underline{\mu}) = \ell(v), \ \forall v \in X_h \end{array}\right\} \Rightarrow a(u(\underline{\mu}) - u_h(\underline{\mu}), v; \underline{\mu}) = 0 \Rightarrow a(e(\underline{\mu}), v; \underline{\mu}) = 0, \ \forall v \in X_h.$$

Therefore for $v = u_h(\underline{\mu}) \in X_h$, and using symmetry of the bilinear form

$$a(e(\underline{\mu}), u_h(\underline{\mu})) = 0 \Rightarrow a(u_h(\underline{\mu}), e(\underline{\mu})) = 0. \tag{18}$$

From (6), choosing $v = e(\underline{\mu}) \in Y$

$$a(u(\underline{\mu}), e(\underline{\mu}); \underline{\mu}) = \ell(e(\underline{\mu})) \Rightarrow T_{\text{root}}(\underline{\mu}) - T_{\text{root }h}(\underline{\mu}) = a(u(\underline{\mu}), e(\underline{\mu}); \underline{\mu})$$

From the two equations above using linearity

$$T_{\text{root}}(\underline{\mu}) - T_{\text{root }h}(\underline{\mu}) = a(u(\underline{\mu}), e(\underline{\mu}); \underline{\mu}) - \underbrace{a(u_h(\underline{\mu}), e(\underline{\mu}))}_{0} = a(e(\underline{\mu}), e(\underline{\mu}); \underline{\mu}) \tag{19}$$

We then notice that

$$T_{\text{root}}(\underline{\mu}) - T_{\text{root }h}(\underline{\mu}) = a(e(\underline{\mu}), e(\underline{\mu}); \underline{\mu}) = |||e(\underline{\mu})|||^2 \leq Ch^2; \tag{20}$$

since we assume that the solution $u \in H^2(\Omega; \mathcal{T}_h)$. For the last part of (20), *a priori* theory for the energy norm were used (given in the notes). So we expect at least an $h^2$ convergence from the method. In practice since the conductivities at each domain are discontinuous, the solution may not be in $H^2(\Omega; \mathcal{T}_h)$. To estimate the convergence rate for our problem, we do the following test.

$$\left.\begin{array}{rcl} (T_{\text{root}})_{h_{\text{fine}}} - (T_{\text{root}})_{2h_{\text{fine}}=h_{\text{medium}}} & = & C(2h_{\text{fine}})^b \\ (T_{\text{root}})_{h_{\text{fine}}} - (T_{\text{root}})_{4h_{\text{fine}}=h_{\text{coarse}}} & = & C(4h_{\text{fine}})^b \end{array}\right\} \Rightarrow$$

$$b = \frac{1}{\log 2} \log \left( \frac{(T_{\text{root}})_{h_{\text{fine}}} - (T_{\text{root}})_{h_{\text{coarse}}}}{(T_{\text{root}})_{h_{\text{fine}}} - (T_{\text{root}})_{h_{\text{medium}}}} \right)$$

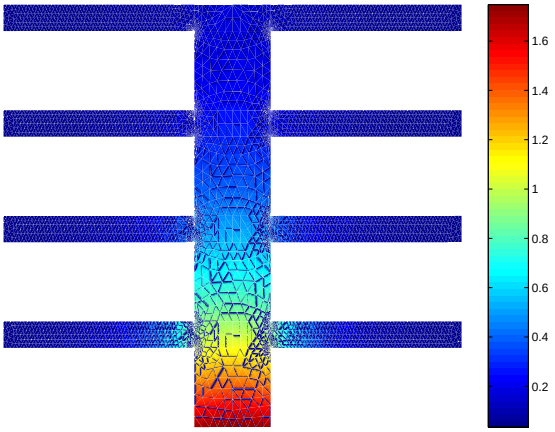


Figure 2: Temperature distribution for $\underline{\mu_0}$

The evaluations are done for $\underline{\mu}_0 = \{0.4, 0.6, 0.8, 1.2, 0.1\}$.

$$\begin{array}{rcl} T_{\text{root}_{h_{\text{fine}}}} & = & 1.7350 \\ T_{\text{root}_{h_{\text{medium}}}} & = & 1.7342 \\ T_{\text{root}_{h_{\text{coarse}}}} & = & 1.7313 \end{array}$$

The exponent $b$ is then computed

$$b = 2.19. \tag{21}$$

As we can see for this example $b$ is approximately 2, so the presence of geometric singularities does not affect the convergence rate.

**Part 2 - Reduced-Basis Approximation**

For the development of the reduced-basis approximation we introduce a sample in parameter space,

$$S_N = \{\underline{\mu}^1, \underline{\mu}^2, \ldots, \underline{\mu}^N\}$$

with $N \ll n$. Each $\underline{\mu}^i,\ i = 1, \ldots, N$, belongs in the parameter set $\mathcal{D}$. We then introduce the reduced-basis space as

$$W_N = \text{span}\{u_h(\underline{\mu}^1), u_h(\underline{\mu}^2), \ldots, u_h(\underline{\mu}^N)\} \tag{22}$$

where $u_h(\underline{\mu}^i)$ is the finite-element solution for $\underline{\mu} = \underline{\mu}^i$. To simplify the notation, we define $\zeta^i \in X$ as

$$\zeta^i = u_h(\underline{\mu}^i), \quad i = 1, \ldots, N;$$

we can then write $W_N = \text{span}\{\zeta^i,\ i = 1, \ldots, N\}$. Any member $v_N$ of $W_N$ can be represented as

$$v_N = \sum_{j=1}^{N} \beta^j \zeta^j, \tag{23}$$

for some unique choice of $\beta^j \in \mathbb{R},\ j = 1, \ldots, N$. (We implicitly assume that the $\zeta^i,\ i = 1, \ldots, N$, are linearly independent; it follows that $W_N$ is an $N$-dimensional subspace of $X$.)

In the reduced-basis approach we look for an approximation $u_N(\underline{\mu})$ to $u_h(\underline{\mu})$ (which for our purposes here we presume is *arbitrarily close* to $u(\underline{\mu})$) in $W_N$; in particular, we express $u_N(\underline{\mu})$ as

$$u_N(\underline{\mu}) = \sum_{j=1}^{N} u_N^j\, \zeta^j; \tag{24}$$

we denote by $\underline{u}_N(\underline{\mu}) \in \mathbb{R}^N$ the coefficient vector $(u_N^1, \ldots, u_N^N)^T$. The energy principle is crucial here (though more generally the weak form would suffice). To wit, we apply the classical Rayleigh-Ritz procedure to define

$$u_N(\underline{\mu}) = \arg \min_{w_N \in W_N} J(w_N); \tag{25}$$

alternatively we can apply Galerkin projection to obtain the equivalent statement

$$a(u_N(\underline{\mu}), v; \underline{\mu}) = \ell(v),\ \forall v \in W_N. \tag{26}$$

The output can then be calculated from

$$T_{\text{root } N}(\underline{\mu}) = \ell^O(u_N(\underline{\mu})). \tag{27}$$

$\alpha$) We want to prove that in the energy norm

$$|||u(\underline{\mu}) - u_N(\underline{\mu})||| \leq |||u(\underline{\mu}) - w_N|||,\ \forall w_N \in W_N. \tag{28}$$

Any $w_N$ which is a member of $W_N$, can be written as $w_N = u_N(\underline{\mu}) + v_N$, where $u_N(\underline{\mu}),\ v_N \in W_N$. We then have

$$\begin{aligned} |||u(\underline{\mu}) - w_N|||^2 &= a(u(\underline{\mu}) - w_N, u(\underline{\mu}) - w_N) \\ &= a(u(\underline{\mu}) - u_N(\underline{\mu}) - v_N, u(\underline{\mu}) - u_N(\underline{\mu}) - v_N) \\ &= a(u(\underline{\mu}) - u_N(\underline{\mu}), u(\underline{\mu}) - u_N(\underline{\mu})) - 2a(v_N, u(\underline{\mu}) - u_N(\underline{\mu})) + a(v_N, v_N); \end{aligned}$$

where we have used, the definition of the energy norm, bi-linearity and symmetry of $a(\cdot, \cdot)$. By virtue of the Galerkin orthogonality in $W_N$

$$a(v_N, u(\underline{\mu}) - u_N(\underline{\mu})) = 0.$$

Therefore,

$$\begin{aligned} |||u(\underline{\mu}) - w_N|||^2 &= a(u(\underline{\mu}) - u_N(\underline{\mu}), u(\underline{\mu}) - u_N(\underline{\mu})) + a(v_N, v_N) \\ &= |||u(\underline{\mu}) - u_N(\underline{\mu})|||^2 + |||v_N|||^2 \end{aligned}$$

And the desired result (28) is readily proved, since $|||v_N||| \geq 0$. Note that we didn't make any assumption on $w_N$ other than it belongs to the reduced-basis space $W_N$, this result is valid for all $w_N \in W_N$.

$\beta$) The proof of this result is identical to the proof given in Part 1 - Question $\epsilon$. We only need replace $u_h(\underline{\mu}) \rightarrow u_N(\underline{\mu})$ and $X_h \rightarrow W_N$.

$$T_{\text{root}}(\underline{\mu}) - T_{\text{root }N}(\underline{\mu}) = |||u(\underline{\mu}) - u_N(\underline{\mu})|||^2. \tag{29}$$

$\gamma$) Since $u_N(\underline{\mu}) \in W_N$, from (24)

$$u_N(\underline{\mu}) = \sum_{\beta=1}^{N} u_N^\beta \, \zeta^\beta;$$

Therefore from (26)

$$a(u_N(\underline{\mu}), v; \underline{\mu}) = \ell(v), \ \forall v \in W_N \Rightarrow \sum_{\beta=1}^{N} u_N^\beta a(\zeta^\beta, v; \underline{\mu}) = \ell(v), \forall v \in W_N.$$

Choosing $v = \zeta^\alpha, \ \alpha = 1, \ldots, N$, each entry of $\underline{A}_N(\underline{\mu})$ can be obtained from

$$\underline{A}_N(\underline{\mu})_{\alpha\,\beta} = a(\zeta^\beta, \zeta^\alpha), \ \forall \alpha, \ \beta \in \{1, \ldots, N\}. \tag{30}$$

Since, $\zeta^\alpha$ is the finite element solution for a particular configuration $\underline{\mu}_\alpha$, we can write $\zeta^\alpha$ in term of the nodal basis functions $\varphi_i, \ i = 1, \ldots, n$ ($n \equiv$ dimension of the finite element space),

$$\zeta^\alpha = \sum_{i=1}^{n} \zeta_i^\alpha \varphi_i.$$

Then each element of $\underline{A}_N(\underline{\mu})$ is calculated as follows:

$$\begin{aligned}
\underline{A}_N(\underline{\mu})_{\alpha\,\beta} &= a(\zeta^\beta, \zeta^\alpha) \\
&= a(\sum_{i=1}^{n} \zeta_i^\alpha \varphi_i, \sum_{j=1}^{n} \zeta_j^\beta \varphi_j) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \zeta_i^\alpha \zeta_j^\beta a(\varphi_i, \varphi_j) \\
&= (\underline{\zeta}^\alpha)^T \underline{A}_h(\underline{\mu})(\underline{\zeta}^\beta), \ \forall \alpha, \beta \in \{1, \ldots, N\}
\end{aligned}$$

The above can be written succinctly in terms of $\underline{Z}$ an $n \times N$ matrix, the $j^{th}$ column of which is $\underline{u}_h(\underline{\mu}^j)$ (the nodal values of $u_h(\underline{\mu}^j)$),

$$\underline{A}_N(\underline{\mu}) = \underline{Z}^T \underline{A}_h(\underline{\mu}) \underline{Z}. \tag{31}$$

A similar procedure can be used, for the load vector $\underline{F}_N$. The steps are outlined below ($\alpha = \{1, \ldots, N\}$).

$$\begin{aligned}
F_{N\,\alpha} &= \ell(\zeta^\alpha) \\
&= \ell(\sum_{i=1}^{n} \zeta_i^\alpha \varphi_i) \\
&= \sum_{i=1}^{n} \zeta_i^\alpha \ell(\varphi_i) \\
&= (\underline{\zeta}^\alpha)^T \underline{F}_h, \ \alpha = \{1, \ldots, N\}.
\end{aligned}$$

Therefore

$$\underline{F}_N = \underline{Z}^T \underline{F}_h. \tag{32}$$

Finally since $\ell^O(v) \equiv \ell(v)$,

$$\underline{F}_N = \underline{L}_N$$

$\delta$) The bilinear form is

$$a(w,v;\underline{\mu}) = \sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla w \cdot \nabla v \, dA + \mathrm{Bi} \int_{\Gamma\setminus\Gamma_{\mathrm{root}}} wv \, dS. \tag{33}$$

Choosing now

$$\begin{array}{ll} \sigma^1(\underline{\mu}) = k^1 & a^1(w,v) = \int_{\Omega^1} \nabla w \cdot \nabla v \, dA \\ \sigma^2(\underline{\mu}) = k^2 & a^2(w,v) = \int_{\Omega^2} \nabla w \cdot \nabla v \, dA \\ \sigma^3(\underline{\mu}) = k^3 & a^3(w,v) = \int_{\Omega^3} \nabla w \cdot \nabla v \, dA \\ \sigma^4(\underline{\mu}) = k^4 & a^4(w,v) = \int_{\Omega^4} \nabla w \cdot \nabla v \, dA \\ \sigma^5(\underline{\mu}) = k^0 & a^5(w,v) = \int_{\Omega^0} \nabla w \cdot \nabla v \, dA \\ \sigma^6(\underline{\mu}) = \mathrm{Bi} & a^6(w,v) = \int_{\Gamma\setminus\Gamma_{\mathrm{root}}} wv \, dS \end{array} \tag{34}$$

we can verify that the bilinear form $a(w,v;\underline{\mu})$ can be decomposed as

$$a(w,v;\underline{\mu}) = \sum_{q=1}^{Q} \sigma^q(\underline{\mu}) a^q(w,v), \ \forall w,v \in X, \ \forall \underline{\mu} \in \mathcal{D}, \tag{35}$$

for $Q = 6$.

We now prove that for the discrete form of the problem, both for the finite element matrix $\underline{A}_h(\underline{\mu})$ and the reduced-basis matrix $\underline{A}_N(\underline{\mu})$, a similar decomposition exists. Since

$$\begin{aligned} A_{h\,\alpha\,\beta}(\underline{\mu}) &= a(\varphi_\beta, \varphi_\alpha) \\ &= \sum_{q=1}^{Q} \sigma^q(\underline{\mu}) a^q(\varphi_\beta, \varphi_\alpha) \\ &= \sum_{q=1}^{Q} \sigma^q(\underline{\mu}) A^q_{h\,\alpha\,\beta}, \ \alpha,\beta \in \{1,\ldots,n\} \end{aligned}$$

and in matrix notation it becomes

$$\underline{A}_h(\underline{\mu}) = \sum_{q=1}^{Q} \sigma^q(\underline{\mu}) \underline{A}^q_h. \tag{36}$$

For the reduced-basis matrix,

$$\begin{aligned} A_{N\,\alpha\,\beta}(\underline{\mu}) &= a(\zeta^\beta, \zeta^\alpha) \\ &= \sum_{q=1}^{Q} \sigma^q(\underline{\mu}) a^q(\zeta^\beta, \zeta^\alpha) \\ &= \sum_{q=1}^{Q} \sigma^q(\underline{\mu}) A^q_{N\,\alpha\,\beta}, \ \alpha,\beta \in \{1,\ldots,N\} \end{aligned}$$

and in matrix notation it becomes

$$\underline{A}_N(\underline{\mu}) = \sum_{q=1}^{Q} \sigma^q(\underline{\mu}) \underline{A}^q_N. \tag{37}$$

$\epsilon$) The algorithm for the off-line/on-line version of the reduced-basis approximation is indicated below.

- *Off-line*

  **1.** Choose $N$.

  **2.** Choose the sample $S_N$.

**3.** Construct $\underline{Z}$.

**4.** Construct $\underline{A}_N^q,\ q = 1, \ldots, Q$; $\underline{F}_N$; and $\underline{L}_N$.

- *On-line*

**1.** Form $\underline{A}_N(\underline{\mu})$ from (37)

**2.** Solve $A_N(\underline{\mu})\underline{u}_N(\underline{\mu}) = \underline{F}_N$.

**3.** Evaluate the output $T_{\text{root }N}(\underline{\mu})$ from (4).

The implementation of this algorithm (files `offline.m` and `online.m`) can be found in Appendix 1. For $N = 10$, and the sample set $S_N$ given in the datafile `sn.dat` we compute for

$$\begin{array}{lll} \underline{\mu}_0 = \{0.4,\ 0.6,\ 0.8,\ 1.2,\ 0.1\} & \rightarrow & T_{\text{root }N}(\underline{\mu}_0) = 1.72621, \text{ and for} \\ \underline{\mu}_1 = \{1.8,\ 4.2,\ 5.7,\ 1.9,\ 0.3\} & \rightarrow & T_{\text{root }N}(\underline{\mu}_1) = 1.07496. \end{array}$$

$\zeta$) We next do an operation count for the online part of the algorithm. For simplicity, we assume that addition and multiplication incur the same computational cost.

- Form $\underline{A}_N(\underline{\mu})$ from (37).
  This requires $(Q-1)N^2 = 5N^2$ additions, and $QN^2 = 6N^2$ multiplications, a total of $11N^2$ operations.
- Solve $A_N(\underline{\mu})\underline{u}_N(\underline{\mu}) = \underline{F}_N$.
  Since the resulting system is a dense matrix, a direct solver (Gauss elimination) is usually used. The computational cost is $\frac{2}{3}N^3 + \frac{N^2}{2} + N - 1$ (See lecture notes on solution methods).
- Evaluate the output $T_{\text{root }N}(\underline{\mu})$ from (4).
  This is the cost for the calculation of the inner product, there are $N$ multiplications and $N-1$ additions, a total of $2N - 1$ operations.

So the total cost for the on-line stage for each new $\underline{\mu}$ of interest is

$$\frac{2}{3}N^3 + \frac{34}{3}N^2 + 2N - 2 \qquad (38)$$

operations. As we can see the operation count is independent of $n$.

$\eta$) Finally, we consider the design problem, in which a thermal fin with specified $\{k_1, k_2, k_3, k_4\} = \{0.4, 0.6, 0.8, 1.2\}$ is given, and we need to choose the cooling method (Biot number) that minimizes a cost function

$$C(\text{Bi}) = 0.1\,\text{Bi} + T_{\text{root}}(\text{Bi}). \qquad (39)$$

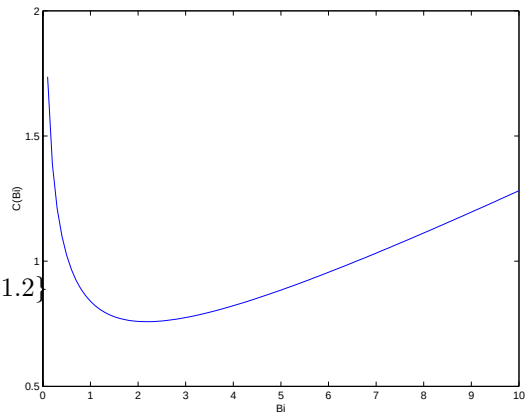


Figure 3: Cost as a function of Biot number

To solve this optimization problem a simple line search is done for Biot numbers in the interval from $[0.1, 10]$. The cost function, as a function of the Biot number is presented in Figure 3. The optimal $Bi = 2.18$ and the cost function is minimized with a value $C(2.18) = 0.7584$.

## Appendix 1 - MATLAB® codes

FE.m

```
function [u,s,A,f]=FE(grid,params,solve)
% Returns the stiffness matrix, right hand side.
% Grid is the grid data structure to use for the matrix
% params Is the parameter vector....
% params= [k1, k2, k3, k4, k0, Bi]';
% solve: Lo

if size(params,1)~= 6
  error(' Check input paramater vector');
end

% Direct stiffness summation for the inner part of the domain...
ind=find(abs(params(1:5))>eps);
A1=zeros(3,3);
A=spalloc(grid.nodes,grid.nodes,10*grid.nodes);
f=zeros(grid.nodes,1);

for i = ind'
  for j = (grid.theta{i})'
    x1=grid.coor(j(1),1);y1=grid.coor(j(1),2);
    x2=grid.coor(j(2),1);y2=grid.coor(j(2),2);
    x3=grid.coor(j(3),1);y3=grid.coor(j(3),2);
    area= -x2*y1 + x3*y1 + x1*y2 - x3*y2 - x1*y3 + x2*y3;
    c11 = y2 - y3;
    c12 = y3 - y1;
    c13 = y1 - y2;
    c21 = x3 - x2;
    c22 = x1 - x3;
    c23 = x2 - x1;
    A1=params(i)/(2*abs(area))*...
       [c11*c11+c21*c21 c11*c12+c21*c22 c11*c13+c21*c23 ;...
        c12*c11+c22*c21 c12*c12+c22*c22 c12*c13+c22*c23 ;...
        c13*c11+c23*c21 c13*c12+c23*c22 c13*c13+c23*c23];
    A(j,j)=A(j,j)+A1;
  end
end

% Robin boundary conditions matrix...
if abs(params(6)>0)
mass=zeros(2,2);
  for j=(grid.theta{6})'
    x1=grid.coor(j(1),1);y1=grid.coor(j(1),2);
    x2=grid.coor(j(2),1);y2=grid.coor(j(2),2);
    dx=sqrt((x2-x1)*(x2-x1)+(y2-y1)*(y2-y1));
    mass=params(6)*[dx/3 dx/6; dx/6 dx/3];
    A(j,j)=A(j,j)+mass;
  end
end


%Right hand side vector...
f=zeros(grid.nodes,1);
for j=(grid.theta{7})'
    x1=grid.coor(j(1),1);y1=grid.coor(j(1),2);
    x2=grid.coor(j(2),1);y2=grid.coor(j(2),2);
    dx=sqrt((x2-x1)*(x2-x1)+(y2-y1)*(y2-y1));
    f(j)=f(j)+[dx/2 dx/2]';
end

u=0.0;
s=0.0;
```

```
if solve~= 0
  % Solve
  u=A\f;
  % ... and calculate output.
  s=dot(u,f);
end
```

---

offline.m

---

```
function [bb]=offline(grid,N,ranges)
% Creates the reduced-basis information,
% and saves it to a datafile...
% This is basically the off-line part of the code...
% Grid: Is the grid data file to be used.
% N: Is the number of basis functions to keep.
% Ranges: Is a Px2 array with the ranges for the parameters
% Datafile: The name of the file that the data related to the
%      reduced basis are going to be saved (Good for having different cases).

% Create the distribution of points
rbpoints=lograndom_point_distribution(ranges,N);
  %rbpoints=random_point_distribution(ranges,N);


% Form the reduced basis
disp('Starting calculation of reduced basis:');
Z=zeros(grid.nodes,N);
for i=1:N
  [Z(:,i) s]=FE(grid, rbpoints(i,:)', 1);
  disp([i rbpoints(i,:)]);
  disp(sprintf('Output functional: %f\n', s));
end

% Preprocessing step to form Aq and the right hand side...
Q=6;
bb.ANq=zeros(Q,N,N);
bb.FN=zeros(N);
bb.N=N;
bb.Q=Q;

for q=1:Q
  ind=zeros(Q,1);ind(q)=1.0;
  [u s A f]=FE(grid, ind, 0);
  bb.ANq(q,:,:)=Z'*A*Z;
end
bb.FN=Z'*f;

function [points]=lograndom_point_distribution(ranges,N)
% Creates the log-distribution of points...
epsilon=0.1;
ran=log(ranges+epsilon);
rand('seed',6510040);
phys=size(ranges,1);
points=zeros(N,phys);

for i=1:phys
  points(:,i)=ran(i,1)+(ran(i,2)-ran(i,1))*rand(N,1);
end

points=exp(points)-epsilon;

function [points]=random_point_distribution(ranges,N)
```

```
% Creates a random range. . . .
rand('seed',6510040);
phys=size(ranges,1);
points=zeros(N,phys);

for i=1:phys
  points(:,i)=ranges(i,1)+(ranges(i,2)-ranges(i,1))*rand(N,1);
end
```

---

online.m

---

```
function [sN]=online(datapoint,bb)
% Datapoint: is the desired point for which we want to evaluate.
% bb: Is the blackbox data structure.

A=zeros(bb.N,bb.N);

for q=1:bb.Q
  A=A+fin(datapoint,q)*squeeze(bb.ANq(q,:,:));
end

uN=A\bb.FN;

sN=dot(uN,bb.FN);


function [Fq]=fin(datapoint,q)
% Returns the coefficient Fq, for the given q.
Fq=datapoint(q);
```

---