Numerical Schemes for Scalar One-Dimensional Conservation Laws

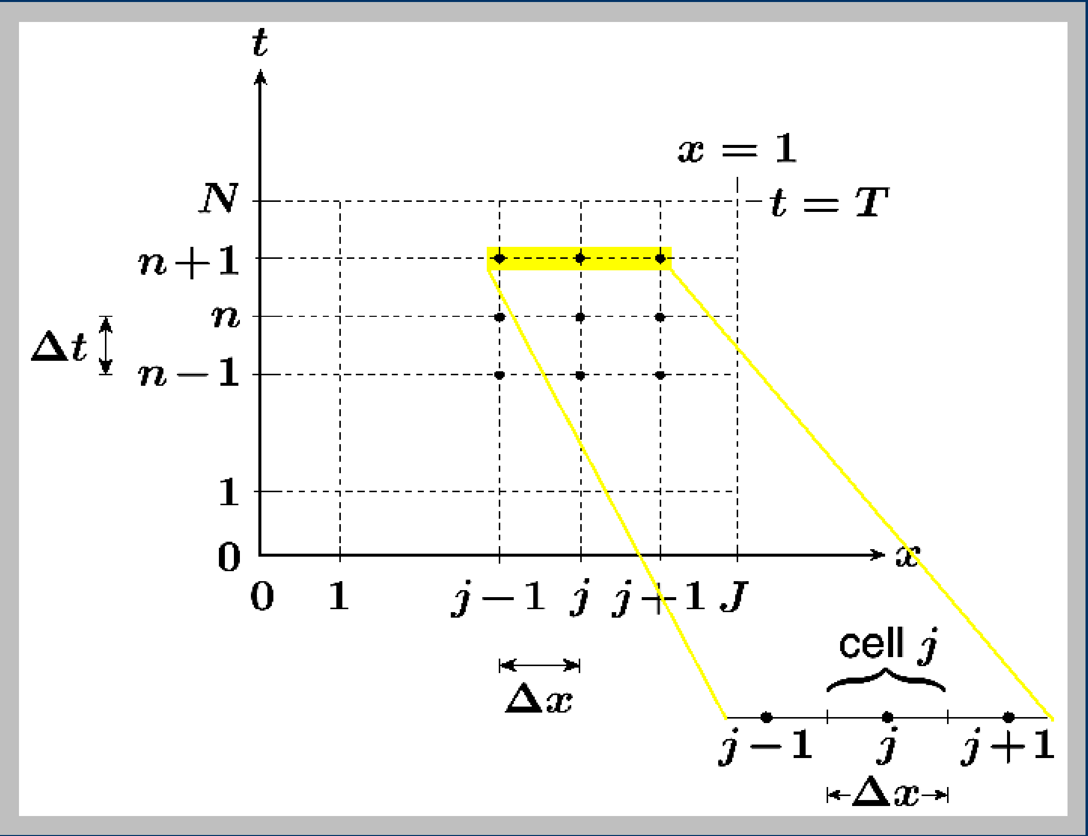
Lecture 12

Finite Volume Discretization

Computational Cells

$$x_j = j\Delta x$$

$$t^n = n\Delta t$$



Finite Volume Discretization

Cell averages

We think of $\hat{u}_j^n$ as representing cell averages

$$\hat{u}_j^n \approx \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t^n)\, dx$$

Conservative Methods

Definition

Applying integral form of conservation law to a cell j

$$\frac{d}{dt}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u\,dx = -\left[f(u(x_{j+\frac{1}{2}},t)) - f(u(x_{j-\frac{1}{2}},t))\right]$$

suggests

$$\frac{\hat{u}_j^{n+1} - \hat{u}_j^n}{\Delta t}\Delta x = -\left(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n\right)$$

$$\Rightarrow \quad \boxed{\hat{u}_j^{n+1} = \hat{u}_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n\right)}$$

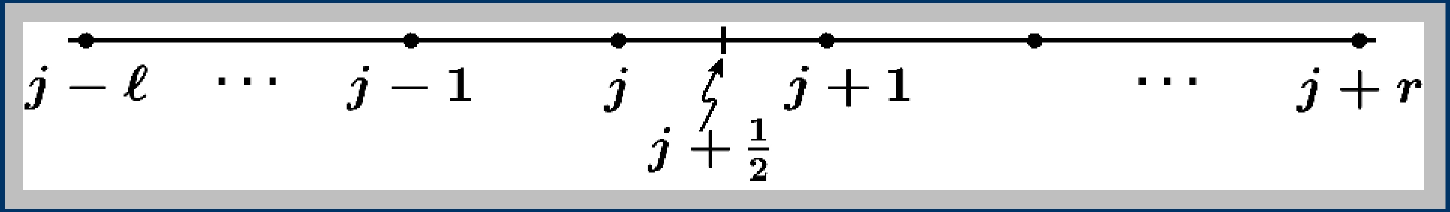
Conservative Methods

Numerical Flux function

$$F_{j+\frac{1}{2}} \equiv F\left(\hat{u}_{j-l}, \hat{u}_{j-l+1}, \ldots, \hat{u}_j, \ldots, \hat{u}_{j+r}\right)$$

and F is a **numerical flux function** of $l + r + 1$ arguments that satisfies the following **consistency** condition

$$\boxed{F(u, u, \ldots, u, u) = f(u)}$$



Conservative Methods

Lax-Wendroff Theorem

If the solution of a **conservative** numerical scheme converges as $\Delta x \to 0$ with $\frac{\Delta t}{\Delta x}$ fixed, then it **converges to a weak solution** of the conservation law.

N1

$\Rightarrow$ **shock capturing** schemes are possible

N2

Conservative Methods

Lax-Wendroff Theorem

Shock Capturing

In the exact problem:

$$\frac{d}{dt}\int_{x_0}^{x_J} u \; dx = -(f_0 - f_J)$$

A conservative numerical scheme satisfies an analogous discrete condition:

N3

$$\frac{\Delta x}{\Delta t}\sum_{j=0}^{J}(\hat{u}_j^{n+1} - \hat{u}_j^{n}) = -\sum_{j=0}^{J}\left(F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}\right)$$

$$= -\left(F_{J+\frac{1}{2}} - F_{-\frac{1}{2}}\right)$$

Conservative Methods

First Order Upwind

Linear Advection Equation...

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \qquad a \text{ constant} > 0$$

$$\hat{u}_j^{n+1} = \hat{u}_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+\frac{1}{2}}^{UP} - F_{j-\frac{1}{2}}^{UP}\right)$$

Let $$F_{j+\frac{1}{2}}^{UP} \equiv a\,\hat{u}_j \qquad \left(F_{j-\frac{1}{2}}^{UP} = a\,\hat{u}_{j-1}\right)$$

$$\Rightarrow \qquad \hat{u}_j^{n+1} = \hat{u}_j^n - \frac{\Delta t\, a}{\Delta x}(\hat{u}_j - \hat{u}_{j-1})$$

Conservative Methods

First Order Upwind

...Linear Advection Equation...

What about $a < 0$?

We can write,

$$\hat{u}_j^{n+1} = \hat{u}_j^n - \frac{a\Delta t}{\Delta x} \begin{cases} \hat{u}_j^n - \hat{u}_{j-1}^n & a > 0 \\ \hat{u}_{j+1}^n - \hat{u}_j^n & a < 0 \end{cases}$$

or

$$\hat{u}_j^{n+1} = \hat{u}_j^n - \frac{a\Delta t}{2\Delta x}\left(\hat{u}_{j+1}^n - \hat{u}_{j-1}^n\right) + \frac{|a|\Delta t}{2\Delta x}\left(\hat{u}_{j+1}^n - 2\hat{u}_j^n + \hat{u}_{j-1}^n\right)$$

Conservative Methods

First Order Upwind

...Linear Advection Equation

In **conservative form**:

$$\hat{u}_j^{n+1} = \hat{u}_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+\frac{1}{2}}^{UP\,n} - F_{j-\frac{1}{2}}^{UP\,n}\right)$$

$$\boxed{F_{j+\frac{1}{2}}^{UP} = \frac{1}{2}a(\hat{u}_{j+1} + \hat{u}_j) - \frac{1}{2}|a|(\hat{u}_{j+1} - \hat{u}_j)}$$

$$F_{j+\frac{1}{2}}^{UP} = a\hat{u}_j \qquad a > 0$$

$$F_{j+\frac{1}{2}}^{UP} = a\hat{u}_{j+1} \qquad a < 0$$

Conservative Methods

First Order Upwind

Nonlinear Case

In the nonlinear case,

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

the flux becomes

N4

$$F^{UP}_{j+\frac{1}{2}} = \frac{1}{2}\left(\hat{f}_{j+1} + \hat{f}_j\right) - \frac{1}{2}|\hat{a}_{j+\frac{1}{2}}|\left(\hat{u}_{j+1} - \hat{u}_j\right)$$

$$\hat{a}_{j+\frac{1}{2}} = \begin{cases} \frac{\hat{f}_{j+1}-\hat{f}_j}{\hat{u}_{j+1}-\hat{u}_j} & \text{if} \quad \hat{u}_{j+1} \neq \hat{u}_j \\ f'(\hat{u}_j) & \text{if} \quad \hat{u}_{j+1} = \hat{u}_j \end{cases}$$

Conservative Methods

Lax-Wendroff

$$F^{LW}_{j+\frac{1}{2}} = \frac{1}{2}\left(\hat{f}_{j+1} + \hat{f}_j\right) - \frac{1}{2}\,\hat{a}^2_{j+\frac{1}{2}}\,\frac{\Delta t}{\Delta x}\left(\hat{u}_{j+1} - \hat{u}_j\right)$$

For the linear equation

$$\hat{u}^{n+1}_j = \hat{u}_j - \frac{C}{2}\left(\hat{u}^n_{j+1} - \hat{u}^n_{j-1}\right) + \frac{C^2}{2}\left(\hat{u}^n_{j+1} - 2\hat{u}^n_j + \hat{u}^n_{j-1}\right)$$

$$C = a\Delta x/\Delta t$$

Conservative Methods

Beam-Warming

$$F^{BW}_{j+\frac{1}{2}} = \frac{1}{4}\left(-\hat{f}_{j+2} + 3\hat{f}_{j+1} + 3\hat{f}_j - \hat{f}_{j-1}\right) - \hat{a}^2_{j+\frac{1}{2}} \frac{\Delta x}{4\Delta t}\left(\hat{u}_{j+2} - \hat{u}_{j+1} + \hat{u}_j - \hat{u}_{j-1}\right)$$
$$- \frac{s_{j+\frac{1}{2}}}{4}\left(-\hat{f}_{j+2} + 3\hat{f}_{j+1} - 3\hat{f}_j + \hat{f}_{j-1}\right) + s_{j+\frac{1}{2}}\, a^2_{j+\frac{1}{2}} \frac{\Delta t}{4\Delta x}\left(\hat{u}_{j+2} - \hat{u}_{j+1} - \hat{u}_j + \hat{u}_{j-1}\right)$$

$$s_{j+\frac{1}{2}} = a_{j+\frac{1}{2}} / |a_{j+\frac{1}{2}}|$$

For the linear equation

$$\hat{u}^{n+1}_j = \hat{u}^n_j - \frac{C}{2}\left(3\hat{u}^n_j - 4\hat{u}^n_{j-1} + \hat{u}^n_{j-2}\right) + \frac{C^2}{2}\left(\hat{u}^n_j - 2\hat{u}^n_{j-1} + \hat{u}^n_{j-2}\right) \qquad a > 0$$

$$\hat{u}^{n+1}_j = \hat{u}^n_j - \frac{C}{2}\left(-3\hat{u}^n_j + 4\hat{u}^n_{j+1} - \hat{u}^n_{j+2}\right) + \frac{C^2}{2}\left(\hat{u}^n_{j+2} - 2\hat{u}^n_{j+1} + \hat{u}^n_j\right) \qquad a < 0$$

Entropy Solutions

Do these schemes converge to the entropy satisfying solution?

EXAMPLE:

Consider a non-physical solution to Burgers' equation:

$$u(x,t) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

i.e. $\hat{u}_j^n$ is either 1 or -1 $\Rightarrow$ $f_j = \frac{1}{2} \quad \forall j$

Conservative Methods

Entropy Solutions

Example

First order upwind:

$$F^{UP}_{j+\frac{1}{2}} = \frac{1}{2}\left(\hat{f}_{j+1} + \hat{f}_j\right) - \frac{1}{2}|\hat{a}_{j+\frac{1}{2}}|\left(\hat{u}_{j+1} - \hat{u}_j\right)$$

Since either $\hat{a}_{j+\frac{1}{2}}$ or $\hat{u}_{j+1} - \hat{u}_j$ is zero $\forall j$

$$\Rightarrow \quad F^{UP}_{j+\frac{1}{2}} = \tfrac{1}{2} \quad \forall j \Rightarrow \quad F^{UP}_{j+\frac{1}{2}} - F^{UP}_{j-\frac{1}{2}} = 0 \quad \forall j$$

$$\Rightarrow \quad \hat{u}^{n+1}_j = \hat{u}^n_j$$

The entropy-violating solution is preserved

N5

Entropy Satisfying Schemes

Monotone Schemes

If a scheme can be written in the form

$$\hat{u}_j^{n+1} = H\left(\hat{u}_{j-l}^n, \hat{u}_{j-l+1}^n, \ldots, \hat{u}_j^n, \ldots, \hat{u}_{j+r}^n\right)$$

with $\dfrac{\partial H}{\partial u_i} \geq 0 \qquad i = j-l, \ldots, j, \ldots, j+r,$

then the scheme is **monotone** and is

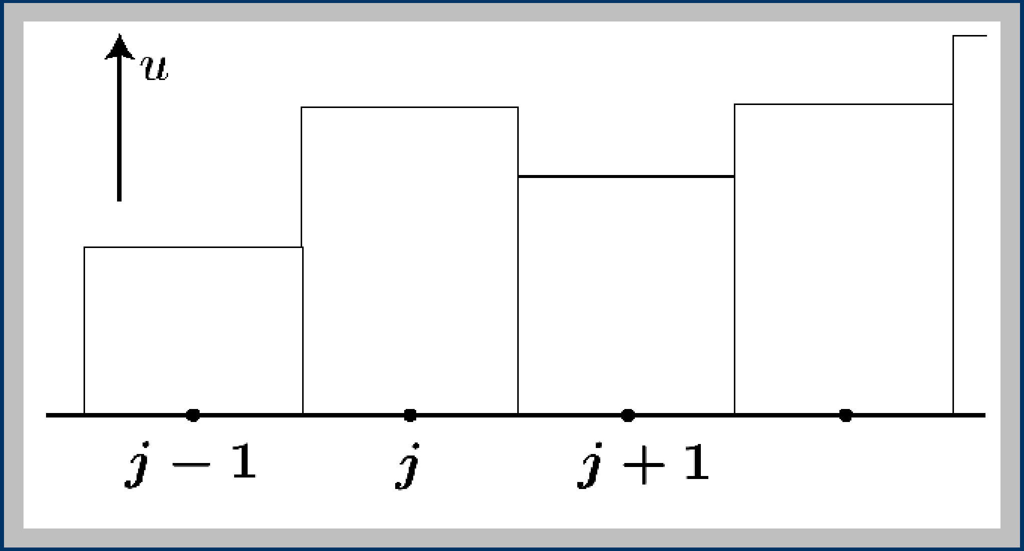
- **entropy satisfying**
- **at most first order accurate**

N6

Entropy Satisfying Schemes

Monotone Schemes

Godunov's Method...



Assume piecewise constant solution over each cell. Compute interface flux by **solving interface** (Riemann) **problem exactly**.

Entropy Satisfying Schemes

Monotone Schemes

...Godunov's Method...

$$F^{Gn}_{j+\frac{1}{2}} = f\left(u(x_{j+\frac{1}{2}}, t^{n+})\right)$$

$$= \begin{cases} \min\limits_{u \in [u_j, u_{j+1}]} f(u) & u_j < u_{j+1} \\ \max\limits_{u \in [u_j, u_{j+1}]} f(u) & u_j > u_{j+1} \end{cases}$$

Then,

$$\hat{u}^{n+1}_j = \hat{u}^n_j - \frac{\Delta t}{\Delta x}\left(F^{Gn}_{j+\frac{1}{2}} - F^{Gn}_{j-\frac{1}{2}}\right)$$

Entropy Satisfying Schemes

Monotone Schemes

...Godunov's Method

Applied to Burgers' equation

$$F^G_{j+\frac{1}{2}} = \begin{cases} \frac{1}{2}\hat{u}^2_{j+1} & \hat{u}_j, \hat{u}_{j+1} < 0 \\ \frac{1}{2}\hat{u}^2_j & \hat{u}_j, \hat{u}_{j+1} > 0 \\ 0 & \hat{u}_j < 0 < \hat{u}_{j+1} \quad \text{(expansion)} \\ \frac{1}{2}\hat{u}^2_j & \hat{u}_j > 0 > \hat{u}_{j+1} \quad \frac{1}{2}(\hat{u}_{j+1} + \hat{u}_j) > 0 \\ \frac{1}{2}\hat{u}^2_{j+1} & \hat{u}_j > 0 > \hat{u}_{j+1} \quad \frac{1}{2}(\hat{u}_{j+1} + \hat{u}_j) < 0 \end{cases}$$

Entropy Satisfying Schemes

E-Schemes

If the numerical flux $F_{j+\frac{1}{2}}$ satisfies

$$\text{sign}(\hat{u}^n_{j+1} - \hat{u}^n_j)(F^n_{j+\frac{1}{2}} - f(u)) \leq 0 \quad \forall u \in [\hat{u}_j, \hat{u}_{j+1}]$$

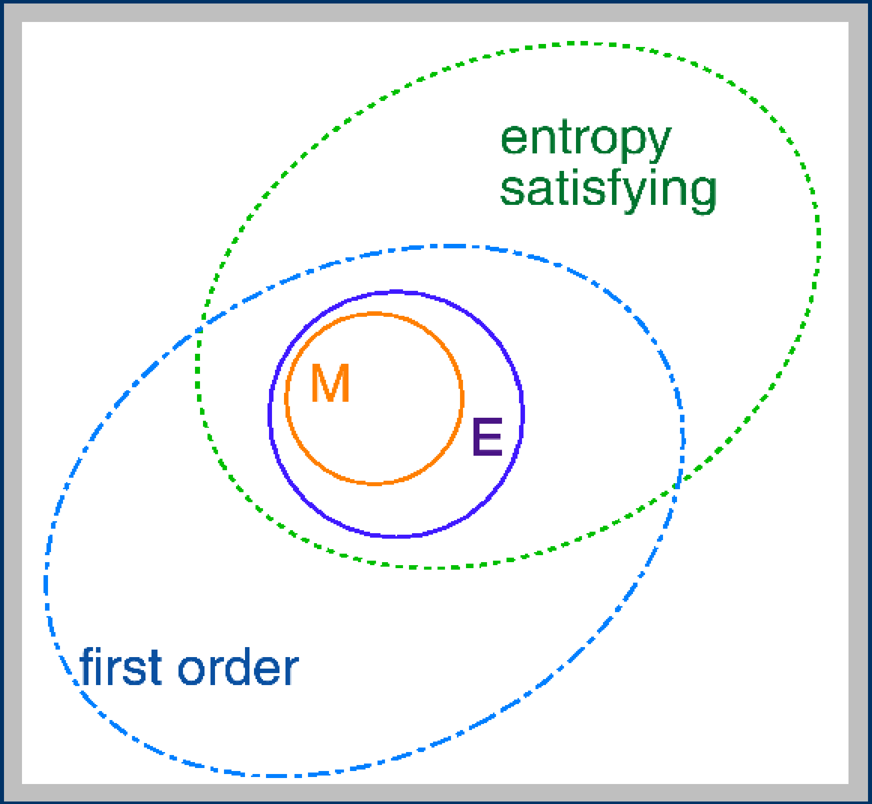
An E-scheme is

- **entropy satisfying**
- **at most first order accurate**

N7

Entropy Satisfying Schemes

Summary



TVD Methods

Motivation

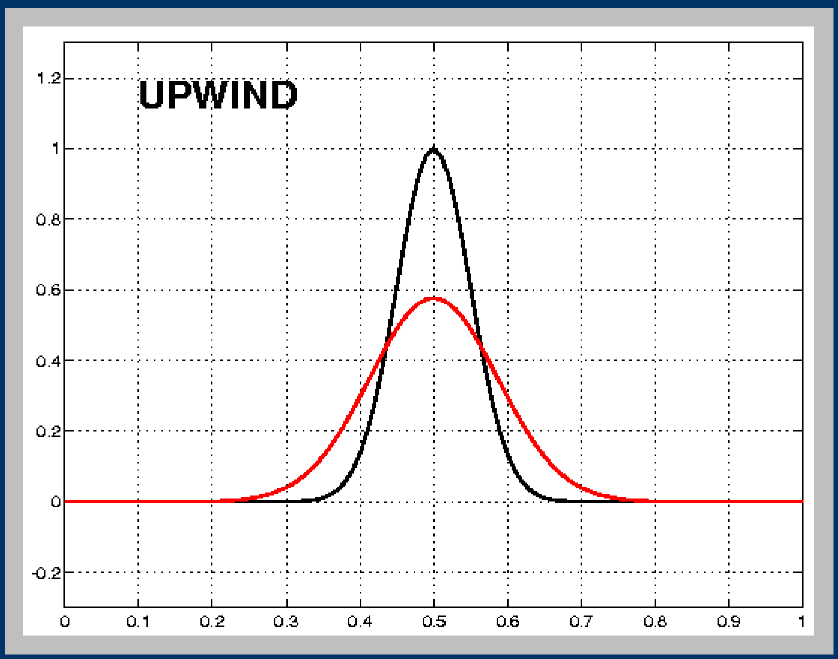
First order schemes give poor resolution but can be made to produce entropy satisfying and **non-oscillatory solutions**

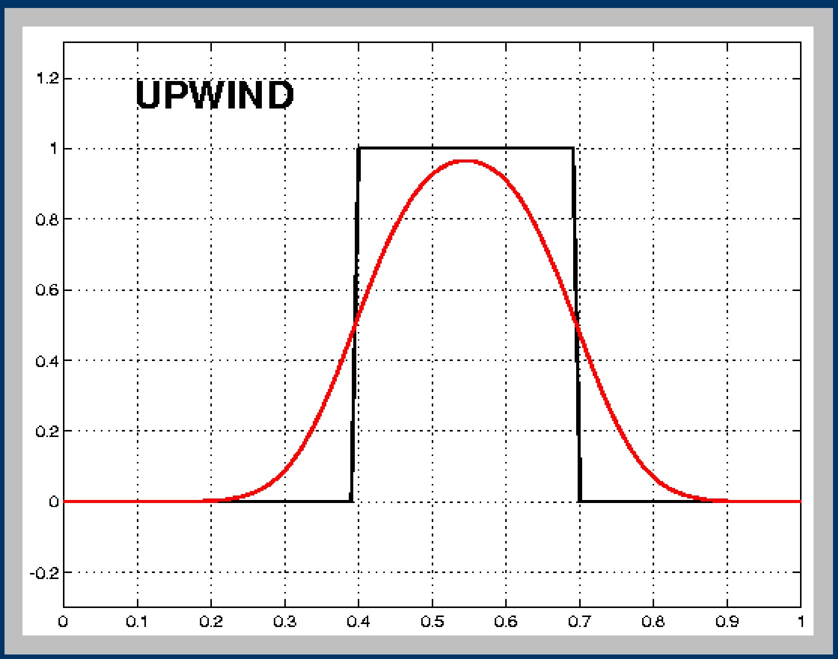
Higher order schemes (at least the ones we have seen so far) produce non-entropy satisfying and **oscillatory solutions**.

Good criterion to design "high order" oscillation free schemes is based on the **Total Variation** of the solution.

TVD Methods

First Order Upwind

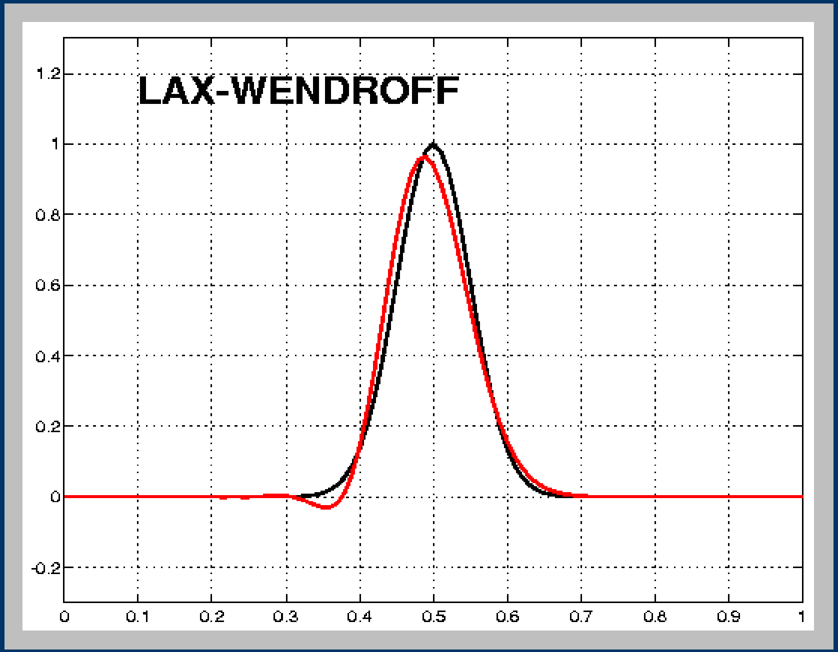


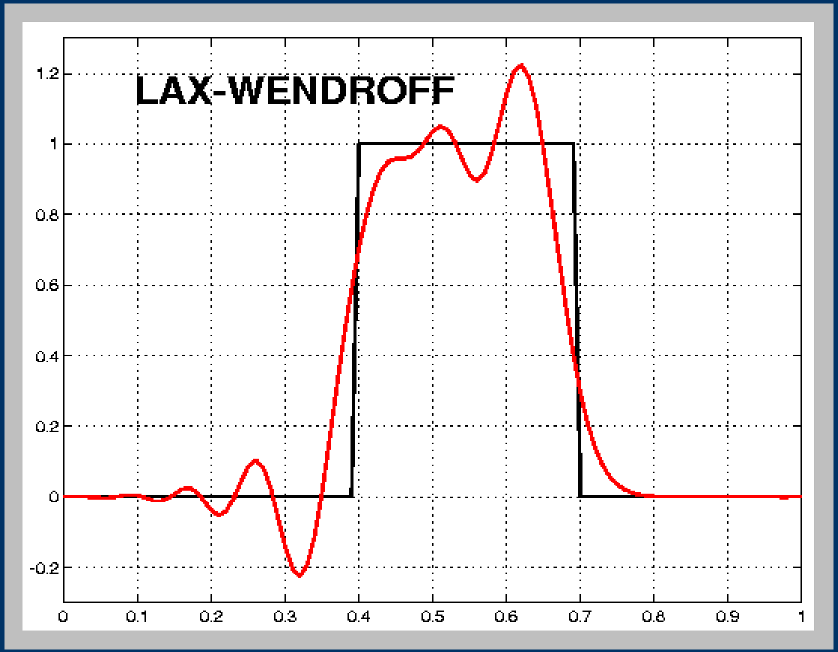


$$J = 100, \ \Delta x = 1/100, \ C = 0.5, \ N = 200$$

TVD Methods

Lax-Wendroff





$$J = 100,\ \Delta x = 1/100,\ C = 0.5,\ N = 200$$

TVD Methods

Definition

Total Variation of the discrete solution

$$TV(\underline{\hat{u}}^n) = \sum_j \left| \hat{u}^n_{j+1} - \hat{u}^n_j \right|$$

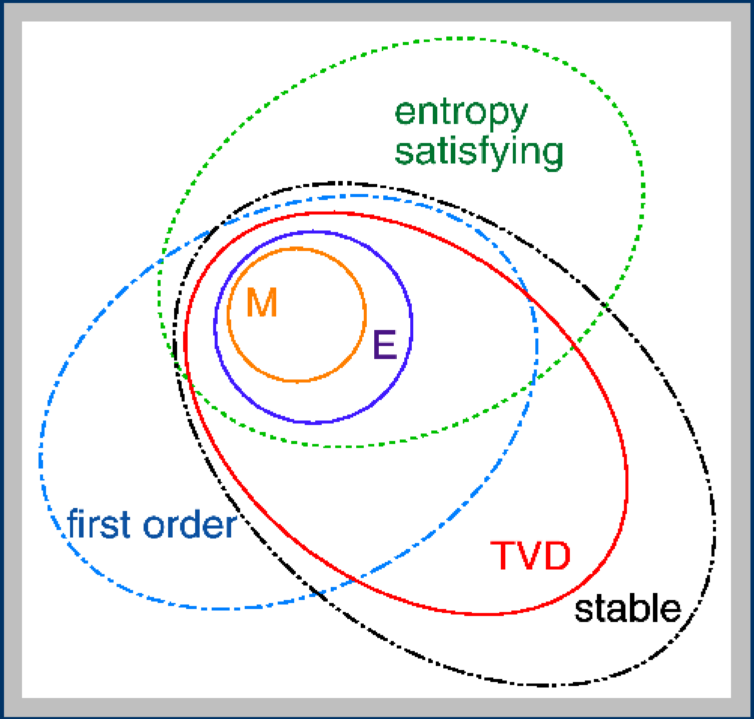
If new extrema are generated $TV(\underline{\hat{u}})$ will increase.

$$\boxed{TV(\underline{\hat{u}}^{n+1}) \leq TV(\underline{\hat{u}}^n)}$$

Total Variation Diminishing Schemes

TVD Methods

Some Properties



- All E-Schemes are TVD
- Conservative TVD Schemes
  $\Rightarrow$ **Converge** to weak solutions

N8

TVD Methods

Conditions for TVD schemes

If a scheme is written in the form

$$\hat{u}_j^{n+1} = \hat{u}_j^n + D_{j+\frac{1}{2}} \Delta \hat{u}_{j+\frac{1}{2}}^n - C_{j-\frac{1}{2}} \Delta \hat{u}_{j-\frac{1}{2}}^n$$

$$\Delta \hat{u}_{j+\frac{1}{2}} = \hat{u}_{j+1} - \hat{u}_j$$

it is TVD iff

$$\begin{aligned} C_{j+\frac{1}{2}} &\geq 0 \\ D_{j+\frac{1}{2}} &\geq 0 \\ C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} &\leq 1 \end{aligned}$$

TVD Methods

Conditions for TVD schemes

Example: Upwind

Upwind scheme for linear equation, $a > 0$:

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right)$$

$$C_{j-\frac{1}{2}} = \tfrac{a\Delta t}{\Delta x}; \qquad D_{j+\frac{1}{2}} = 0$$

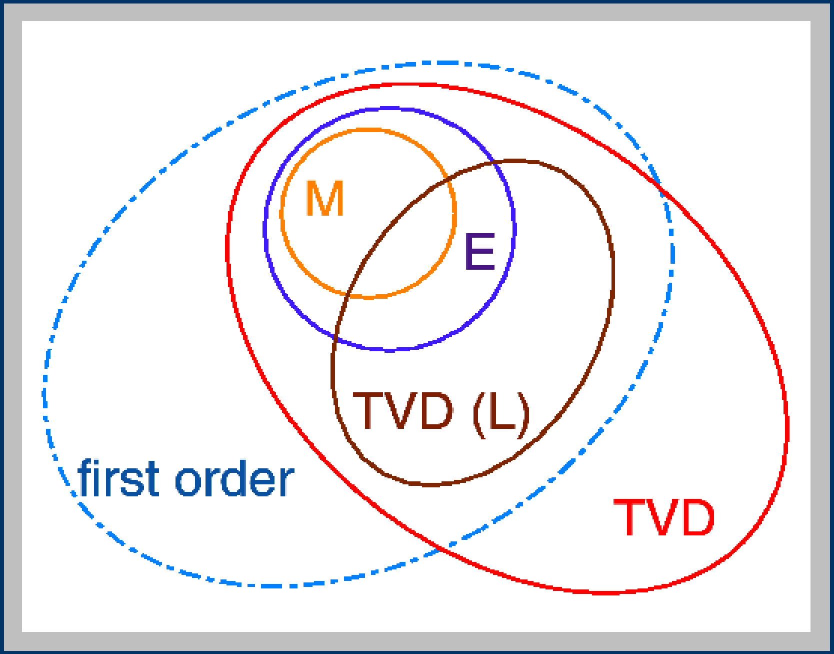
$$C_{j-\frac{1}{2}} = \frac{a\Delta t}{\Delta x} \le 1$$

Stability-like condition !

TVD Methods

Godunov's Theorem

No second or higher order accurate constant coefficient (**linear**) scheme can be **TVD**.



$\Rightarrow$ **Higher order TVD schemes must be non-linear**

TVD Methods

Consider the linear equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \qquad a > 0$$

First order upwind (Godunov) scheme is

$$\hat{u}_j^{n+1} = \hat{u}_j^n - C\left(\hat{u}_j^n - \hat{u}_{j-1}^n\right)$$

$$C = \frac{a\Delta t}{\Delta x}$$

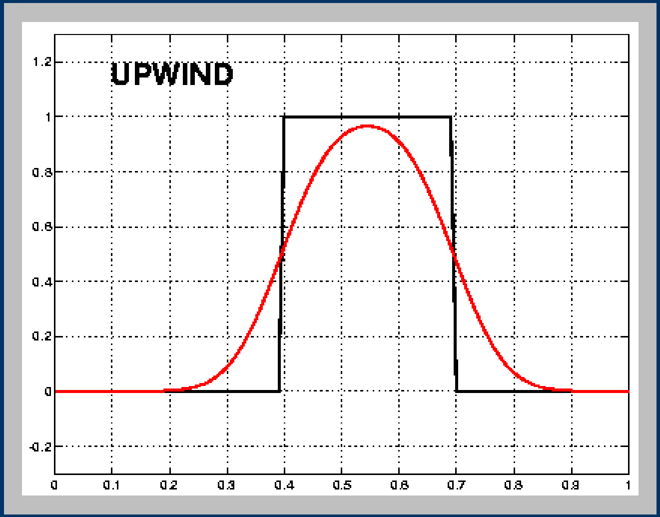
Oscillation free but smeared solutions.

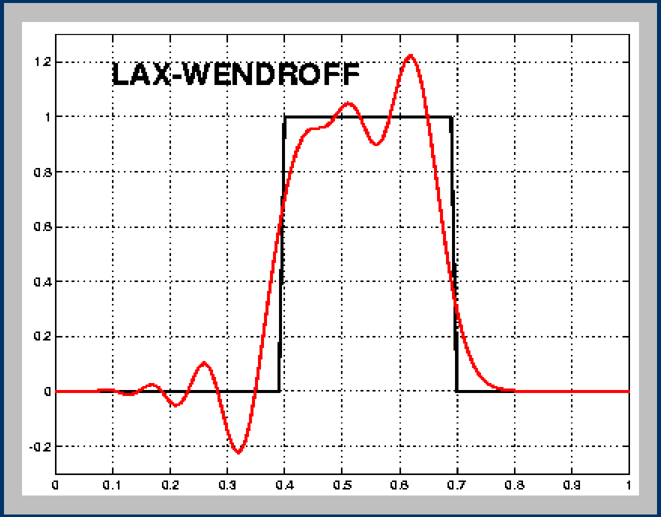
TVD Methods

Lax-Wendroff

$$\hat{u}_j^{n+1} = \hat{u}_j^n - \frac{C}{2}\left(\hat{u}_{j+1}^n - \hat{u}_{j-1}^n\right) + \frac{C^2}{2}\left(\hat{u}_{j+1}^n - 2\hat{u}_j^n + \hat{u}_{j-1}^n\right)$$

Suffers from oscillations.





TVD Methods

High Resolution Schemes

Anti-diffusion

Re-write the Lax-Wendroff scheme :

$$\hat{u}_j^{n+1} = \underbrace{\hat{u}_j^n - C\left(\hat{u}_j^n - \hat{u}_{j-1}^n\right)}_{\text{first order upwind}} - \underbrace{\frac{1}{2}C(1-C)\left(\hat{u}_{j+1}^n - 2\hat{u}_j^n + \hat{u}_{j-1}^n\right)}_{\text{anti-diffusive flux}}$$

$$F_{j+\frac{1}{2}}^{LW} = a\hat{u}_j + \frac{a}{2}(1-C)\left(\hat{u}_{j+1} - \hat{u}_j\right)$$

Introduce **flux limiter** $\phi_{j+\frac{1}{2}}$:

$$F_{j+\frac{1}{2}}^{TVD} = a\hat{u}_j + \frac{a}{2}(1-C)\,\phi_{j+\frac{1}{2}}\left(\hat{u}_{j+1} - \hat{u}_j\right)$$

TVD Methods

High Resolution Schemes

Flux Limiters...

$$\hat{u}_j^{n+1} = \hat{u}_j^n - C\left(\hat{u}_j^n - \hat{u}_{j-1}^n\right)$$

$$- \frac{1}{2}C(1-C)\left[\phi_{j+\frac{1}{2}}\left(\hat{u}_{j+1}^n - \hat{u}_j^n\right) - \phi_{j-\frac{1}{2}}\left(\hat{u}_j^n - \hat{u}_{j-1}^n\right)\right]$$

If $\phi_j = \phi_{j-1} = 1 \Rightarrow$ Lax-Wendroff (not TVD)

If $\phi_j = \phi_{j-1} = 0 \Rightarrow$ Upwind (TVD)

Choose the limiter **as close as possible to 1 but enforcing TVD conditions**

TVD Methods

High Resolution Schemes

...Flux Limiters...

Re-write

$$\hat{u}_j^{n+1} = \hat{u}_j^n - C\Delta\hat{u}_{j-\frac{1}{2}} - \frac{1}{2}C(1-C)(\phi_{j+\frac{1}{2}}\Delta\hat{u}_{j+\frac{1}{2}} - \phi_{j-\frac{1}{2}}\Delta\hat{u}_{j-\frac{1}{2}})$$

$$= u_j^n - C\left\{1 + \frac{1}{2}(1-C)\left[\frac{\phi_{j+\frac{1}{2}}}{r_{j+\frac{1}{2}}} - \phi_{j-\frac{1}{2}}\right]\right\}\Delta\hat{u}_{j-\frac{1}{2}}$$

$$r_{j+\frac{1}{2}} = \Delta\hat{u}_{j-\frac{1}{2}}/\Delta\hat{u}_{j+\frac{1}{2}}$$

Recall the TVD test:

$$\hat{u}_j^{n+1} = \hat{u}_j^n + D_{j+\frac{1}{2}}\Delta\hat{u}_{j+\frac{1}{2}}^n - C_{j-\frac{1}{2}}\Delta\hat{u}_{j-\frac{1}{2}}^n$$

TVD Methods

High Resolution Schemes

...Flux Limiters

Take

$$C_{j+\frac{1}{2}} = C\left\{1 + \frac{1}{2}(1 - C)\left[\frac{\phi_{j+\frac{1}{2}}}{r_{j+\frac{1}{2}}} - \phi_{j-\frac{1}{2}}\right]\right\}$$

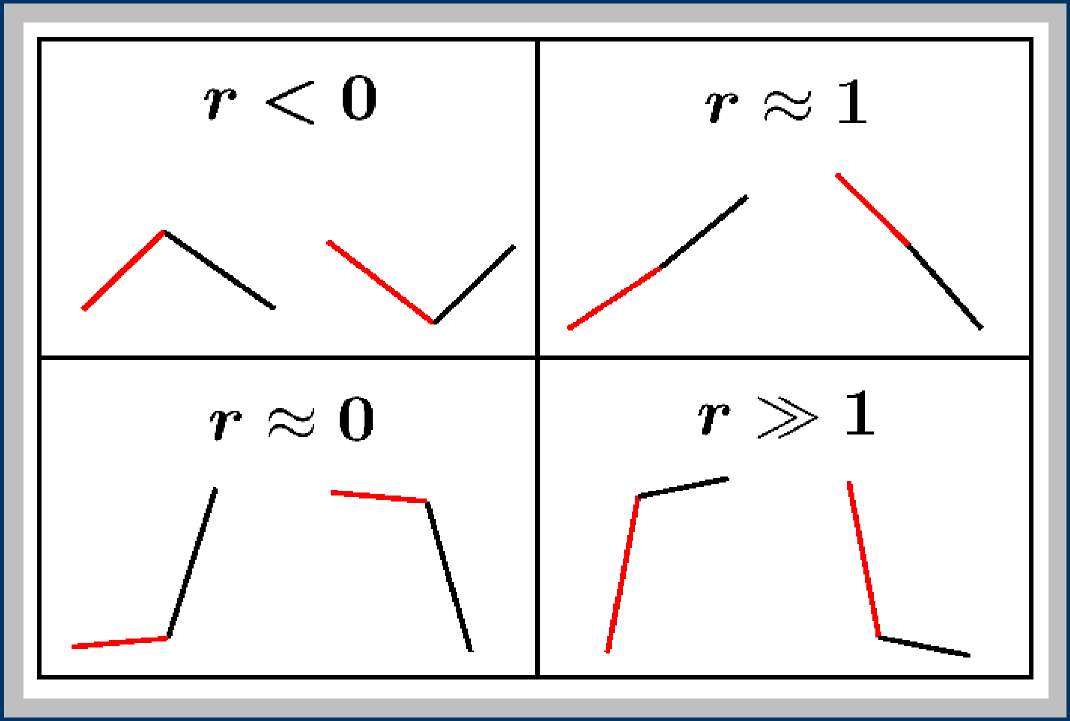
$$D_{j+\frac{1}{2}} = 0$$

TVD criterion $\Rightarrow \quad 0 \leq C_{j+\frac{1}{2}} \leq 1$

TVD Methods

Smoothness Monitor

Choose $\phi_{j+\frac{1}{2}}$ to be function of $r_{j+\frac{1}{2}}$



TVD Methods

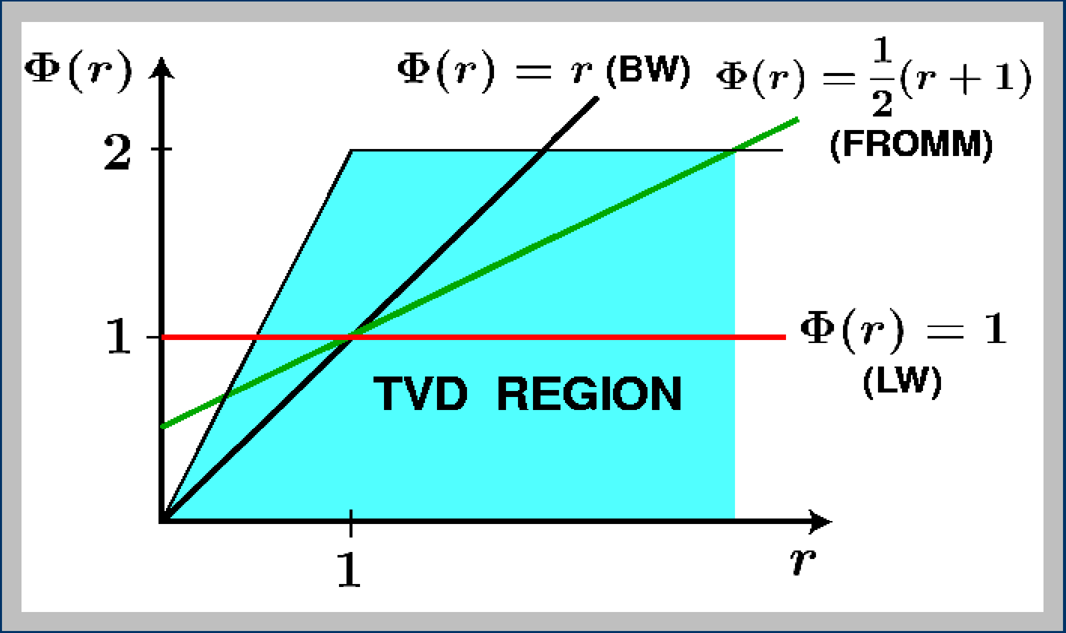
TVD region

It can be seen that the above TVD conditions are satisfied if

$$\phi(r) = 0 \quad r \leq 0$$

$$0 \leq \frac{\phi(r)}{r} \leq 2$$

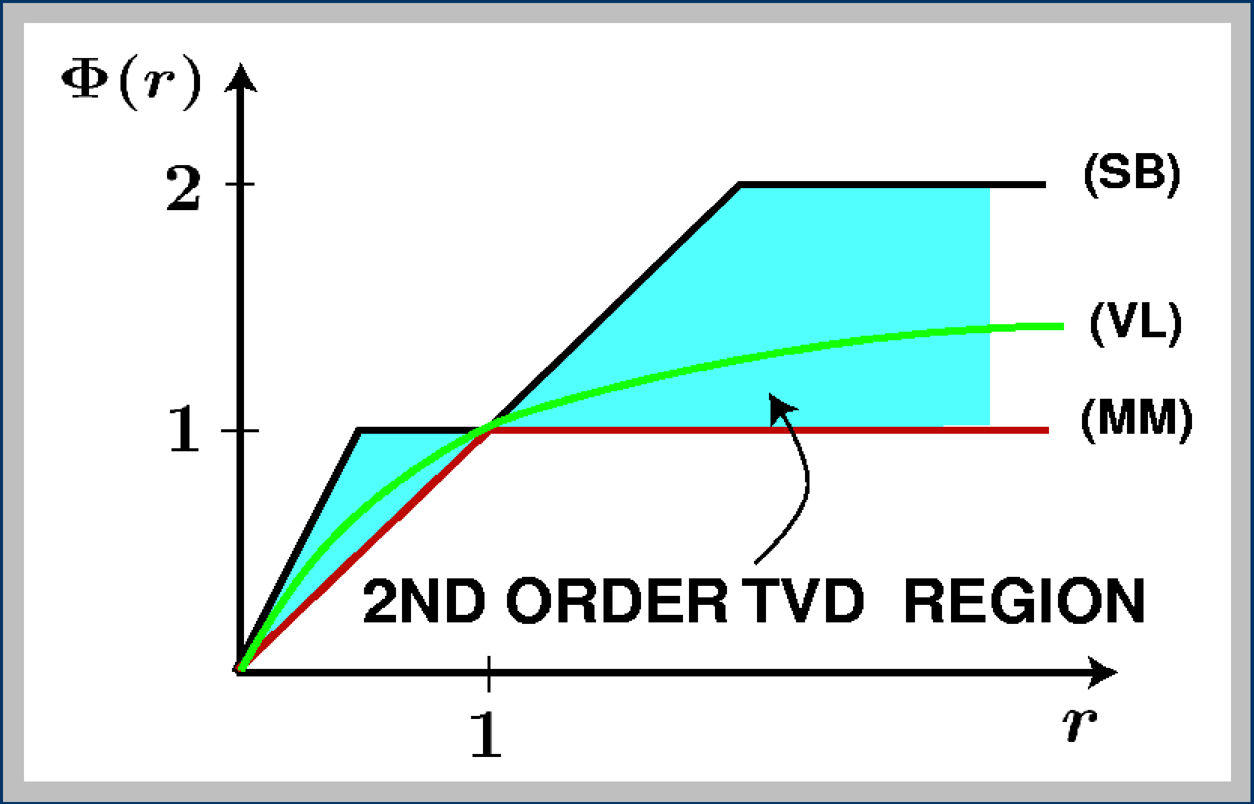
$$0 \leq \phi(r) \leq 2$$



TVD Methods

High Resolution Schemes

2nd Order TVD Region



TVD Methods

Popular Choices

Minmod $\phi(r) = \max(0, \min(1, r))$

Superbee $\phi(r) = \max(0, \min(2r, 1), \min(r, 2))$

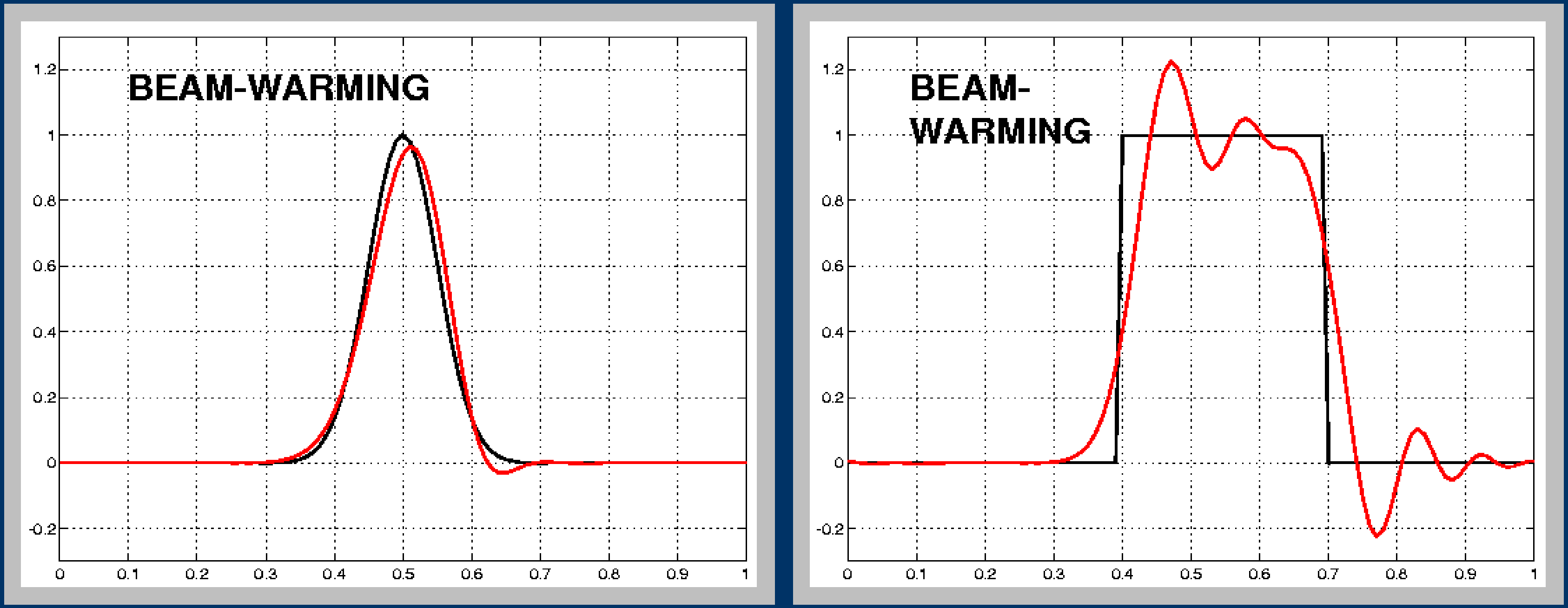
Van Leer $\phi(r) = \dfrac{r + |r|}{1 + |r|}$

All produce **second order** schemes when the solution is smooth, and reduce to **upwind** at **discontinuities**.

TVD Methods

High Resolution Schemes

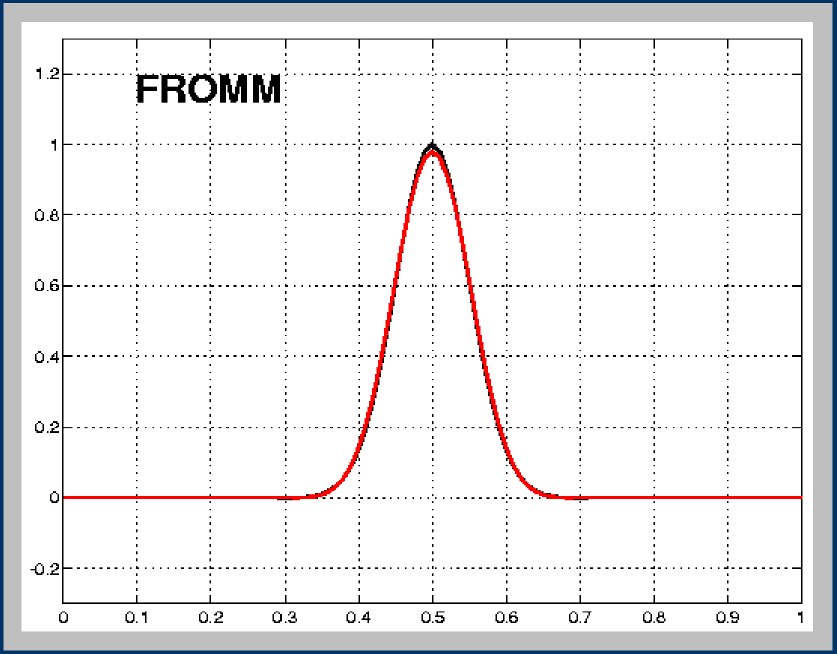
Examples...

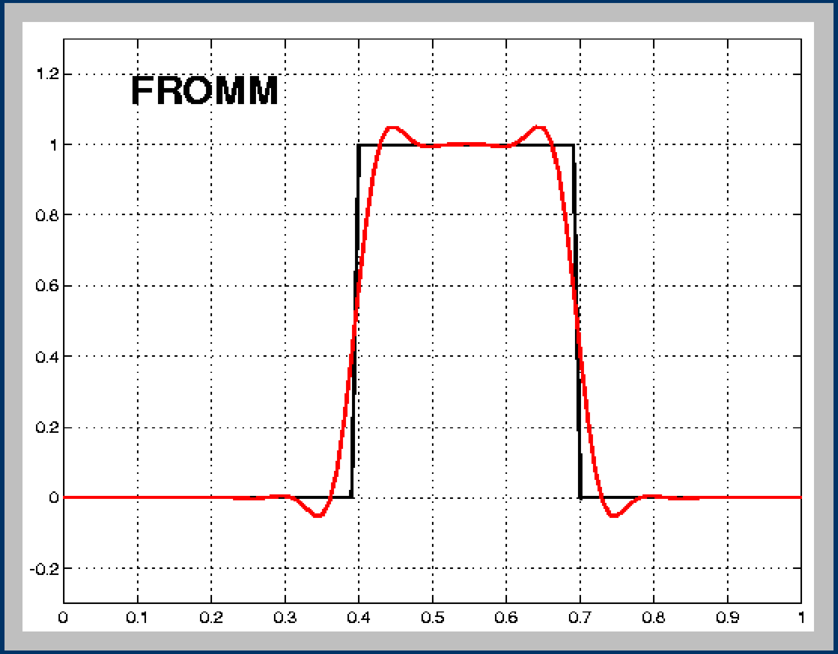


TVD Methods

High Resolution Schemes

...Examples...

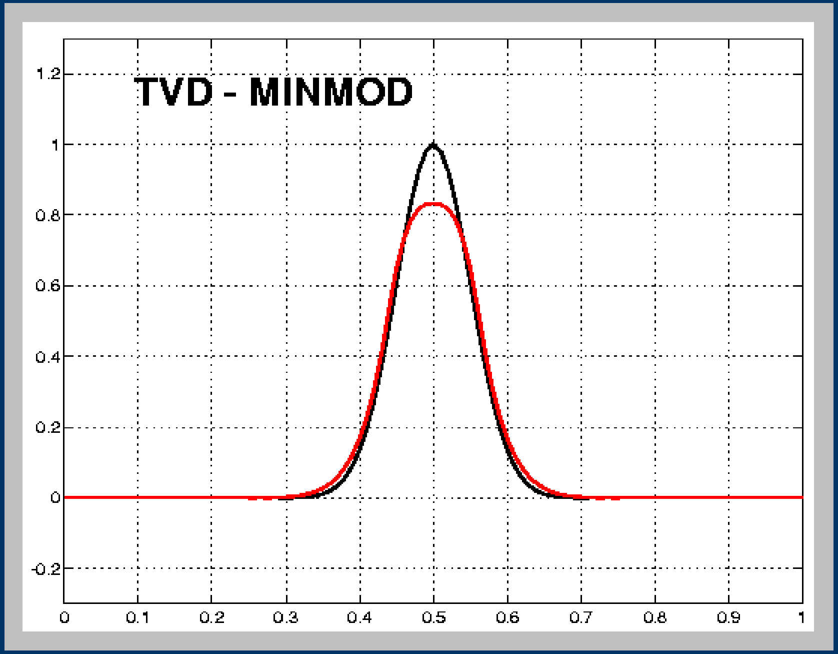


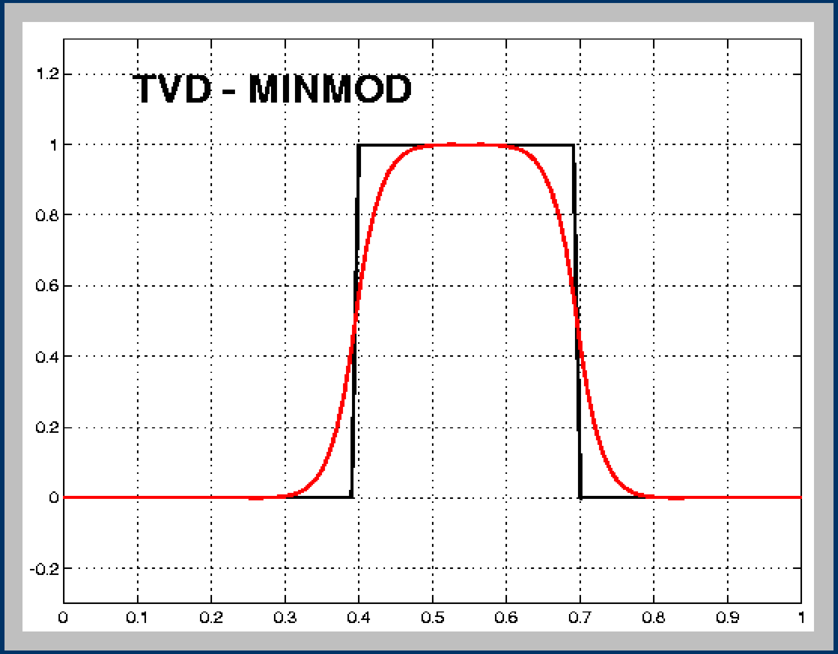


TVD Methods

High Resolution Schemes

...Examples...

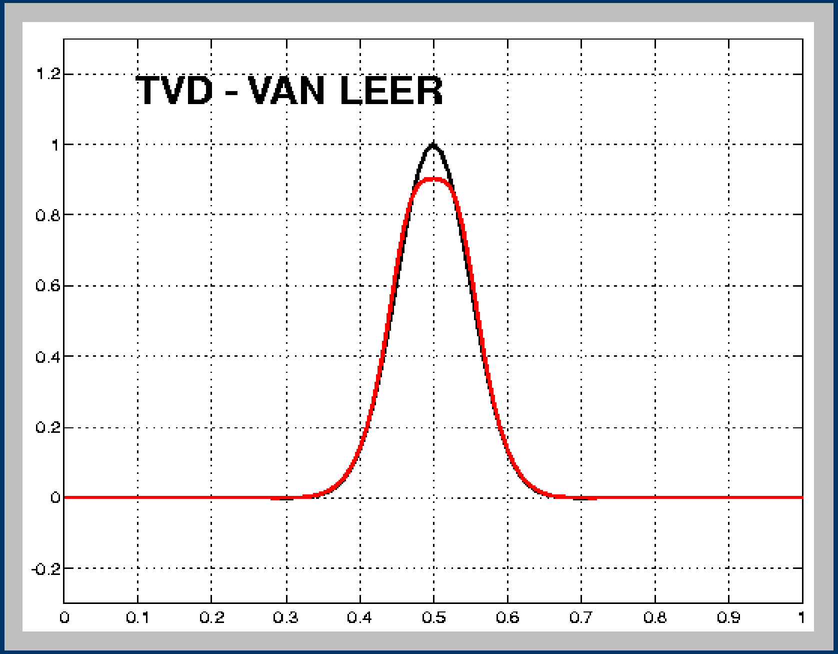


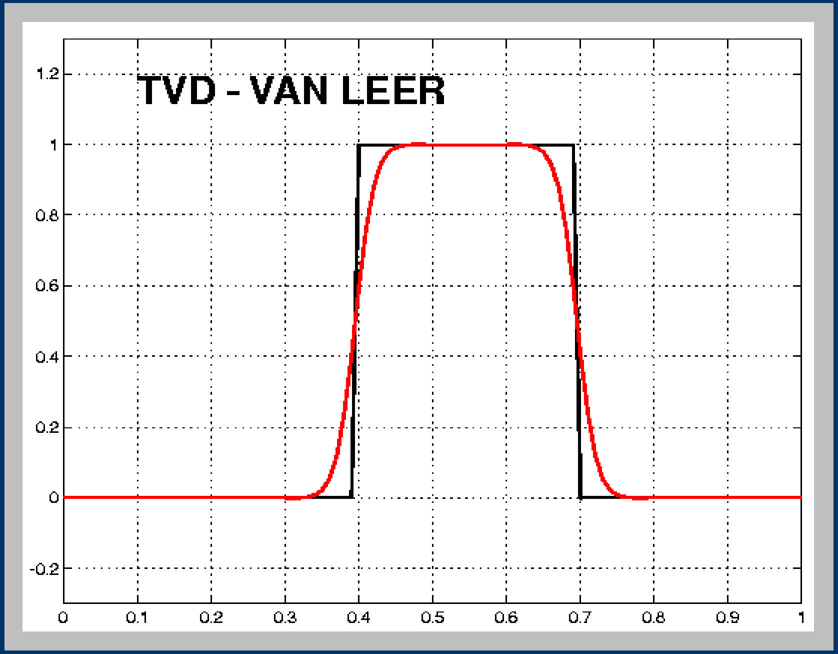


High Resolution Schemes

TVD Methods

...Examples...

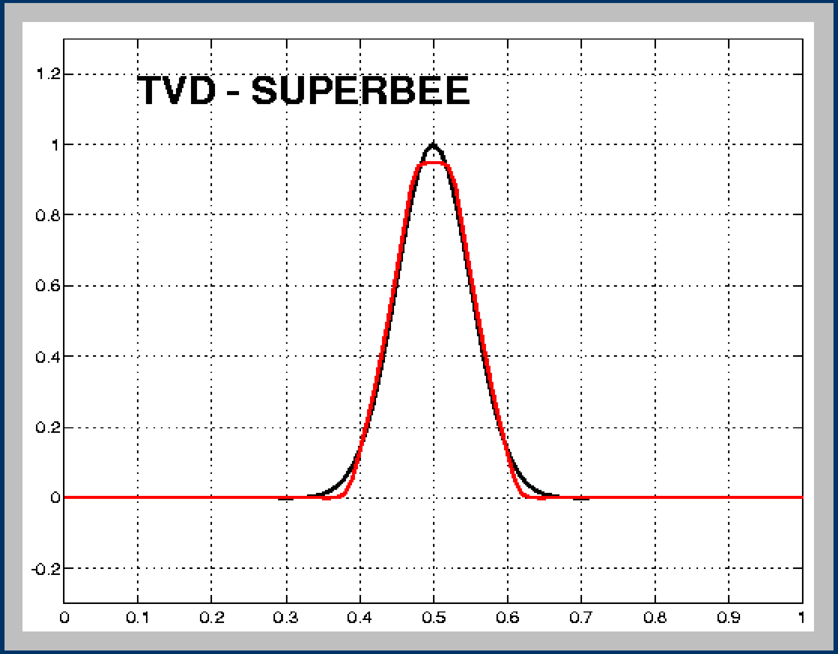


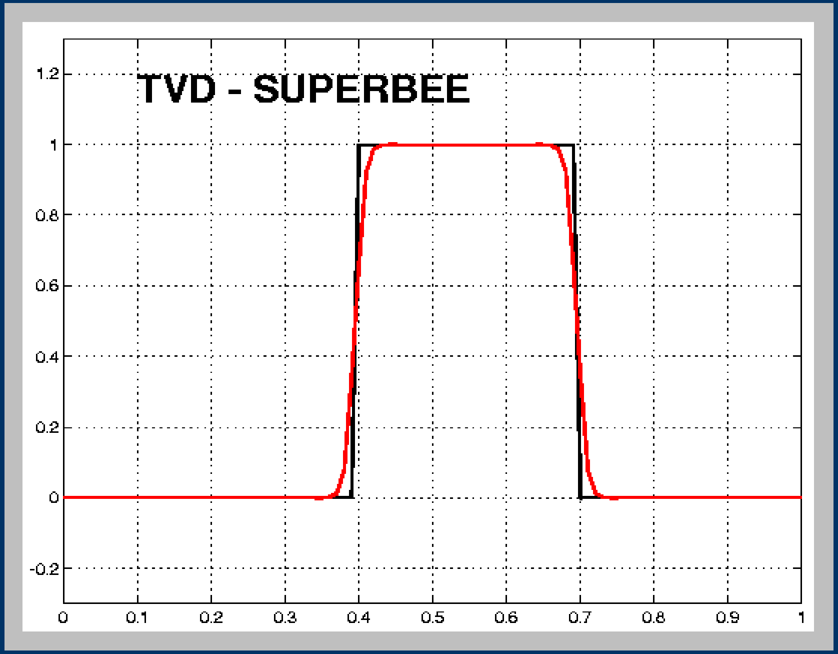


TVD Methods

High Resolution Schemes

...Examples





TVD Methods

Non-linear extension

For a non-linear conservation law the formulation of flux limiters is extended to allow both positive and negative wave speeds