Finite Difference Discretization of Hyperbolic Equations: Linear Problems

Lectures 8, 9 and 10

First Order Wave Equation

INITIAL BOUNDARY VALUE PROBLEM (IBVP)

$$\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} = 0, \quad x \in (0,1)$$

Initial condition: $u(x,0) = u^0(x)$

Boundary conditions: $\begin{cases} u(0,t) = g_0(t) & \text{if } U > 0 \\ u(1,t) = g_1(t) & \text{if } U < 0 \end{cases}$

First Order Wave Equation

Solution

$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx = \left(\frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x}\right)dt$$

If $\frac{dx}{dt} = U \quad \Rightarrow \quad \boxed{x = Ut + \xi}$ Characteristics

$$\Downarrow$$

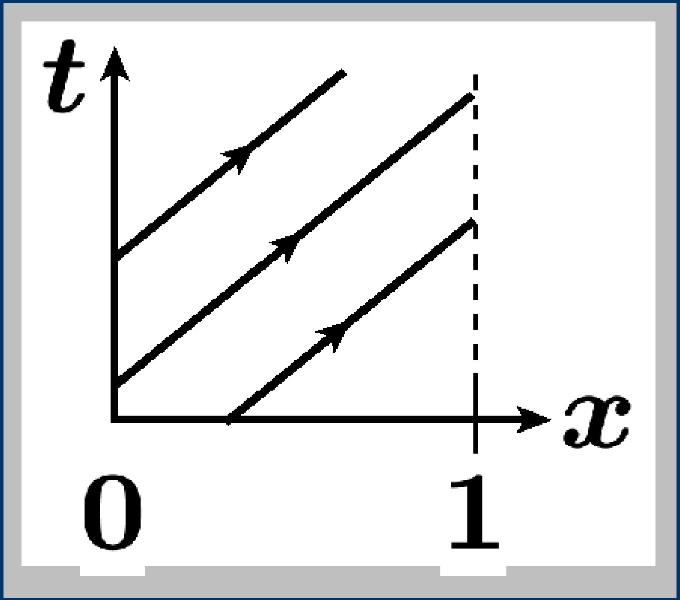
$du = 0, \quad \Rightarrow \quad \boxed{u(x,t) = f(\xi) = f(x - Ut)}$

General solution

First Order Wave Equation

Solution

$U > 0$

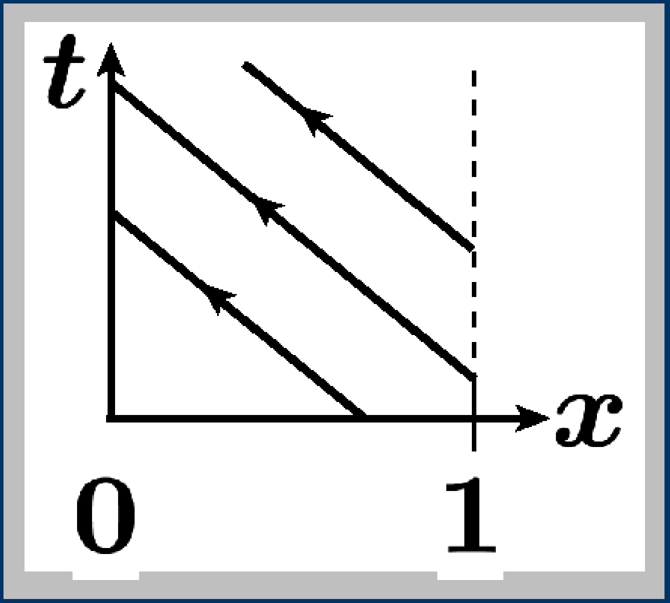


$$u(x,t) = \begin{cases} u^0(x - Ut), & \text{if } x - Ut > 0 \\ g_0(t - x/U), & \text{if } x - Ut < 0 \end{cases}$$

First Order Wave Equation

Solution

$U < 0$



$$u(x,t) = \begin{cases} u^0(x - Ut), & \text{if } x - Ut < 1 \\ g_1(t - x/U), & \text{if } x - Ut > 1 \end{cases}$$

First Order Wave Equation

Stability

$L^2([0,1])$-norm

$$\|u\|_2(t) = \left(\int_0^1 u^2(x,t)\,dx\right)^{\frac{1}{2}}$$

$$\int_0^1 u\left(\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x}\right)\,dx = 0$$

$$\boxed{\frac{d}{dt}\|u\|_2^2 = -U\left(u^2(1,t) - u^2(0,t)\right)}$$

Model Problem

$$\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} = 0, \quad x \in (0, 1)$$

Initial condition: $u(x, 0) = u^0(x)$

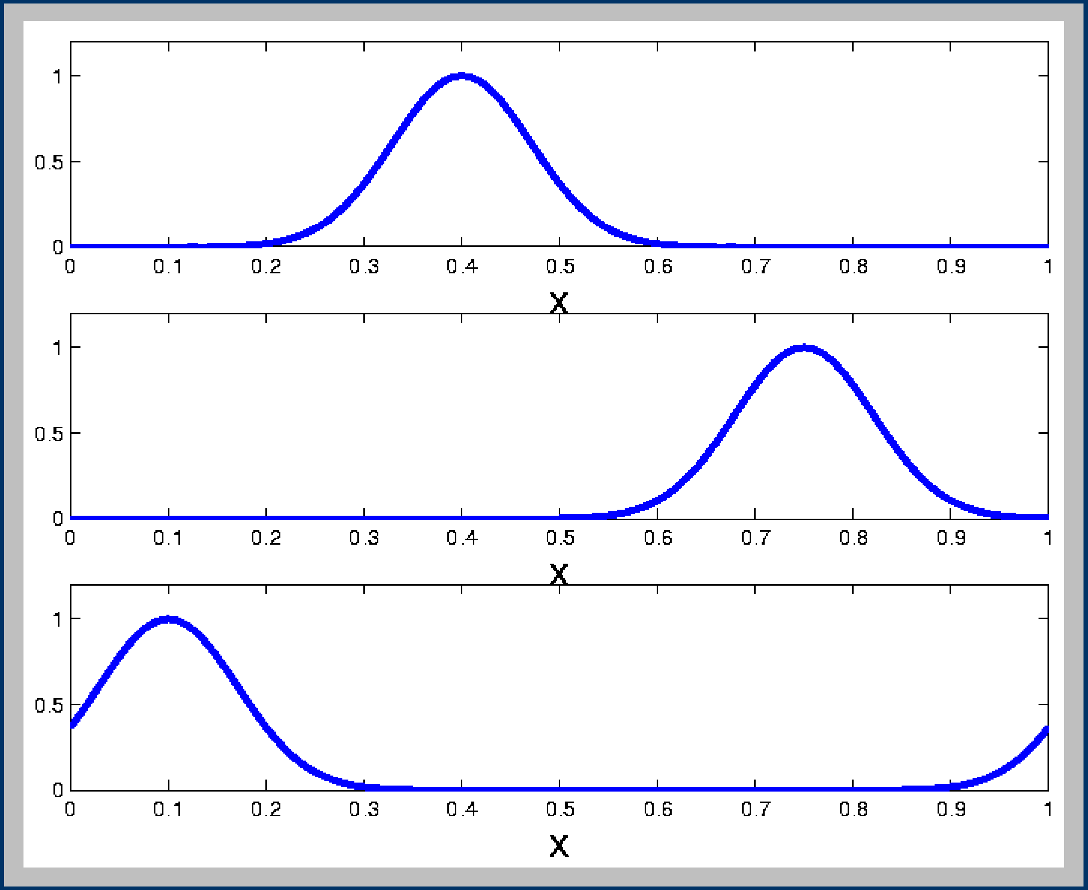
Periodic Boundary conditions: $u(0, t) = u(1, t)$

$$\frac{d}{dt} \|u\|_2^2 = 0 \quad \Rightarrow \quad \|u\|_2(t) = \|u^0\|_2 = \text{constant}$$

Model Problem

Example

Periodic Solution ($U > 0$)



$t = 0$

$t = T$

$t = 2T$

Finite Difference Solution

Discretization

Discretize $(0,1)$ into J equal intervals Δx

$$\Delta x = \frac{1}{J}, \qquad x_j = j\Delta x$$

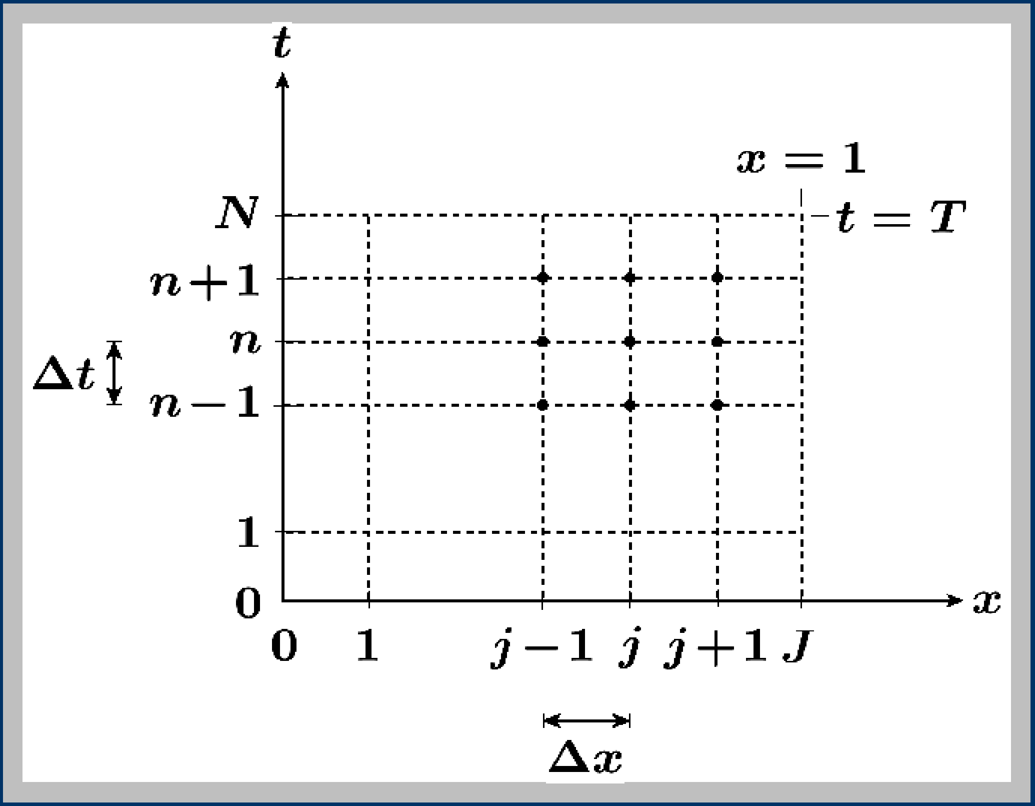
and $(0,T)$ into N equal intervals Δt

$$\Delta t = \frac{T}{N}, \qquad t^n = n\Delta t$$

$$\hat{u}_j^n \approx u_j^n \equiv u(x_j, t^n), \qquad \text{for } \begin{cases} 0 \le j \le J \\ 0 \le n \le N \end{cases}$$

Finite Difference Solution

Discretization



Finite Difference Solution

Discretization

NOTATION:

- $\hat{v}_j^n$ approximation to $v(x_j, t^n) \equiv v_j^n$
- $\underline{\hat{v}}^n \in \mathbb{R}^J$ vector of approximate values at time n;

$$\underline{\hat{v}}^n = \{\hat{v}_j^n\}_{j=1}^J$$

- $\underline{v}^n \in \mathbb{R}^J$ vector of exact values at time n;

$$\underline{v}^n = \{v(x_j, t^n)\}_{j=1}^J$$

Finite Difference Solution

Approximation

For example . . . (for $U > 0$)

$$\left.\frac{\partial v}{\partial x}\right|_j^n \approx \frac{v(x_j, t^n) - v(x_{j-1}, t^n)}{\Delta x} = \frac{v_j^n - v_{j-1}^n}{\Delta x}$$

$$\left.\frac{\partial v}{\partial t}\right|_j^n \approx \frac{v(x_j, t^{n+1}) - v(x_j, t^n)}{\Delta t} = \frac{v_j^{n+1} - v_j^n}{\Delta t}$$

Forward in Time Backward (Upwind) in Space

Finite Difference Solution

First Order Upwind Scheme

$u_t + U u_x = 0$ suggests . . .

$$\frac{\hat{u}_j^{n+1} - \hat{u}_j^n}{\Delta t} + U \frac{\hat{u}_j^n - \hat{u}_{j-1}^n}{\Delta x} = 0 \quad \Rightarrow$$

$$\hat{u}_j^{n+1} = \hat{u}_j^n - C(\hat{u}_j^n - \hat{u}_{j-1}^n) \qquad \begin{cases} 1 \le j \le J \\ 0 \le n \le N \end{cases}$$

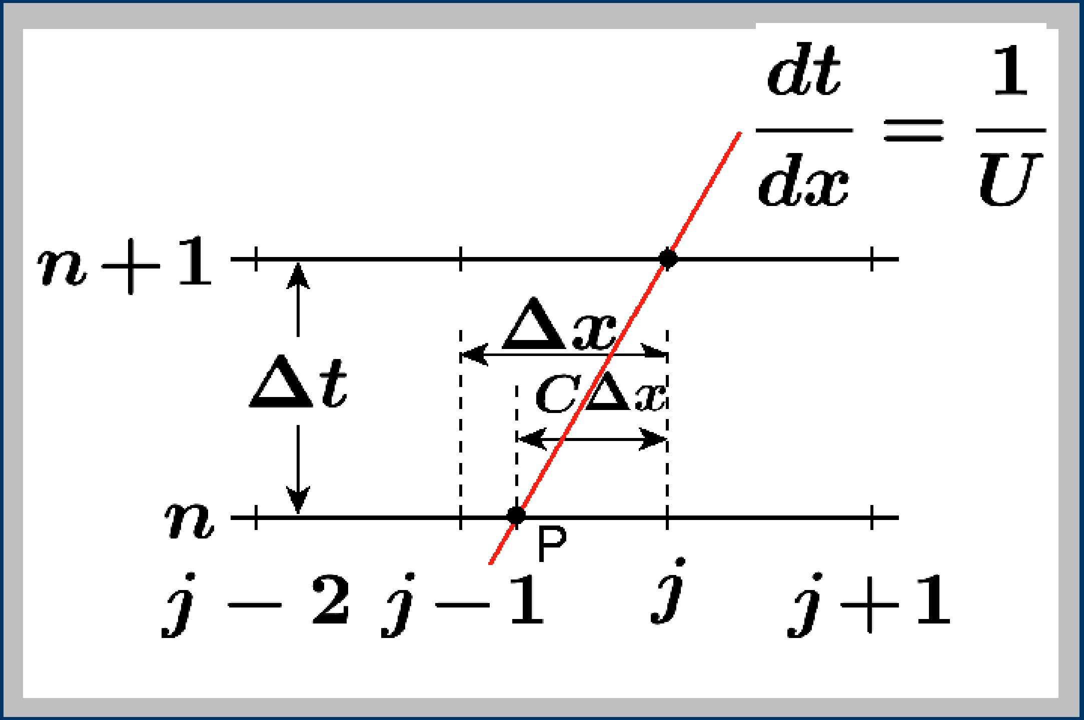
$$\hat{u}_0^n = \hat{u}_J^n \qquad 0 \le n \le N$$

Courant number $C = U\Delta t / \Delta x$

Finite Difference Solution

First Order Upwind Scheme

Interpretation



$$u_j^{n+1} = u_P$$

Use Linear Interpolation

$j-1,\ j$

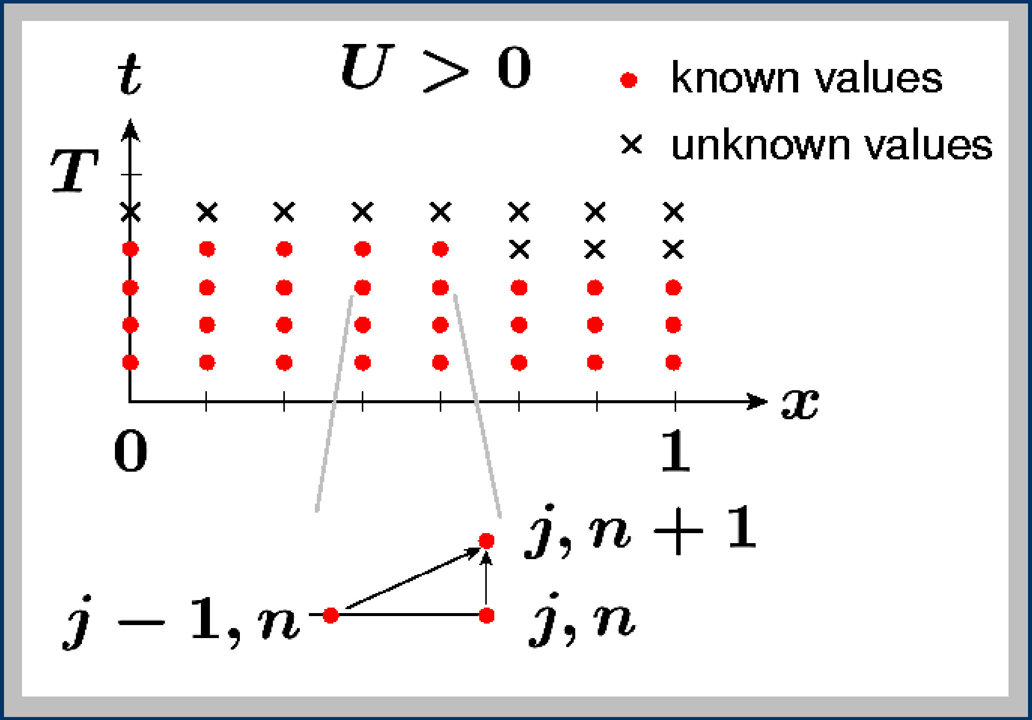
N1

$$u_P \approx C\hat{u}_{j-1}^n + (1-C)\hat{u}_j^n$$

Finite Difference Solution

First Order Upwind Scheme

Explicit Solution



no matrix inversion

$\hat{u}^n$ **exists** and is **unique**

$$\hat{u}_j^{n+1} = \hat{u}_j^n - C(\hat{u}_j^n - \hat{u}_{j-1}^n)$$

Finite Difference Solution

First Order Upwind Scheme

Matrix Form

We can write

$$\underline{\hat{u}}^n = \hat{S}\,\underline{\hat{u}}^{n-1} = \hat{S}^n\,\underline{\hat{u}}^0$$

$$\underline{\hat{u}}^0 \equiv \underline{u}^0$$

$$\underbrace{\begin{pmatrix} (1-C) & 0 & 0 & \cdots & C \\ C & (1-C) & 0 & \ddots & 0 \\ 0 & \ddots & \ddots & \vdots & 0 \\ \vdots & \ddots & C & (1-C) & 0 \\ 0 & \cdots & 0 & C & (1-C) \end{pmatrix}}_{\hat{S}}$$

Finite Difference Solution

First Order Upwind Scheme

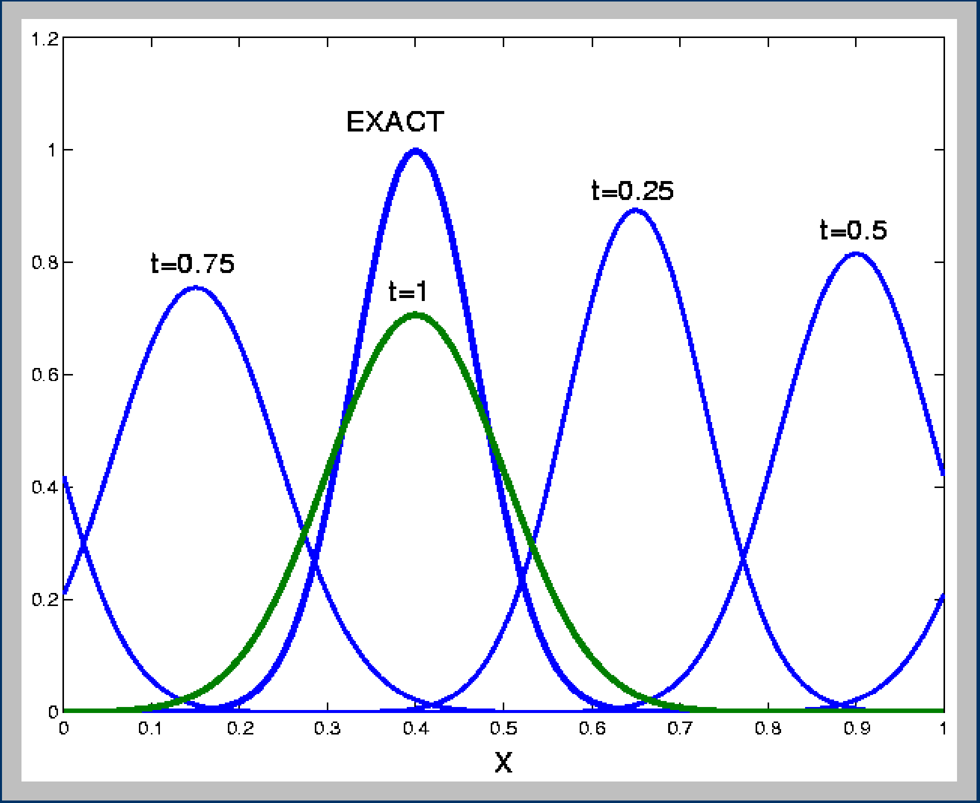
Example

$$u_t + u_x = 0$$

$$\Delta x = \frac{1}{100}$$

$$C = \frac{\Delta t}{\Delta x} = 0.5$$

$$T = 1 \Rightarrow N = 200$$



Convergence

Definition

The finite difference algorithm **converges** if

$$\lim_{\substack{\Delta x, \Delta t \to 0 \\ N\Delta t = T \\ J\Delta x = 1}} \|\underline{\hat{u}}^n - \underline{u}^n\| = 0, \qquad 1 \leq n \leq N$$

for **any** initial condition $u^0(x)$.

$$\|\underline{v}\| = \left(\Delta x \sum_{j=1}^{J} v_j^2\right)^{1/2} = \sqrt{\Delta x}\, \|\underline{v}\|_2$$

N2

Consistency

Definition

The difference scheme $\hat{\mathcal{L}}\underline{\hat{u}}^n = \mathbf{0}$,

is **consistent** with the differential equation $\mathcal{L}u = 0$

If:

For **all** smooth functions v

$$(\hat{\mathcal{L}}\,\underline{v}^n)_j - (\mathcal{L}\,v)_j^n \to 0, \quad \text{for} \begin{cases} 1 \leq j \leq J \\ 1 \leq n \leq N \end{cases}$$

when $\Delta x, \Delta t \to 0$.

Consistency

First Order Upwind Scheme

Difference operator

$$\hat{\mathcal{L}}\underline{v}^n = \frac{1}{\Delta t}\{\underline{v}^{n+1} - \hat{\mathcal{S}}\underline{v}^n\}$$

Differential operator

$$\mathcal{L}v \equiv \frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x}$$

Consistency

First Order Upwind Scheme

$$(\hat{\mathcal{L}}\underline{v}^n)_j \equiv \frac{v_j^{n+1} - v_j^n}{\Delta t} + U\frac{v_j^n - v_{j-1}^n}{\Delta x}$$

$$= (v_t + Uv_x)_j^n + \frac{\Delta t}{2}(v_{tt})_j^n + U\frac{\Delta x}{2}(v_{xx})^n + \ldots$$

$$(\mathcal{L}v)_j^n \equiv (v_t + Uv_x)_j^n$$

$$\boxed{(\hat{\mathcal{L}}\,\underline{v}^n)_j - (\mathcal{L}\,v)_j^n = \mathcal{O}(\Delta x, \Delta t)}$$

$\Rightarrow$ First order accurate in space and time.

Truncation Error

Insert exact solution $\underline{u}$ into difference scheme

$$(\hat{\mathcal{L}}\,\underline{u})_j^n - \underbrace{(\mathcal{L}\,\underline{u})_j^n}_{=0} = \tau_j^n, \quad \text{for } \begin{cases} 1 \le j \le J \\ 1 \le n \le N \end{cases}$$

$$\underline{u}^{n+1} = \hat{\mathcal{S}}\underline{u}^n + \Delta t\, \underline{\tau}^n$$

Consistency $\Rightarrow \|\underline{\tau}^n\| = \mathcal{O}(\Delta x, \Delta t), \;\; 1 \le n \le N$

Stability

Definition

The difference scheme $\underline{\hat{u}}^{n+1} = \hat{S}\underline{\hat{u}}^n$ is **stable** if:

there exists C_T such that

$$\|\underline{v}^n\| = \|\hat{S}^n\,\underline{v}^0\| \leq C_T\|\underline{v}^0\|$$

for all $\underline{v}^0$; and n, Δt such that $0 \leq n\Delta t \leq T$

Above condition can be written as

$$\|\hat{S}\,\underline{v}\| \leq (1 + \mathcal{O}(\Delta t))\,\|\underline{v}\|$$

Stability

First Order Upwind Scheme

$$\hat{u}_j^{n+1} = \hat{u}_j^n - C(\hat{u}_j^n - \hat{u}_{j-1}^n)$$

$$= (1 - C)\,\hat{u}_j^n + C\,\hat{u}_{j-1}^n$$

$$= \alpha\,\hat{u}_j^n + \beta\,\hat{u}_{j-1}^n$$

First Order Upwind Scheme

Stability

$$\sum_{j=1}^{J} |\hat{u}_j^{n+1}|^2 = \sum_{j=1}^{J} |\alpha \hat{u}_j^n + \beta \hat{u}_{j-1}^n|^2$$

$$\leq \sum_{j=1}^{J} |\alpha|^2 |\hat{u}_j^n|^2 + 2|\alpha||\beta||\hat{u}_j^n||\hat{u}_{j-1}^n| + |\beta|^2 |\hat{u}_{j-1}^n|^2$$

$$\leq \sum_{j=1}^{J} |\alpha|^2 |\hat{u}_j^n|^2 + |\alpha||\beta|(|\hat{u}_j^n|^2 + |\hat{u}_{j-1}^n|^2) + |\beta|^2 |\hat{u}_{j-1}^n|^2$$

$$= \sum_{j=1}^{J} (|\alpha|^2 + 2|\alpha||\beta| + |\beta|^2) |\hat{u}_j^n|^2 = (|\alpha| + |\beta|)^2 \sum_{j=1}^{J} |\hat{u}_j^n|^2$$

Stability

$$\|\underline{u}^{n+1}\|_2^2 \leq (|\alpha| + |\beta|)^2 \|\underline{u}^n\|_2^2$$

Stability if

$$|\alpha| + |\beta| \leq 1, \quad \Rightarrow$$

$$|1 - C| + |C| \leq 1, \qquad 0 \leq C \leq 1$$

Upwind scheme is **stable** provided

$$\boxed{U > 0, \qquad \Delta t \leq \frac{\Delta x}{U}}$$

Lax Equivalence Theorem

A **consistent** finite difference scheme for a partial differential equation for which the initial value problem is well-posed is **convergent** if and only if it is **stable**.

Lax Equivalence Theorem

Proof

$$\|\underline{\hat{u}}^n - \underline{u}^n\| = \|\hat{\mathcal{S}}\underline{\hat{u}}^{n-1} - \hat{\mathcal{S}}\underline{u}^{n-1} + \Delta t\, \underline{\tau}^{n-1}\|$$

$$\leq \|\hat{\mathcal{S}}(\underline{\hat{u}}^{n-1} - \underline{u}^{n-1})\| + \Delta t\, \mathcal{O}(\Delta x, \Delta t)$$

$$\leq \|\underline{\hat{u}}^{n-1} - \underline{u}^{n-1}\| + \Delta t\, \mathcal{O}(\Delta x, \Delta t)$$

$$\vdots$$

$$\leq \underbrace{\|\underline{\hat{u}}^0 - \underline{u}^0\|}_{=0} + \underbrace{n\Delta t}_{\leq T}\, \mathcal{O}(\Delta x, \Delta t)$$

$$\leq \mathcal{O}(\Delta x, \Delta t)$$ (first order in Δx, Δt)

Lax Equivalence Theorem

First Order Upwind Scheme

- Consistency: $\|\underline{\tau}\| = \mathcal{O}(\Delta x, \Delta t)$
- Stability: $\|\underline{\hat{u}}^{n+1}\| \leq \|\underline{\hat{u}}^{n}\|$ for $C \equiv U\Delta t/\Delta x \leq 1$
- $\Rightarrow$ **Convergence**

$\underline{e} = \underline{u} - \underline{\hat{u}}$

$$\|\underline{e}^n\| \leq (C_x \Delta x + C_t \Delta t), \; 1 \leq n \leq N$$

or $$|e_j^n| \leq (C_x \Delta x + C_t \Delta t), \begin{cases} 1 \leq j \leq J, \\ 1 \leq n \leq N \end{cases}$$

C_x and C_t are constants independent of Δx, Δt

Lax Equivalence Theorem

First Order Upwind Scheme

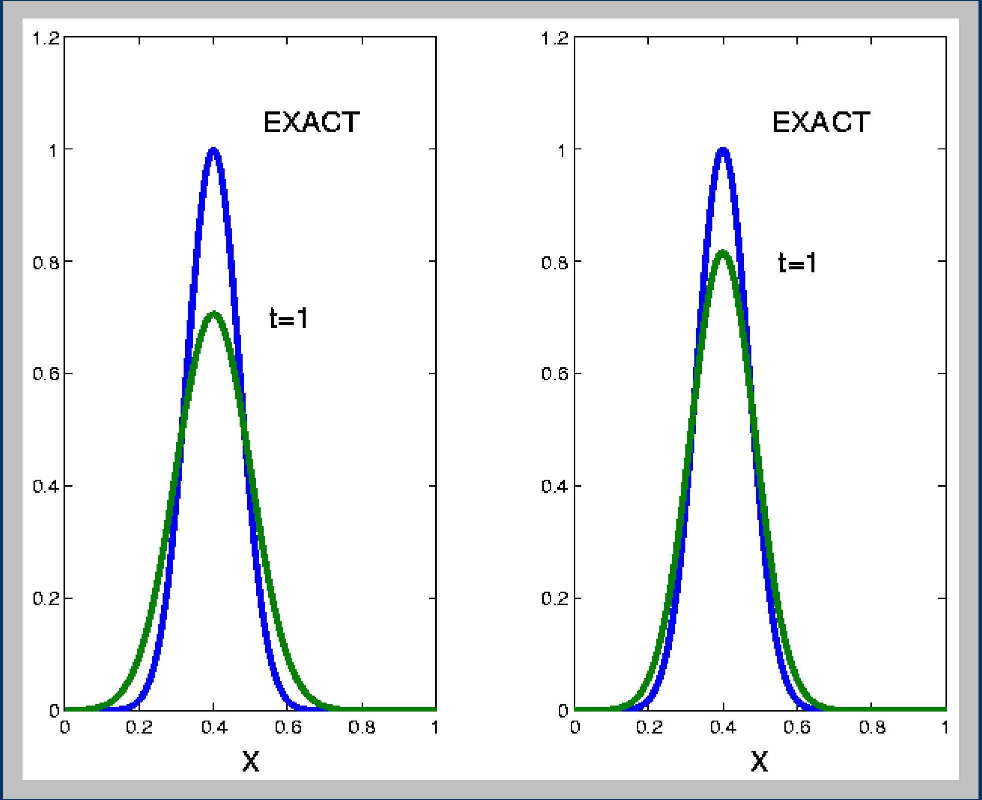
Example

Solutions for:

$$C = 0.5$$

$\Delta x = 1/100$ (left)
$\Delta x = 1/200$ (right)

Convergence is slow !!



Domains of Dependence

Mathematical Domain of Dependence of $u(x_j, t^N)$

Set of points in (x, t) where the initial or boundary data may have some effect on $u(x_j, t^N)$.

Numerical Domain of Dependence of $\hat{u}_j^N$

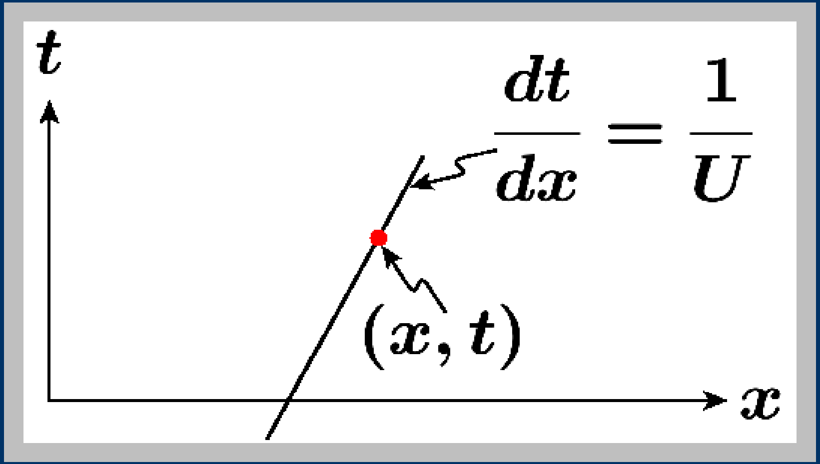
Set of points x_k, t^n where the initial or boundary data may have some effect on $\hat{u}_j^N$.

N3

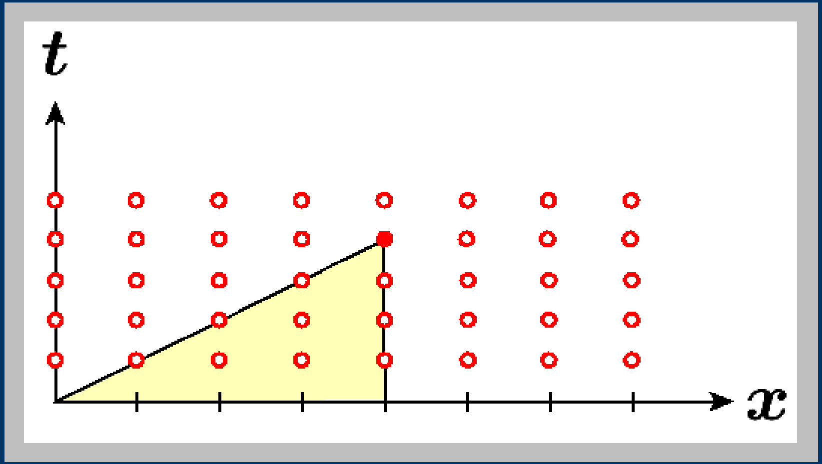
CFL Condition

Domains of Dependence

First Order Upwind Scheme



Analytical



Numerical $(U > 0)$

CFL Condition

CFL Condition

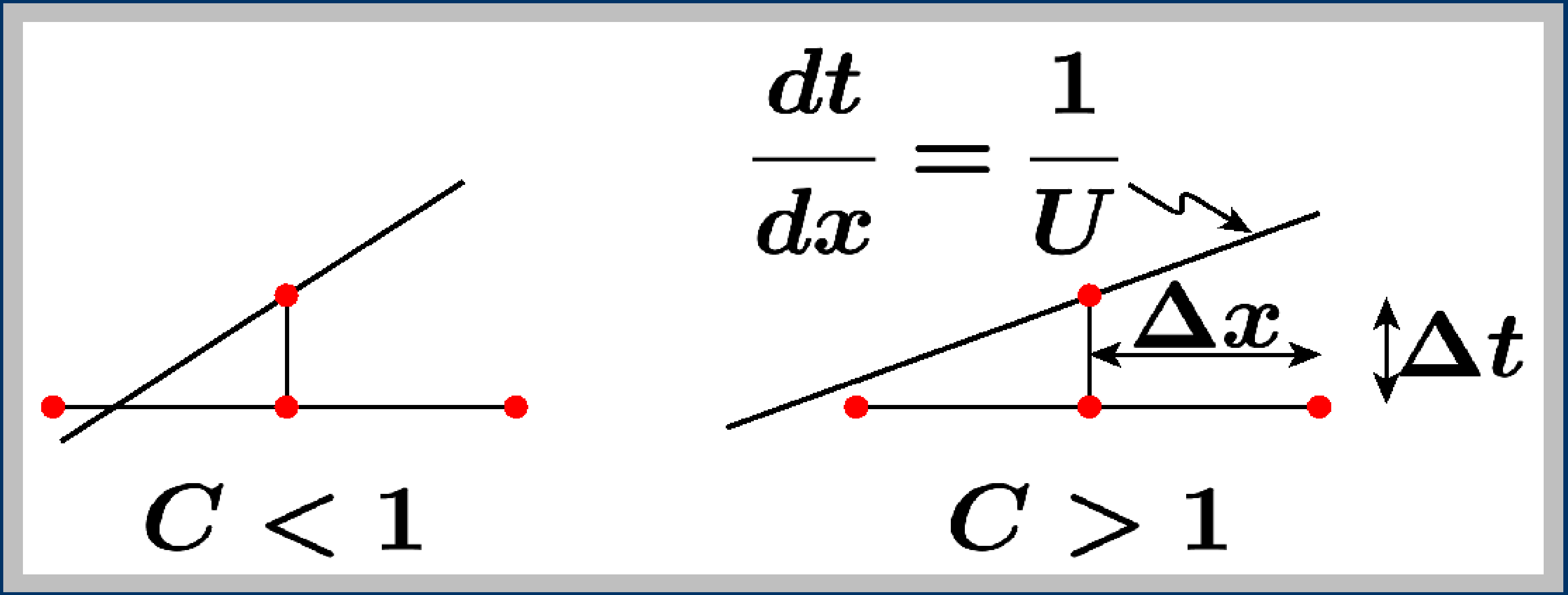
For each (x_j, t^N) the **mathematical domain of dependence** is **contained** in the **numerical domain of dependence**.

CFL Theorem

The **CFL condition** is a **necessary** condition for the **convergence** of a numerical approximation of a partial differential equation, linear or nonlinear.

CFL Condition

CFL Theorem



Stable **Unstable**

Fourier Analysis

- Provides a systematic method for determining stability $\rightarrow$ von Neumann Stability Analysis
- Provides insight into discretization errors

Fourier Analysis

Continuous Problem

Fourier Modes and Properties...

Fourier mode: $\Phi_k(x) = e^{i2\pi kx}, \quad k \in \mathbb{Z}$ (integer)

- Periodic (period = 1)
- Orthogonality

$$\int_0^1 \Phi_k(x)\Phi_{-k'}(x)\,dx = \delta_{kk'}$$

- Eigenfunction of $\frac{\partial^m}{\partial x^m}$

$$\frac{\partial^m}{\partial x^m}\Phi_k(x) = (i2\pi k)^m\,\Phi_k(x)$$

Fourier Analysis

Continuous Problem

...Fourier Modes and Properties

- Form a basis for periodic functions in $L^2([0,1])$

$$v(x) = \sum_{k=-\infty}^{\infty} \mathbb{V}_k \Phi_k(x) = \sum_{k=-\infty}^{\infty} \mathbb{V}_k e^{i2\pi kx}$$

- Parseval's theorem

$$\|v\|_2^2 = \sum_{k=-\infty}^{\infty} |\mathbb{V}_k|^2$$

Fourier Analysis

Continuous Problem

Wave Equation

$$u(x,t) = \sum_{k=-\infty}^{\infty} \mathbb{U}_k(t)\Phi_k(x) = \sum_{k=-\infty}^{\infty} \mathbb{U}_k(t)e^{i2\pi kx}$$

$$u_t + U u_x = 0 \quad \Rightarrow \quad \sum_{k=-\infty}^{\infty} \left(\frac{d\mathbb{U}_k}{dt} + i2\pi kU\,\mathbb{U}_k\right) e^{i2\pi kx} = 0$$

$$u^0(x) = \sum_{k=-\infty}^{\infty} \mathbb{U}_k^0 e^{i2\pi kx} \quad \Rightarrow \quad \mathbb{U}_k(t) = \mathbb{U}_k^0\, e^{-i2\pi kUt}$$

Fourier Analysis

Discrete Problem

Fourier Modes and Properties...

Fourier mode: $\underline{\Phi}_k = \{\Phi_k(x_j)\}_{j=0}^{J-1}$,

k (integer) $\in (-J/2+1,\ J/2)$

$$\Phi_k(x_j) = e^{i2\pi kj\Delta x} \equiv e^{ij\theta} = \Phi_{\theta j}, \qquad \boxed{\theta = 2\pi k\Delta x}$$

$$k \in (-J/2+1,\ J/2) \ \Rightarrow\ \theta \in (-\pi + 2\pi\Delta x,\ \pi)$$

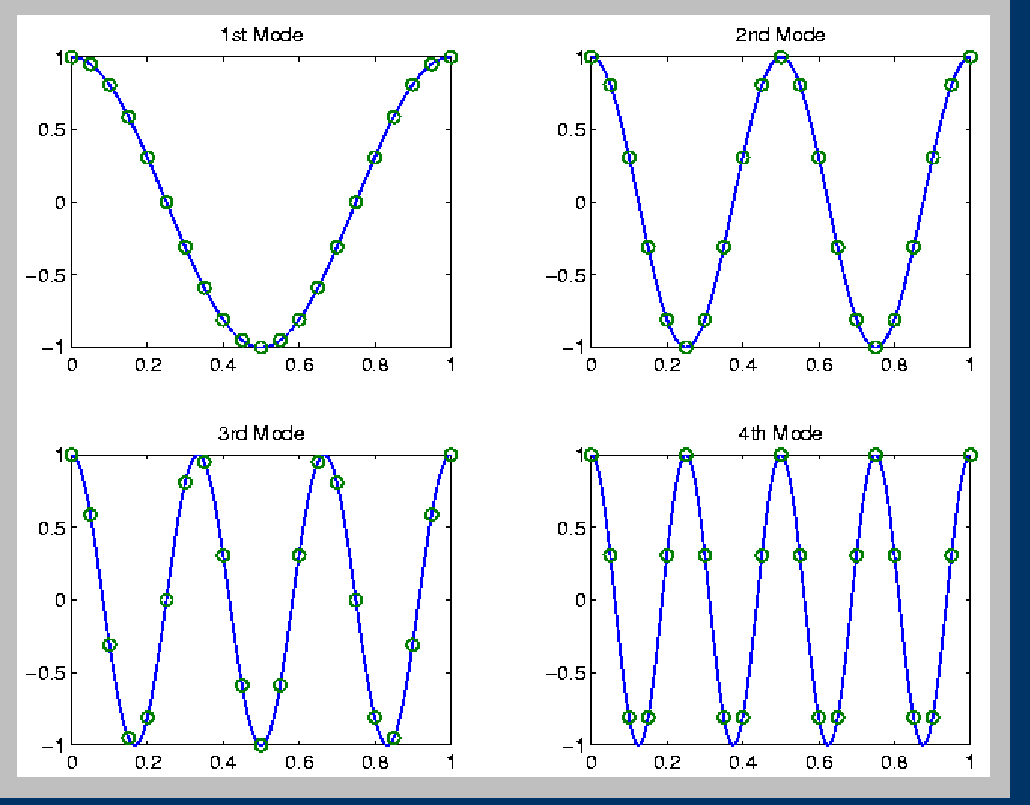
Fourier Analysis

Discrete Problem

...Fourier Modes and Properties...

Real part of first 4 Fourier modes

$$\Delta x = 1/20$$



Fourier Analysis

Discrete Problem

...Fourier Modes and Properties...

- Periodic (period = J)
- Orthogonality

N4

$$\frac{1}{J}\underline{\Phi}_{\theta}^{T}\underline{\Phi}_{-\theta'} = \frac{1}{J}\sum_{j=0}^{J-1} e^{i2\pi kj\Delta x} e^{-i2\pi k'j\Delta x} = \delta_{kk'}$$

$$= \frac{1}{J}\sum_{j=0}^{J-1} e^{ij\theta} e^{-ij\theta'} = \begin{cases} 1 \text{ if } \theta = \theta' \\ 0 \text{ if } \theta \neq \theta' \end{cases}$$

Fourier Analysis

Discrete Problem

...Fourier Modes and Properties...

- Eigenfunctions of difference operators e.g., **N5**
  - $\delta_{2x}\underline{v}|_j = v_{j+1} - v_{j-1}$

$$\delta_{2x}\,\underline{\Phi}_\theta = i2\sin(\theta)\,\underline{\Phi}_\theta$$

  - $\delta_x^2\underline{v}|_j = v_{j+1} - 2v_j + v_{j-1}$

$$\delta_x^2\,\underline{\Phi}_\theta = -4\sin^2(\theta/2)\,\underline{\Phi}_\theta$$

  - $\Delta_x^-\underline{v}|_j = v_j - v_{j-1}$

$$\Delta_x^-\,\underline{\Phi}_\theta = (1 - e^{-i\theta})\,\underline{\Phi}_\theta$$

Fourier Analysis

Discrete Problem

...Fourier Modes and Properties

- Basis for periodic (discrete) functions $\underline{v} = \{v_j\}_{j=1}^{J}$

$$\underline{v} = \sum_{\substack{\theta = -\pi \\ +2\pi\Delta x}}^{\pi} \mathbb{V}_\theta \, \underline{\Phi}_\theta \quad \rightarrow \quad v_j = \sum_{\substack{\theta = -\pi \\ +2\pi\Delta x}}^{\pi} \mathbb{V}_\theta \, e^{ij\theta}$$

- Parseval's theorem

$$\|\underline{v}\|^2 \equiv \underbrace{\Delta x}_{1/J} \|\underline{v}\|_2^2 = \sum_{\substack{\theta = -\pi \\ +2\pi\Delta x}}^{\pi} |\mathbb{V}_\theta|^2$$

Fourier Analysis

von Neumann Stability Criterion

Write $\underline{\hat{u}}^{n+1} = \sum_\theta \hat{\mathbb{U}}_\theta^{n+1} \underline{\Phi}_\theta, \quad \underline{\hat{u}}^n = \sum_\theta \hat{\mathbb{U}}_\theta^n \underline{\Phi}_\theta$

Stability $\|\underline{\hat{u}}^{n+1}\| \leq (1 + \mathcal{O}(\Delta t))\|\underline{\hat{u}}^n\|$

$$\Rightarrow \sum_\theta |\hat{\mathbb{U}}_\theta^{n+1}|^2 \leq (1 + \mathcal{O}(\Delta t)) \sum_\theta |\hat{\mathbb{U}}_\theta^n|^2$$

Stability for all data $\Rightarrow$

$$\boxed{|\hat{\mathbb{U}}_\theta^{n+1}| \leq (1 + \mathcal{O}(\Delta t))\, |\hat{\mathbb{U}}_\theta^n|, \quad \forall \theta}$$

Fourier Analysis

von Neumann Stability Criterion

First Order Upwind Scheme...

$$\boxed{\hat{u}_j^n = \sum_{\theta} \hat{\mathbb{U}}_{\theta}^n \; \Phi_{\theta j} = \sum_{\theta} \hat{\mathbb{U}}_{\theta}^n \; e^{ij\theta}}$$

$$\hat{u}_j^{n+1} - \hat{u}_j^n + C(\hat{u}_j^n - \hat{u}_{j-1}^n) = 0, \quad \forall j \;\Rightarrow$$

$$\sum_{\theta} (\hat{\mathbb{U}}_{\theta}^{n+1} - \hat{\mathbb{U}}_{\theta}^n + C(1 - e^{-i\theta})\hat{\mathbb{U}}_{\theta}^n) e^{ij\theta} = 0, \quad \forall j \;\Rightarrow$$

Fourier Analysis

von Neumann Stability Criterion

...First Order Upwind Scheme...

$$\hat{\mathbb{U}}_{\theta}^{n+1} = \underbrace{\left((1-C) + Ce^{-i\theta}\right)}_{g(C,\theta)} \hat{\mathbb{U}}_{\theta}^{n} = g(C,\theta)\, \hat{\mathbb{U}}_{\theta}^{n}$$

amplification factor

Stability if $|\hat{\mathbb{U}}_{\theta}^{n+1}| \leq |\hat{\mathbb{U}}_{\theta}^{n}|, \quad \forall \theta$ which implies

$$\boxed{|g(C,\theta)| \leq 1, \qquad \forall \theta}$$

Fourier Analysis

von Neumann Stability Criterion

...First Order Upwind Scheme

$$|g(C,\theta)|^2 = |(1-C) + Ce^{-i\theta}|^2$$

$$= (1 - C + C\cos(\theta))^2 + C^2\sin^2(\theta)$$

$$= (1 - 2C\sin^2(\theta/2))^2 + 4C^2\sin^2(\theta/2)\cos^2(\theta/2)$$

$$= 1 - 4C(1-C)\sin^2(\theta/2)$$

Stability if:

$$|g(C,\theta)| \leq 1 \;\Rightarrow 0 \leq C \equiv \frac{U\Delta t}{\Delta x} \leq 1$$

Fourier Analysis

von Neumann Stability Criterion

FTCS Scheme...

$$\frac{\hat{u}_j^{n+1} - \hat{u}_j^n}{\Delta t} + U\,\frac{\hat{u}_{j+1}^n - \hat{u}_{j-1}^n}{2\Delta x} = 0$$

$$\Rightarrow \; \underline{\hat{u}}^{n+1} = \underline{\hat{u}}^n - \frac{C}{2}\,\delta_{2x}\underline{\hat{u}}^n$$

Fourier Decomposition: $u_j^n = \sum_\theta \hat{\mathbb{U}}_\theta^n e^{ij\theta}$

$$\Rightarrow \; \sum_\theta \left(\hat{\mathbb{U}}_\theta^{n+1} - \hat{\mathbb{U}}_\theta^n + iC\sin(\theta)\hat{\mathbb{U}}_\theta^n\right) e^{ij\theta} = 0$$

Fourier Analysis

von Neumann Stability Criterion

...FTCS Scheme

$$\hat{\mathbb{U}}_\theta^{n+1} = \underbrace{(1 - iC\sin(\theta))}_{g(C,\theta)} \hat{\mathbb{U}}_\theta^{n} = g(C,\theta)\,\hat{\mathbb{U}}_\theta^{n}$$

amplification factor

$$|g(C,\theta)|^2 = 1 + C^2 \sin^2(\theta) \geq 1, \quad \text{for } C \neq 0$$

$\Rightarrow$ **Unconditionally Unstable** $\Rightarrow$ **Not Convergent**

Lax-Wendroff Scheme

Write a Taylor series expansion in time about t^n

$$u(x,t^{n+1}) = u(x,t^n) + \Delta t \left.\frac{\partial u}{\partial t}\right|^n + \frac{\Delta t^2}{2} \left.\frac{\partial^2 u}{\partial t^2}\right|^n + \ldots$$

But . . .

$$\frac{\partial u}{\partial t} = -U\frac{\partial u}{\partial x} \qquad \text{(from } u_t + U u_x = 0\text{)}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(-U\frac{\partial u}{\partial x}\right) = -U\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = U^2\frac{\partial u^2}{\partial x^2}$$

Lax-Wendroff Scheme

Spatial Approximation

$$u(x,t^{n+1}) = u(x,t^n) - U\Delta t \left.\frac{\partial u}{\partial x}\right|^n + \frac{U^2 \Delta t^2}{2} \left.\frac{\partial^2 u}{\partial x^2}\right|^n + \ldots$$

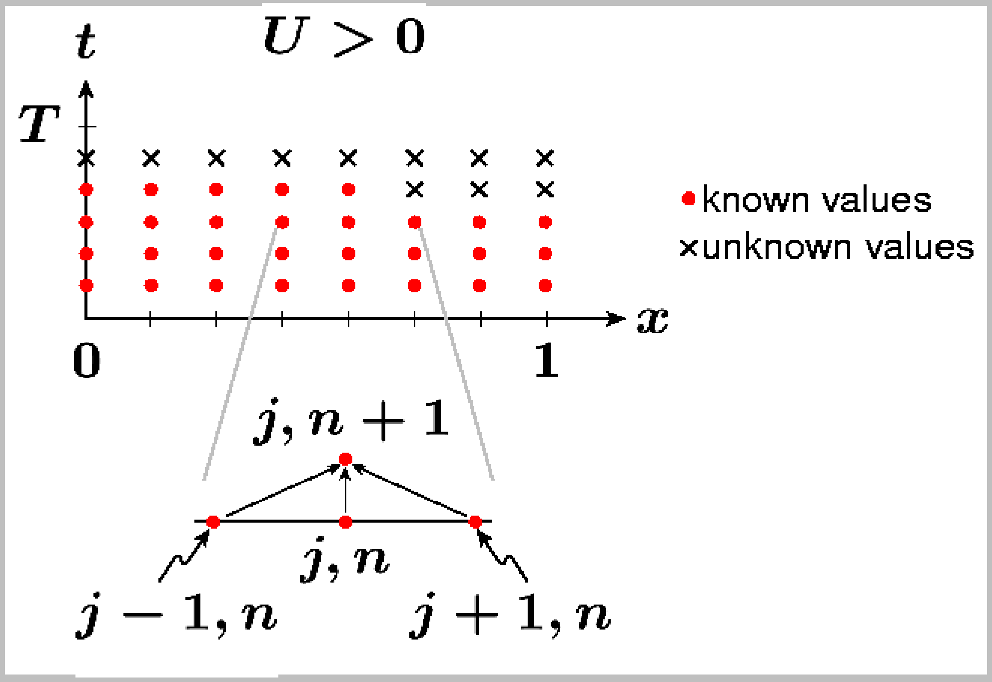
Approximate spatial derivatives

$$\left.\frac{\partial v}{\partial x}\right|_j \approx \frac{1}{2\Delta x}\, \delta_{2x}\, \underline{v}|_j = \frac{v_{j+1} - v_{j-1}}{2\Delta x}$$

$$\left.\frac{\partial^2 v}{\partial x^2}\right|_j \approx \frac{1}{\Delta x^2}\, \delta_x^2\, \underline{v}|_j = \frac{v_{j+1} - 2v_j + v_{j-1}}{\Delta x^2}$$

Lax-Wendroff Scheme

Equations



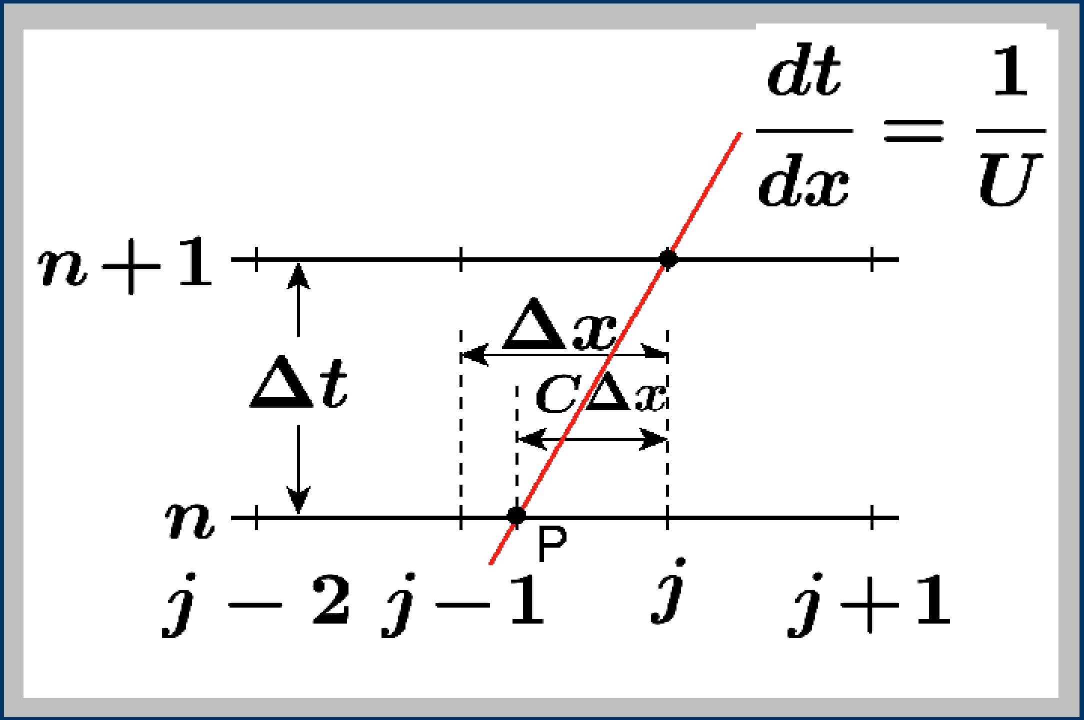
no matrix inversion

$\underline{\hat{u}}^n$ **exists** and is **unique**

$$\hat{u}_j^{n+1} = \hat{u}_j^n - \frac{C}{2}(\hat{u}_{j+1}^n - \hat{u}_{j-1}^n) + \frac{C^2}{2}(\hat{u}_{j+1}^n - 2\hat{u}_j^n + \hat{u}_{j-1}^n)$$

Lax-Wendroff Scheme

Interpretation



$$u_j^{n+1} = u_P$$

Use Quadratic Interpolation

$j-1,\ j,\ j+1$

$$u_P \approx \frac{C}{2}(1+C)\hat{u}_{j-1}^n + (1+C)(1-C)\hat{u}_j^n - \frac{C}{2}(1-C)\hat{u}_{j+1}^n$$

Lax-Wendroff Scheme

Analysis

Consistency

$$(\hat{\mathcal{L}}\underline{v}^n)_j \equiv \frac{v_j^{n+1}-v_j^n}{\Delta t} + U\frac{v_{j+1}^n - v_{j-1}^n}{2\Delta x} - \frac{U^2\Delta t}{2}\frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{\Delta x^2}$$

$$= (v_t + U v_x)_j^n + \underbrace{\frac{\Delta t}{2}\left(v_{tt}|_j^n - U^2\, v_{xx}^n|_j^n\right)}_{=0 \;\; (\text{for } \; v=u)} + \ldots$$

$$(\mathcal{L}v)_j^n \equiv (v_t + U v_x)_j^n$$

$$\boxed{(\hat{\mathcal{L}}\,\underline{v}^n)_j - (\mathcal{L}\,v)_j^n = \mathcal{O}(\Delta x^2, \Delta t^2)}$$

$\Rightarrow$ Second order accurate in space and time.

Lax-Wendroff Scheme

Analysis

Truncation Error

Insert exact solution $\underline{u}$ into difference scheme

$$(\hat{\mathcal{L}}\,\underline{u})_j^n - \underbrace{(\mathcal{L}\,\underline{u})_j^n}_{=0} = \tau_j^n, \quad \text{for} \begin{cases} 1 \le j \le J \\ 1 \le n \le N \end{cases}$$

$$\underline{u}^{n+1} = \hat{\mathcal{S}}\underline{u}^n + \Delta t\,\underline{\tau}^n$$

Consistency $\Rightarrow \|\underline{\tau}^n\| = \mathcal{O}(\Delta x^2, \Delta t^2), \;\; 1 \le n \le N$

Lax-Wendroff Scheme

Analysis

Stability

$$\underline{\hat{u}}^{n+1} = \underline{\hat{u}}^{n} - \frac{C}{2}\,\delta_{2x}\,\underline{\hat{u}}^{n} + \frac{C^2}{2}\,\delta_x^2\,\underline{\hat{u}}^{n}$$

$$\Rightarrow \quad \hat{\mathbb{U}}_\theta^{n+1} = \hat{\mathbb{U}}_\theta^{n} - iC\sin(\theta)\,\hat{\mathbb{U}}_\theta^{n} - C^2(1-\cos(\theta))\,\hat{\mathbb{U}}_\theta^{n}$$

$$= \underbrace{\Big(1 - 2C^2\sin^2(\theta/2) - iC\sin(\theta)\Big)}_{g(C,\theta)}\,\hat{\mathbb{U}}_\theta^{n}$$

$$|g(C,\theta)|^2 = 1 - 4C^2(1-C^2)\sin^4(\theta/2)$$

Stability if: $|g(C,\theta)| \leq 1 \;\Rightarrow |C| \equiv |U|\Delta t/\Delta x \leq 1$

Lax-Wendroff Scheme

Analysis

Convergence

- Consistency: $\|\underline{\tau}\| = \mathcal{O}(\Delta x^2, \Delta t^2)$
- Stability: $\|\underline{\hat{u}}^{n+1}\| \leq \|\underline{\hat{u}}^{n}\|$ for $C \equiv U\Delta t/\Delta x \leq 1$
- $\Rightarrow$ **Convergence**

$\underline{e} = \underline{u} - \underline{\hat{u}}$

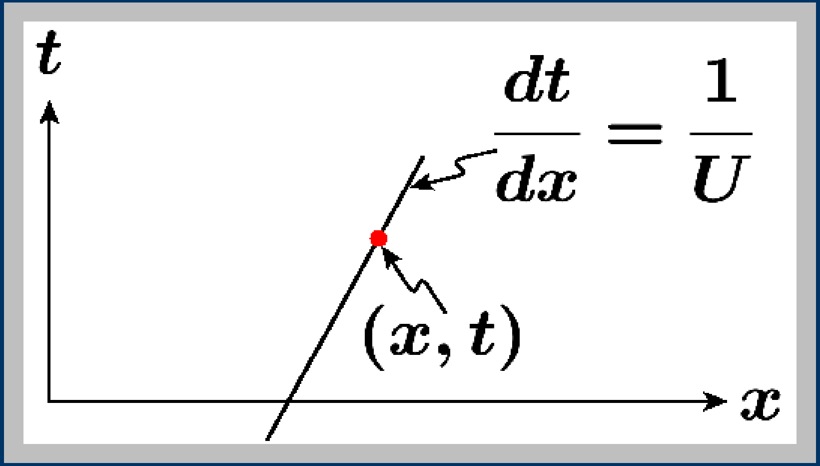
$$\|\underline{e}^n\| \leq (C_x \Delta x^2 + C_t \Delta t^2),\ 1 \leq n \leq N$$

or $$|e_j^n| \leq (C_x \Delta x^2 + C_t \Delta t^2), \begin{cases} 1 \leq j \leq J, \\ 1 \leq n \leq N \end{cases}$$

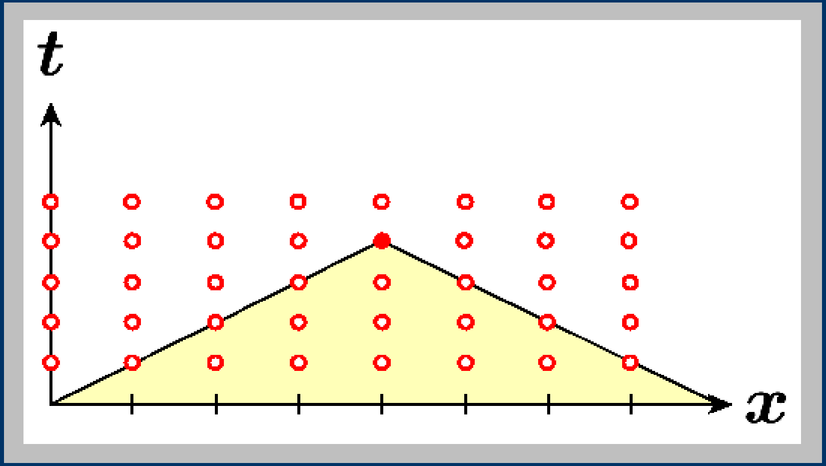
C_x and C_t are constants independent of Δx, Δt

Lax-Wendroff Scheme

Domains of Dependence



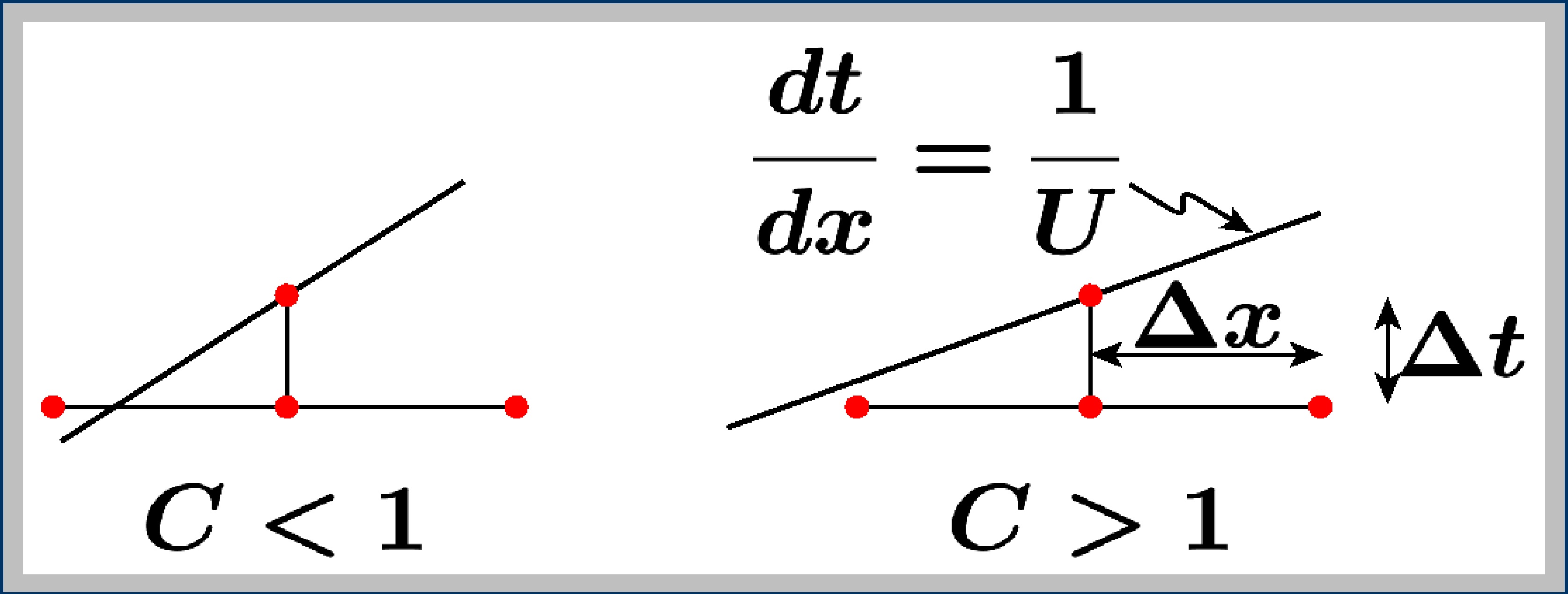
Analytical



Numerical

Lax-Wendroff Scheme

CFL Condition



Stable **Unstable**

Lax-Wendroff Scheme

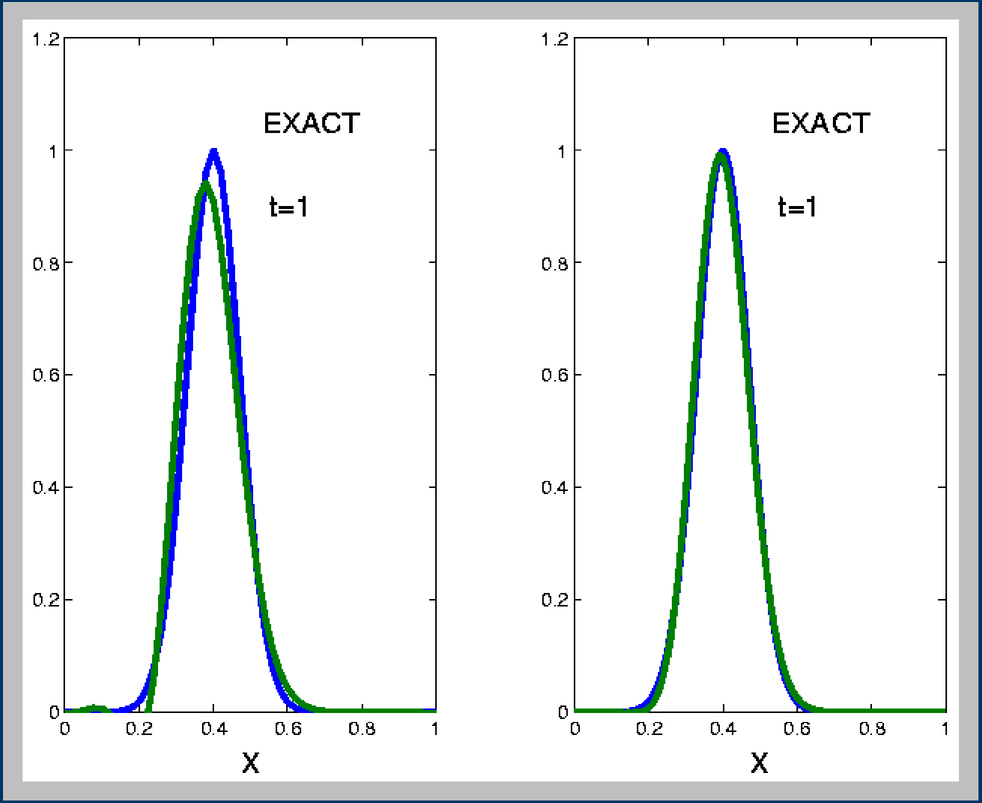
Example

Solutions for:

$$C = 0.5$$

$\Delta x = 1/50$ (left)

$\Delta x = 1/100$ (right)



Lax-Wendroff Scheme

Example

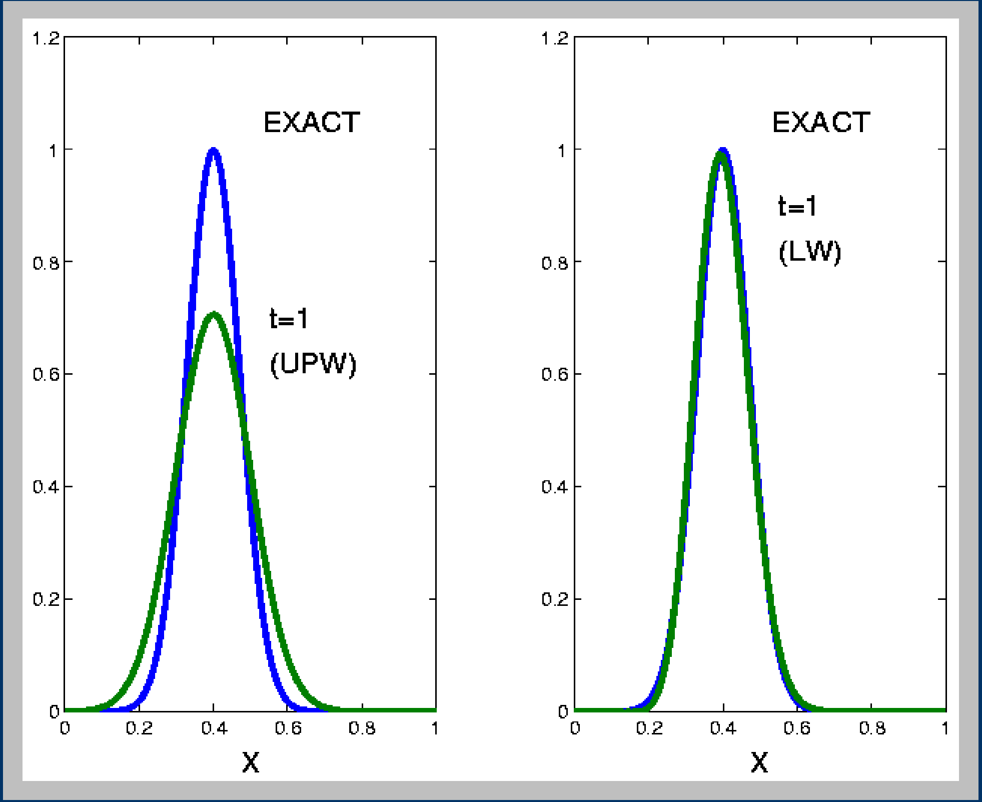
$$\Delta x = 1/100$$

$$C = 0.5$$

Upwind (left)

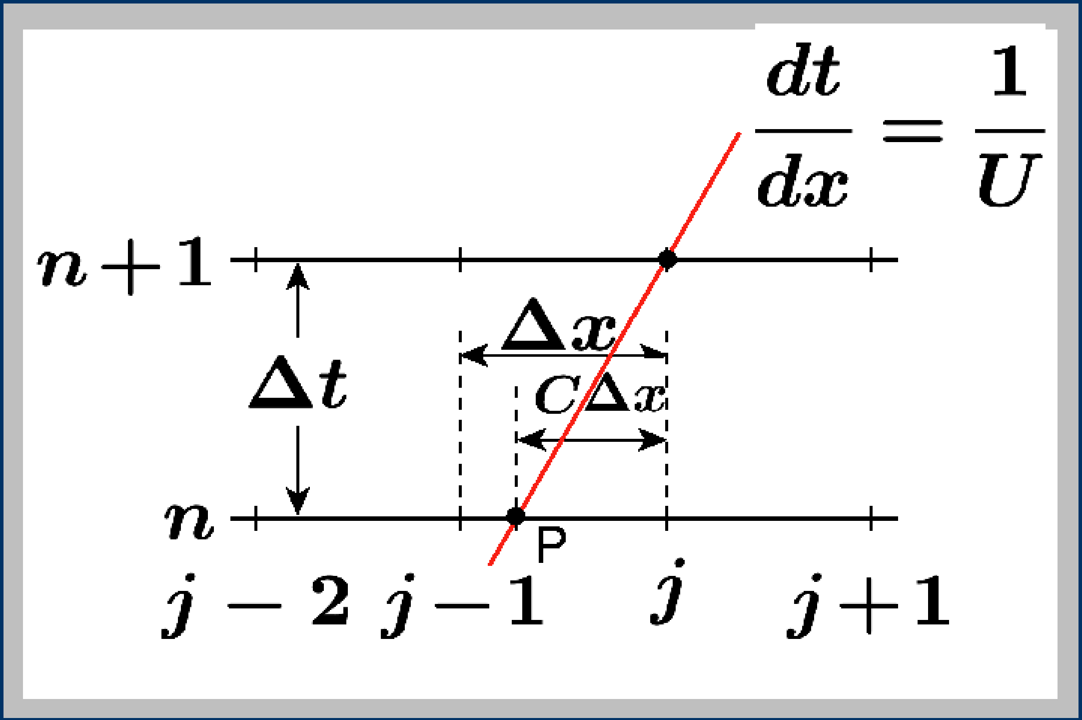
vs.

Lax-Wendroff (right)



Beam-Warming Scheme

Derivation



$$u_j^{n+1} = u_P$$

Use Quadratic Interpolation

$j-2,\ j-1,\ j$

$$u_P \approx -\tfrac{C}{2}(1-C)\hat{u}_{j-2}^n + C(2-C)\ \hat{u}_{j-1}^n + \tfrac{1}{2}(1-C)(2-C)\hat{u}_j^n$$

Beam-Warming Scheme

Consistency and Stability

$$\hat{u}_j^{n+1} = \hat{u}_j^n - \frac{C}{2}(3\hat{u}_j^n - 4\hat{u}_{j-1}^n + \hat{u}_{j-2}^n) + \frac{C^2}{2}(\hat{u}_j^n - 2\hat{u}_{j-1}^n + \hat{u}_{j-2}^n)$$

- Consistency, $\|\underline{\tau}\| \sim \mathcal{O}(\Delta x^2, \Delta t^2)$
- Stability

$$|g(C,\theta)|^2 = 1 - 4C(1-C)^2(2-C)\sin^4(\theta/2)$$

$$|g(C,\theta)| < 1 \quad \Rightarrow \quad \boxed{0 \leq C \leq 2}$$

Method of Lines

Generally applicable to time evolution PDE's

- **Spatial discretization**
  $\Rightarrow$ Semi-discrete scheme (system of coupled ODE's)
- **Time discretization** (using ODE techniques)
  $\Rightarrow$ Discrete scheme

By studying the semi-discrete scheme we can better understand spatial and temporal discretization errors

Method of Lines

NOTATION:

- $\overline{v}_j(t)$ approximation to $v(x_j, t) \equiv v_j(t)$
- $\underline{\overline{v}}(t)$ vector of semi-discrete approximations;

$$\underline{\overline{v}}(t) = \{\overline{v}_j(t)\}_{j=1}^{J}$$

Method of Lines

Spatial Discretization

$$\frac{\partial u}{\partial t} + U\,\frac{\partial u}{\partial x} = 0$$

Central differences...(for example)

$$\frac{d\overline{u}_j}{dt} + \frac{U}{2\Delta x}\left(\overline{u}_{j+1} - \overline{u}_{j-1}\right) = 0, \qquad 1 \le j \le J$$

or, in vector form,

N6

$$\frac{d\underline{u}}{dt} + \frac{U}{2\Delta x}\,\delta_{2x}\underline{u} = 0$$

Method of Lines

Spatial Discretization

Fourier Analysis...

Write semi-discrete approximation as

$$\overline{u}_j(t) = \sum_{\substack{\theta = -\pi \\ +2\pi\Delta x}}^{\pi} \overline{\mathbb{U}}_\theta(t)\, e^{ij\theta}$$

Inserting into semi-discrete equation

$$\sum_\theta \left(\frac{d\,\overline{\mathbb{U}}_\theta}{dt} + i\frac{U}{\Delta x}\sin(\theta)\,\overline{\mathbb{U}}_\theta\right) e^{ij\theta} = 0, \quad 1 \leq j \leq J$$

Method of Lines

Spatial Discretization

...Fourier Analysis...

For each θ, we have a **scalar** ODE

$$\frac{d\,\overline{\mathbb{U}}_\theta}{dt} + i\frac{U}{\Delta x}\sin(\theta)\,\overline{\mathbb{U}}_\theta = 0$$

$$\Rightarrow \quad \overline{\mathbb{U}}_\theta(t) = \overline{\mathbb{U}}_\theta^0 e^{-i\frac{U}{\Delta x}\sin(\theta)t}$$

$|\overline{\mathbb{U}}_\theta(t)| = |\overline{\mathbb{U}}_\theta^0|$ Neutrally stable

Method of Lines

Spatial Discretization

...Fourier Analysis...

Exact solution

$$u_j(t) = \sum_k \mathbb{U}_k^0 e^{i2\pi(kx_j - kU\,t)}$$

$$\boxed{\omega_{EX} = kU}$$

Semi-discrete solution

$$\overline{u}_j(t) = \sum_\theta \overline{\mathbb{U}}_\theta^0 e^{ij\theta} e^{-i\frac{U}{\Delta x}\sin(\theta)t}$$

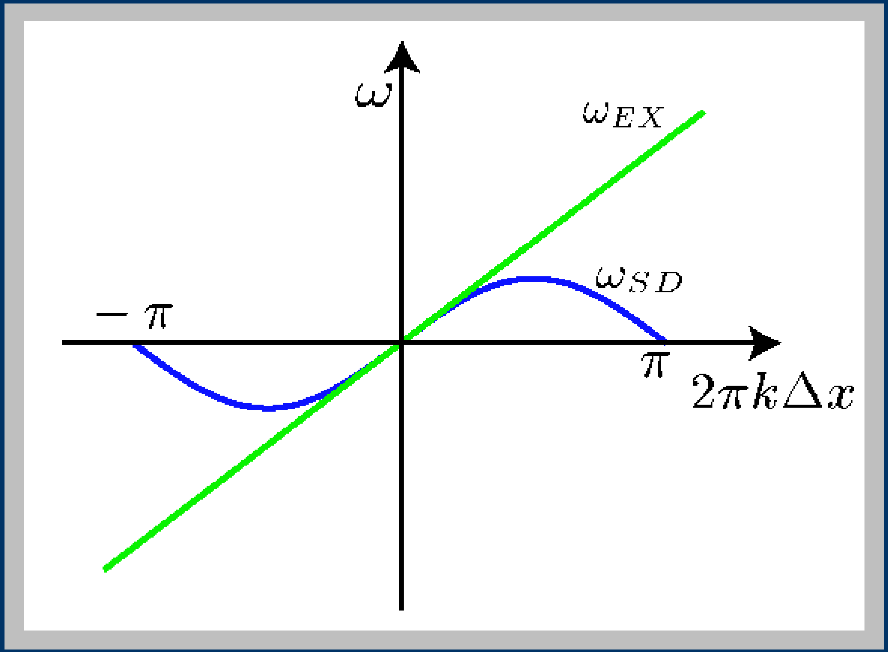
$$= \sum_k \overline{\mathbb{U}}_k^0 e^{i2\pi(kx_j - \frac{U}{2\pi\Delta x}\sin(2\pi k\Delta x)\,t)}$$

$$\boxed{\omega_{SD} = \frac{U}{2\pi\Delta x}\sin(2\pi k\Delta x)}$$

Method of Lines

Spatial Discretization

...Fourier Analysis



$$\omega_{EX} = kU \quad \text{vs.} \quad \omega_{SD} = \frac{U}{2\pi\Delta x}\sin(2\pi k\Delta x)$$

Method of Lines

Time Discretization

Predictor/Corrector Algorithm...

Model ODE

$$\frac{du}{dt} = \lambda u$$

$$\hat{u}^p = \hat{u}^n + \Delta t \lambda \, \hat{u}^n \qquad \text{Predictor}$$

$$\hat{u}^{n+1} = \hat{u}^n + \Delta t \lambda \, \hat{u}^p \qquad \text{Corrector}$$

Combining the two steps we have

$$z = \Delta t \lambda$$

$$\hat{u}^{n+1} = \hat{u}^n + \Delta t \lambda \, \hat{u}^n + \Delta t^2 \lambda^2 \, \hat{u}^n = (1 + z + z^2) \, \hat{u}^n$$

Method of Lines

Time Discretization

...Predictor/Corrector Algorithm

Semi-discrete equation

$$\frac{d\underline{\overline{u}}}{dt} + \frac{U}{2\Delta x}\,\delta_{2x}\underline{\overline{u}} = 0$$

$$\underline{\hat{u}}^p = \underline{\hat{u}}^n + \tfrac{C}{2}\,\delta_{2x}\underline{\hat{u}}^n \qquad \text{Predictor}$$

$$\underline{\hat{u}}^{n+1} = \underline{\hat{u}}^n + \tfrac{C}{2}\,\delta_{2x}\underline{\hat{u}}^p \qquad \text{Corrector}$$

Combining the two steps we have

$$\underline{\hat{u}}^{n+1} = \underline{\hat{u}}^n + \frac{C}{2}\,\delta_{2x}\underline{\hat{u}}^n + \frac{C^2}{4}\,\delta_{2x}^2\underline{\hat{u}}^n$$

Method of Lines

Fourier Stability Analysis

$$\underline{\hat{u}}^{n+1} = \underline{\hat{u}}^{n} + \frac{C}{2}\,\delta_{2x}\underline{\hat{u}}^{n} + \frac{C^2}{4}\,\delta_{2x}^{2}\underline{\hat{u}}^{n}$$

Fourier transform

$$\Downarrow$$

$$\hat{\mathbb{U}}_{\theta}^{n+1} = \hat{\mathbb{U}}_{\theta}^{n} - iC\sin(\theta)\,\hat{\mathbb{U}}_{\theta}^{n} - C^2\sin^2(\theta)\,\hat{\mathbb{U}}_{\theta}^{n}$$

$$= (1 + z_\theta + z_\theta^2)\,\hat{\mathbb{U}}_{\theta}^{n}, \quad \forall\theta$$

$$z_\theta = -iC\sin(\theta)$$

Amplification factor

$$g(C,\theta) = 1 + z_\theta + z_\theta^2$$

$z_\theta = i\alpha_\theta$ with $\alpha_\theta \in \mathbb{R}$

$$|g(C,\theta)|^2 = (1-\alpha_\theta^2)^2 + \alpha_\theta^2 = 1 - \alpha_\theta^2(1-\alpha_\theta^2)$$

Stability $\Rightarrow \ \alpha_\theta^2 \leq 1 \ \ \forall\theta \ \ \Rightarrow \ C \leq 1$

Method of Lines

Fourier Stability Analysis

PDE $u(x,t)$

$\Downarrow$

Semi-discrete $\overline{u}_j(t)$	**A** $\rightarrow$	**Discrete** $\hat{u}_j^n$
$\downarrow$ **B**		$\downarrow$ **A**
Semi-discrete Fourier $\overline{\mathbb{U}}_\theta(t)$	**B** $\rightarrow$	**Discrete Fourier** $\hat{\mathbb{U}}_\theta^n$

Method of Lines

Fourier Stability Analysis

Path B...

Semi-discrete

$$\frac{d\overline{u}}{dt} + \frac{U}{2\Delta x}\,\delta_{2x}\overline{u} = 0$$

Fourier semi-discrete

$$\frac{d\,\overline{\mathbb{U}}_\theta}{dt} + i\frac{U}{\Delta x}\sin(\theta)\,\overline{\mathbb{U}}_\theta = 0$$

Predictor

$$\hat{\mathbb{U}}^p = \hat{\mathbb{U}}^n - iC\sin(\theta)\hat{\mathbb{U}}^n$$

Corrector

$$\hat{\mathbb{U}}^{n+1} = \hat{\mathbb{U}}^n - iC\sin(\theta)\hat{\mathbb{U}}^p$$

Discrete

$$\boxed{\hat{\mathbb{U}}_\theta^{n+1} = \left(1 + z_\theta + z_\theta^2\right)\hat{\mathbb{U}}_\theta^n}$$

Method of Lines

Fourier Stability Analysis

...Path B

- Gives the same discrete Fourier equation
- Simpler
- "Decouples" spatial and temporal discretizations

  For each θ, the discrete Fourier equation is the result of discretizing the **scalar** semi-discrete **ODE** for the θ Fourier mode

Method of Lines

Model Equation: $$\frac{du}{dt} = \lambda u$$ u, λ complex-valued

Discretization

$$\frac{\hat{u}^{n+1} - \hat{u}^n}{\Delta t} = \lambda \hat{u}^n$$ EF

$$\frac{\hat{u}^{n+1} - \hat{u}^n}{\Delta t} = \lambda \hat{u}^{n+1}$$ EB

$$\frac{\hat{u}^{n+1} - \hat{u}^n}{\Delta t} = \frac{1}{2} \lambda (\hat{u}^n + \hat{u}^{n+1})$$ CN

Method of Lines

Absolute Stability Diagrams...

Given $\dfrac{du}{dt} = \lambda u$ and u, λ complex-valued

$$\frac{\hat{u}^{n+1} - \hat{u}^n}{\Delta t} = \lambda \hat{u}^n \text{ (EF) or } \lambda \hat{u}^{n+1} \text{ (EB) or } \ldots;$$

$\mathcal{R}^{abs}_{EF}$ or ... $\in \mathbb{C}$ is defined such that

N7 N8

$$z \equiv \Delta t \lambda \in \mathcal{R}^{abs}_{EF} \text{ or...} \Leftrightarrow |\hat{u}^{n+1}| < |\hat{u}^n|$$

$$\Rightarrow |\hat{u}^n| \to 0 \text{ as } n \to \infty$$

Method of Lines

Methods for ODE's

...Absolute Stability Diagrams...

$$\hat{u}^{n+1} - \hat{u}^n = \Delta t \lambda \hat{u}^n \qquad \text{EF}$$

$$\Rightarrow \qquad \hat{u}^{n+1} = (1+z)\,\hat{u}^n$$

$$\hat{u}^{n+1} - \hat{u}^n = \Delta t \lambda \hat{u}^{n+1} \qquad \text{EB}$$

$$\Rightarrow \qquad \hat{u}^{n+1} = \frac{1}{1-z}\,\hat{u}^n$$

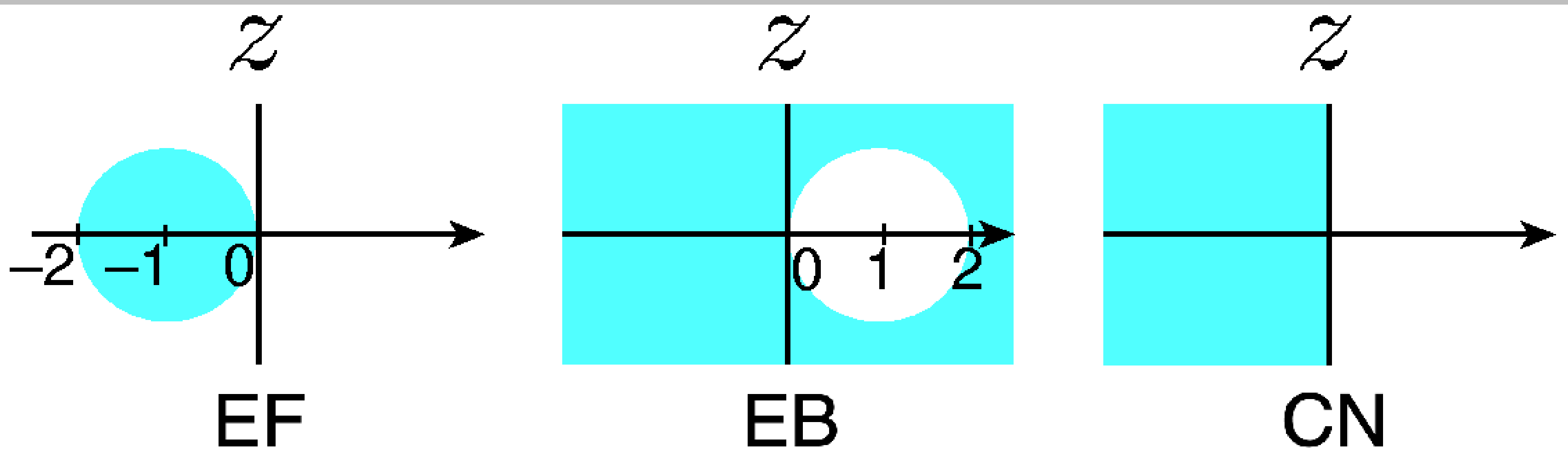
$$\hat{u}^{n+1} - \hat{u}^n = \tfrac{1}{2}\Delta t \lambda (\hat{u}^n + \hat{u}^{n+1}) \qquad \text{CN}$$

$$\Rightarrow \qquad \hat{u}^{n+1} = \frac{1+z/2}{1-z/2}\,\hat{u}^n$$

Method of Lines

Methods for ODE's

...Absolute Stability Diagrams



N9

Method of Lines

For each θ

$$\frac{d\,\overline{\mathbb{U}}_\theta}{dt} + i\frac{U}{\Delta x}\sin(\theta)\,\overline{\mathbb{U}}_\theta = 0, \quad \text{or} \quad \boxed{\frac{d\,\overline{\mathbb{U}}_\theta}{dt} = \lambda_\theta \overline{\mathbb{U}}_\theta}$$

Thus,

$$\lambda_\theta = -i\frac{U}{\Delta x}\sin(\theta)$$

- λ_θ (and $z_\theta = \Delta t \lambda_\theta$) is purely imaginary
- $\lambda_\theta \to \infty$ for $\Delta x \to 0$

Method of Lines

Methods for ODE's

...Application to the Wave Equation...

$$\frac{d\,\overline{\mathbb{U}}_\theta}{dt} = \lambda_\theta \overline{\mathbb{U}}_\theta$$

⇒ **EF** is unconditionally **unstable**

⇒ **EB** is unconditionally **stable**

⇒ **CN** is unconditionally **stable**

Method of Lines

Methods for ODE's

...Application to the Wave Equation...

Stable schemes can be obtained by:

1) Selecting explicit time stepping algorithms which have some stability on the imaginary axis

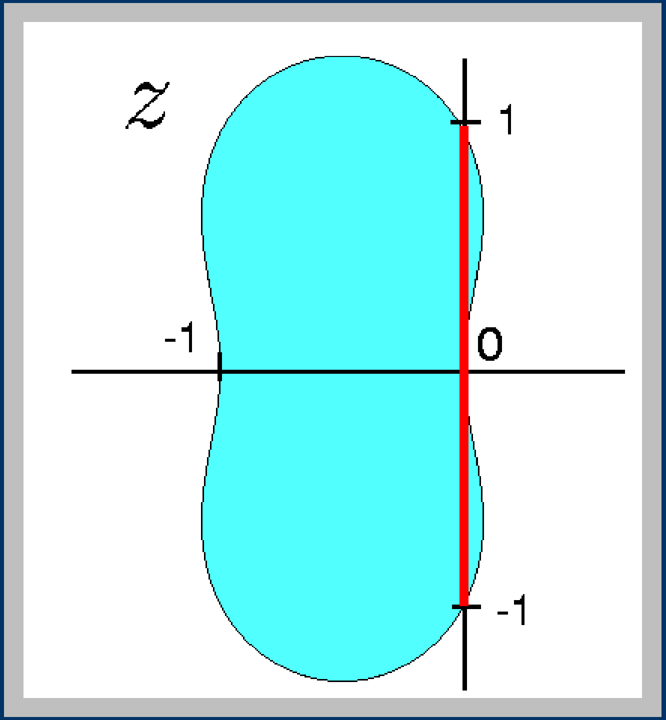
2) Modifying the original equation by adding "artificial viscosity" $\Rightarrow \quad \Re(\lambda_\theta) < 0$

Method of Lines

Methods for ODE's

...Application to the Wave Equation...

Explict Time stepping Schemes



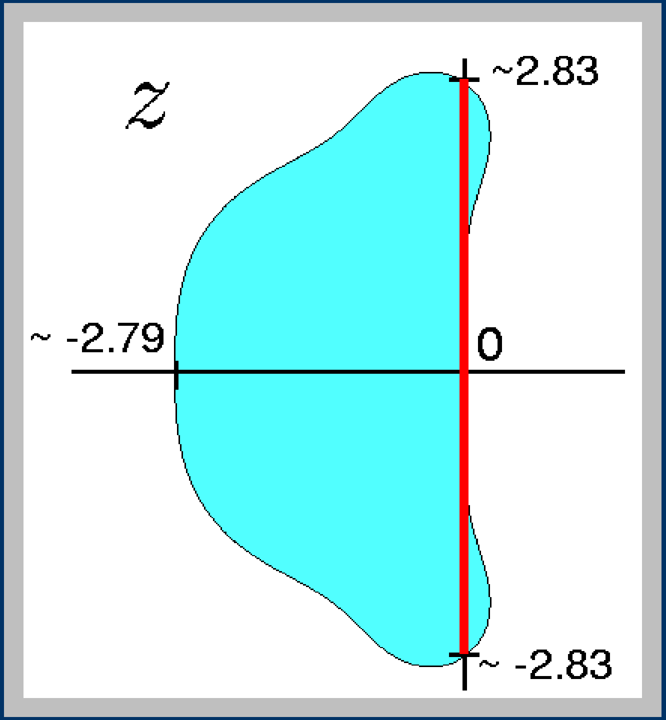
Predictor/Corrector

$$\hat{u}^{n+1} = (1 + z + z^2)\hat{u}^n$$

$$z_\theta = iC\sin(\theta)$$

$$\Rightarrow \quad C \leq 1$$

Explicit Time stepping Schemes



4 Stage Runge-Kutta

$$\hat{u}^{n+1} = \left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}\right)\hat{u}^n$$

$$z_\theta = iC\sin(\theta)$$

$$\Rightarrow \quad C \leq 2\sqrt{2} \sim 2.83$$

Method of Lines

Adding Artificial Viscosity

$$\frac{d\overline{u}}{dt} + \frac{U}{2\Delta x}\,\delta_{2x}\,\overline{u} - \underbrace{\mu\frac{U}{2\Delta x}\,\delta_x^2\,\overline{u}}_{\text{Additional Term}} = 0$$

EF Time $+\ \mu = 1 \quad \Rightarrow$ First Order Upwind

EF Time $+\ \mu = C \quad \Rightarrow$ Lax-Wendroff

Method of Lines

Adding Artificial Viscosity

For each Fourier mode θ,

$$\frac{d\,\overline{\mathbb{U}}_\theta}{dt} + \{i\frac{U}{\Delta x}\sin(\theta) \underbrace{-2\mu\frac{U}{\Delta x}\sin^2(\theta/2)}_{\text{Additional Term}}\}\,\overline{\mathbb{U}}_\theta = 0$$

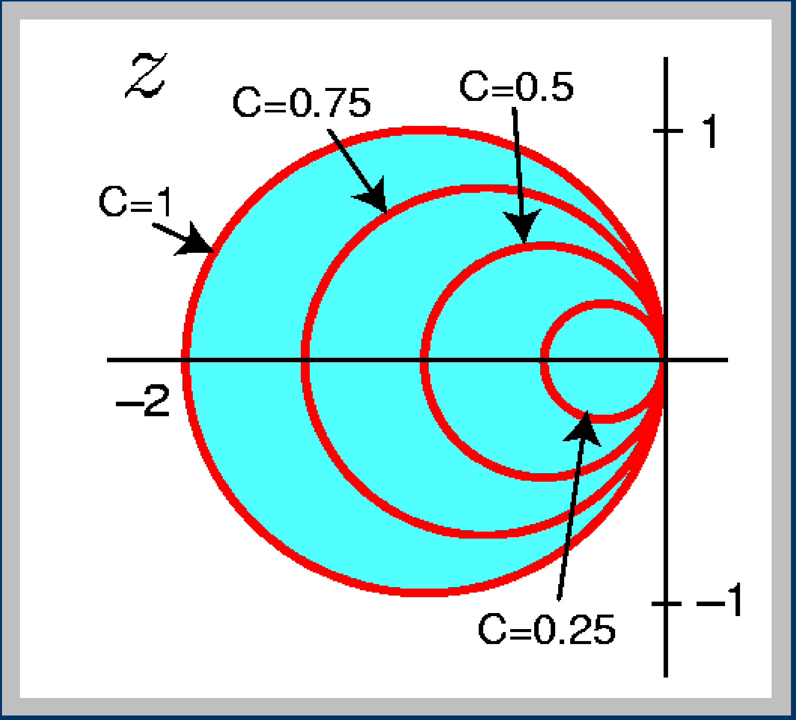
$$z_\theta = -2\mu C \sin^2(\theta/2) - iC\sin(\theta)$$

Method of Lines

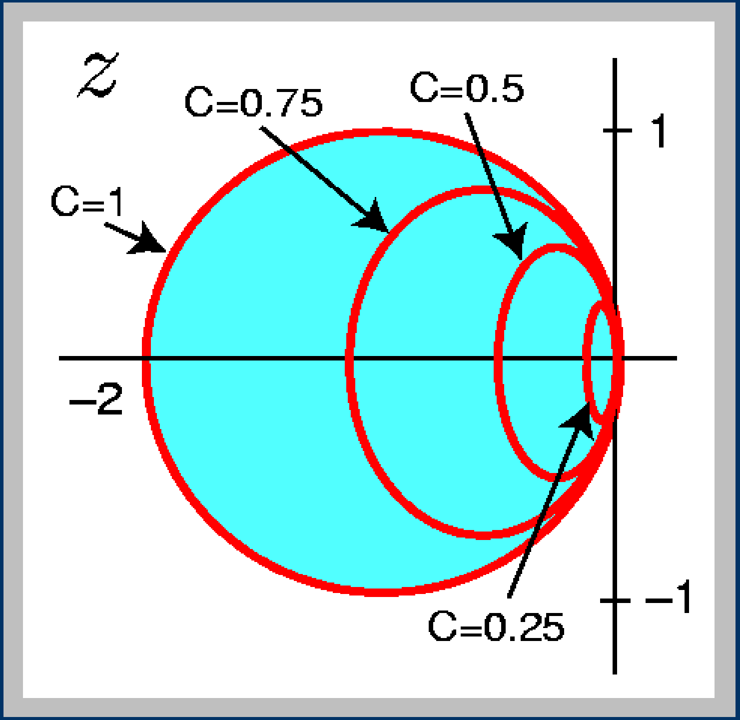
Methods for ODE's

...Application to the Wave Equation...

First Order Upwind Scheme $\mu = 1$



Lax-Wendroff Scheme $\mu = C$



Dissipation and Dispersion

Model Problem

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = \kappa\frac{\partial^2 u}{\partial x^2} - a\frac{\partial^3 u}{\partial x^3}, \quad x \in (0,1)$$

with $u(x,0) = u^0(x)$ and periodic boundary conditions.

Solution

$$u(x,t) = \sum_{k=-\infty}^{k=\infty} \mathbb{U}_k^0 \, e^{-4\pi^2\sigma(k)t} \; e^{i2\pi(kx-\omega(k)t)}$$

$$\sigma(k) = \kappa k^2, \qquad \omega(k) = Uk - a4\pi^2 k^3$$

Dissipation and Dispersion

Model Problem

$e^{-4\pi^2\sigma(k)t}$ represents **Decay**

$\sigma(k)$ dissipation relation

$e^{i2\pi(kx-\omega(k)t)}$ represents **Propagation**

$\omega(k)$ dispersion relation

For the exact solution of $u_t + U u_x = 0$

$\sigma = 0$ **no dissipation**

$\omega = kU$, or $\omega/k = U$ (constant) **no dispersion**

Dissipation and Dispersion

Modified Equation

First Order Upwind

$$u_t + U u_x = \frac{U \Delta x}{2}(1 - C) u_{xx} - \frac{U \Delta x^2}{6}(1 - C^2) u_{xxx}$$

Lax-Wendroff

$$u_t + U u_x = -\frac{U \Delta x^2}{6}(1 - C^2) u_{xxx}$$

Beam-Warming

$$u_t + U u_x = \frac{U \Delta x^2}{6}(2 - C)(1 - C) u_{xxx}$$

Dissipation and Dispersion

Modified Equation

- For the upwind scheme **dissipation** dominates over dispersion $\Rightarrow$ **Smooth** solutions
- For Lax-Wendroff and Beam-Warming **dispersion** is the leading error effect $\Rightarrow$ **Oscillatory** solutions (if not well resolved)
- Lax-Wendroff has a **negative** phase error
- Beam-Warming has (for $C < 1$) a **positive** phase error

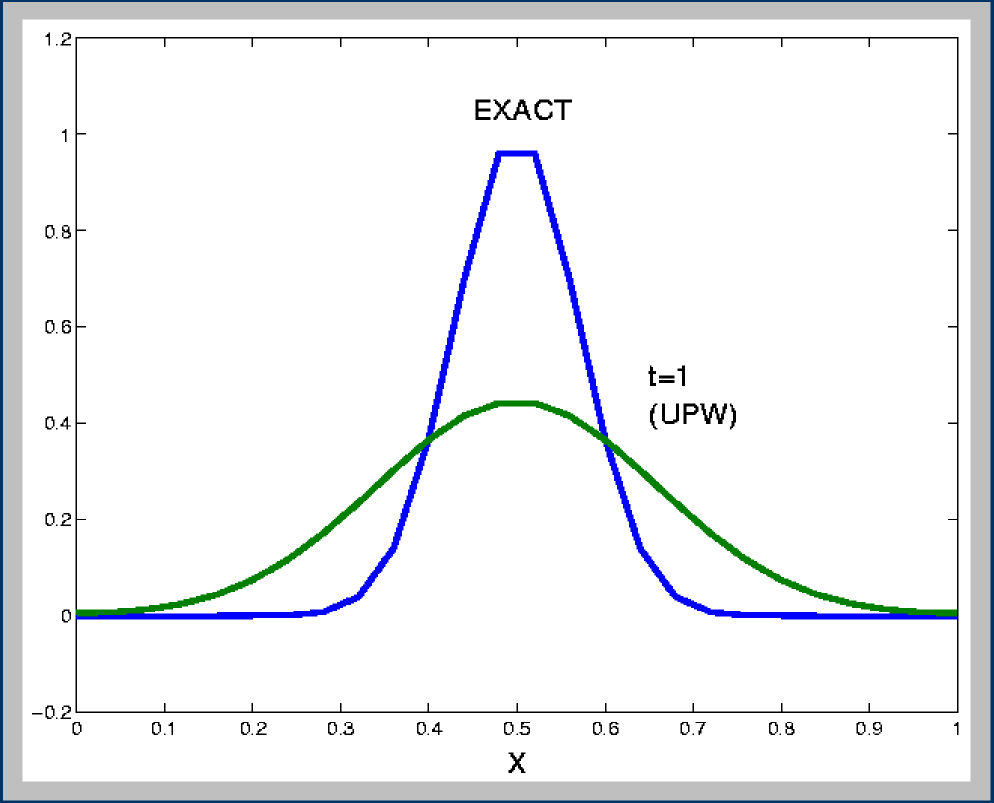
Dissipation and Dispersion

Examples

$$\Delta x = 1/25$$

$$C = 0.5$$

First Order Upwind



Dissipation and Dispersion

Examples

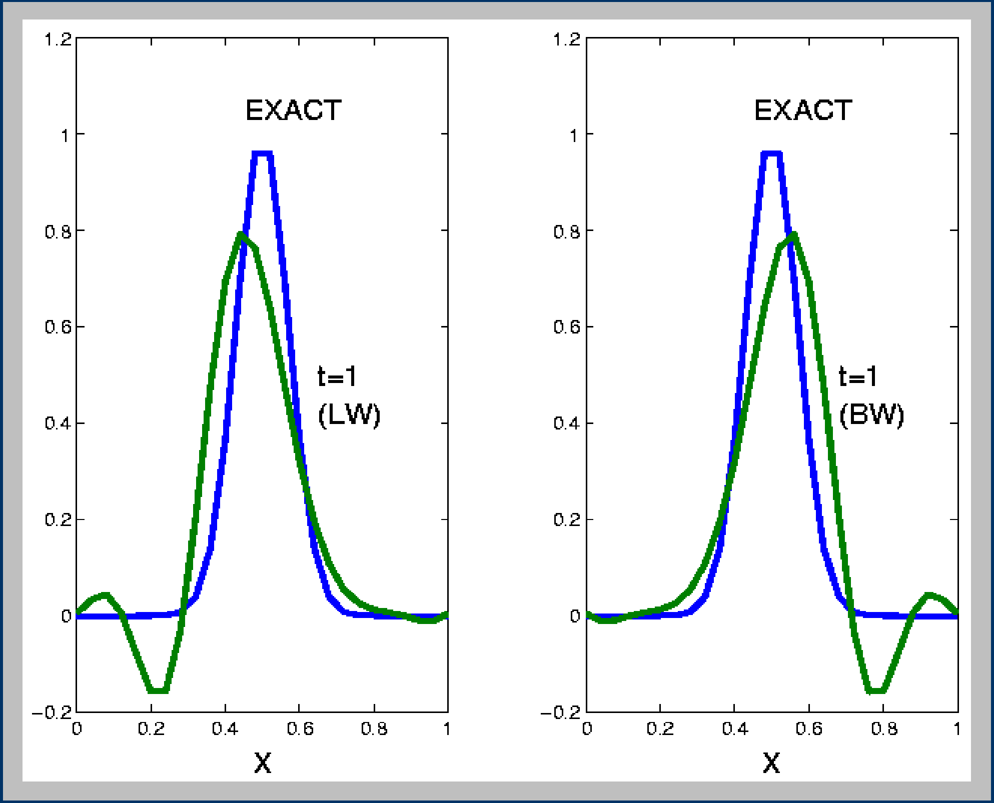
$$\Delta x = 1/25$$

$$C = 0.5$$

Lax-Wendroff (left)

vs.

Beam-Warming (right)



Dissipation and Dispersion

Exact Discrete Relations

For the exact solution

$$\mathbb{U}_\theta^{n+1} = e^{i2\pi kU\Delta t}\,\mathbb{U}_\theta^{n}$$

$$\Rightarrow \quad \omega_{EX} = kU = \theta U/2\pi\Delta x, \text{ and } \sigma_{EX} = 0$$

For the discrete solution

$$\hat{\mathbb{U}}_\theta^{n+1} = g(C,\theta)\,\hat{\mathbb{U}}_\theta^{n}$$

$$g(C,\theta) = e^{-i2\pi\omega(\theta)\Delta t - 4\pi^2\sigma(\theta)\Delta t}$$

$$\Rightarrow \quad \omega(\theta), \text{ and } \sigma(\theta)$$