The repressilator

1. Box 1 of the Elowitz and Leibler paper defines a set of differential equations that model the repressilator. A number of properties of these equations are listed without mathematical derivation. Let's do the math now.

   (a) Find the steady state of the equations satisfying $p_1 = p_2 = p_3 = m_1 = m_2 = m_3 = p$. Show that this symmetric steady state is unique, i.e., that no other steady state is possible.

   (b) Linearize the dynamical equations around the steady state solution. This is done by setting $m_i = p + \delta m_i$ and $p_i = p + \delta p_i$ where δm_i and δp_i are very small and p is the steady state value. Then Taylor expand the right hand side of the dynamical equations to first order in δm_i and δp_i. Define the vector $z^T = (\delta m_1, \delta m_2, \delta m_3, \delta p_1, \delta p_2, \delta p_3)$. Your linearized equations should look like

   $$\frac{dz}{dt} = Az$$

   where

   $$A = \begin{pmatrix} -I & XC \\ \beta I & -\beta I \end{pmatrix}$$

   and X is as defined in the paper. Here I is the 3×3 identity matrix, and C is the cyclic permutation matrix

   $$C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

   (c) Suppose that you are given an eigenvalue λ of A and the corresponding eigenvector $v^T = (\delta m^T, \delta p^T)$. Show that

   $$\beta XC\delta p = (\lambda + \beta)(\lambda + 1)\delta p$$

   In other words, λ is related to the eigenvalues of C. From this fact, find the six eigenvalues of A, and derive the stability condition listed in the paper (warning: there may be a typo in the paper).

   (d) With the help of the stability condition, find a set of parameters for which the repressilator oscillates. Simulate these oscillations using XPP or some other program. Submit your code along with the output of your program.

2. Using the methods I demonstrated in class, construct a stochastic simulation of the repressilator using XPP or some other program. You will need to follow the guidelines sketched in Box 1 of the paper under "Stochastic, discrete approximation." Compare your simulations with Figure 1 of the paper. Submit your code along with the output of the program. Show both oscillatory and nonoscillatory behavior.