1.2.2 Gauss-Jordan Elimination

In the method of Gaussian elimination, starting from a system $A\underline{x} = \underline{b}$ of the general form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ : & : & : & & : \\ : & : & : & & : \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ : \\ : \\ x_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ : \\ : \\ b_n \end{bmatrix} \qquad \textbf{(1.2.2-1)}$$

is converted to an equivalent system $A'\underline{x} = \underline{b}'$ after $\frac{2}{3}N^3$ FLOP's that is of upper triangular form

$$\begin{bmatrix} a'_{11} & a'_{12} & a'_{13} & \dots & a'_{1N} \\ & a'_{22} & a'_{23} & \dots & a'_{2N} \\ & & a'_{33} & \dots & a'_{3N} \\ & & & & a'_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2] \\ : \\ : \\ x_N \end{bmatrix} = \begin{bmatrix} b'_1 \\ b'_2 \\ b'_3 \\ : \\ b'_N \end{bmatrix} \qquad \textbf{(1.2.2-2)}$$

At this point, it is possible, through backward substitution, to solve for the unknowns in the order x_N, x_{N-1}, x_{N-2}, … in $\frac{N^2}{2}$ steps.

In the method of Gauss-Jordan elimination, one continues the work of elimination, placing zeros above the diagonal.
To "zero" the element at (N-1, N), we write the last two equations of **(1.2.2-2)**

$$\left.\begin{aligned} a'_{N-1,N-1}x_{N-1} + a'_{N-1,N}x_N &= b'_{N-1} \\ a'_{N,N}x_N &= b'_N \end{aligned}\right\} \qquad \textbf{(1.2.2-3)}$$

We then define $\lambda_{N-1,N} = \frac{a'_{N-1,N}}{a'_{NN}}$ **(1.2.2-4)**

And replace the N-1st row with the equation obtained after performing the row operation

$$\begin{array}{l} (a'_{N-1,N-1}x_{N-1} + a'_{N-1,N}x_N = b'_{N-1}) \\ \underline{-\lambda_{N-1,N}(a'_{NN}x_N = b'_N)} \\ a'_{N-1,N-1}x_{N-1} + (a'_{N-1,N} - \lambda_{N-1,N}a'_{NN})x_N = b'_{N-1} - b'_N\lambda_{N-1,N} \end{array} \quad \textbf{(1.2.2-5)}$$

Defining

$$b''_{N-1} = b'_{N-1} - b'_N\lambda_{N-1,N} \quad \textbf{(1.2.2-6)}$$

and noting

$$a''_{N-1} = a'_{N-1,N} - \lambda_{N-1,N}a'_{NN} = a'_{N-1,N} - \left(\frac{a'_{N-1,N}}{a'_{NN}}\right)a'_{NN} = 0 \quad \textbf{(1.2.2-7)}$$

After this row operation the set of equations becomes

$$\begin{bmatrix} a'_{11} & a'_{12} & a'_{13} & \dots & a'_{1,N-1} & a'_{1.N} \\ & a'_{22} & a'_{23} & \dots & a'_{2,N-1} & a'_{2,N} \\ & & a'_{33} & \dots & a'_{3,N-1} & a'_{3,N} \\ & & & & : & : \\ & & & & a''_{N-1,N-1} & 0 \\ & & & & & a'_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ : \\ x_{N-1} \\ x_N \end{bmatrix} = \begin{bmatrix} b'_1 \\ b'_2 \\ b'_3 \\ : \\ b''_{N-1} \\ b'_N \end{bmatrix} \quad \textbf{(1.2.2-2)}$$

We can continue this process until the set of equations is in diagonal form

$$\begin{bmatrix} a'''_{11} & & & \\ & a'''_{22} & & \\ & & a'''_{33} & \\ & & & a'''_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ : \\ : \\ x_N \end{bmatrix} = \begin{bmatrix} b'''_1 \\ b'''_2 \\ b'''_3 \\ : \\ b'''_N \end{bmatrix} \quad \textbf{(1.2.2-9)}$$

Dividing each equation by the value of its single coefficient yields

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ : \\ 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ : \\ x_N \end{bmatrix} = \begin{bmatrix} \frac{b_1'''}{a_{11}'''} \\ \frac{b_2'''}{a_{22}'''} \\ \frac{b_3'''}{a_{33}'''} \\ : \\ \frac{b_N'''}{a_{NN}'''} \end{bmatrix} \quad \textbf{(1.2.2-10)}$$

The matrix on the left that has a one everywhere along the principal diagonal and zeros everywhere else is called the identity matrix, and has the property that for any vector $\underline{v}$,

$$I\underline{v} = \underline{v} \quad \textbf{(1.2.2-11)}$$

The form **(1.2.2-10)** therefore immediately gives the solution to the problem.

In practice, we use Gaussian Elimination, stopping at **(1.2.2-2)** to begin backward substitution rather than continue the elimination process because backward substitution is so fast, $N^2 << 2N^3/3$ for all but small problems.

We therefore do not consider the method of Gauss-Jordan Elimination further.