

1.3.1 Interpreting $Ax = b$ as a linear transformation

In Gaussian elimination, we have found a method that returns, if all pivot elements are non-zero, a solution vector $\underline{x} \in \mathbf{R}^N$ to the linear system

$$A\underline{x} = \underline{b} \quad (1.3.1-1)$$

Where $\underline{b} \in \mathbf{R}^N$ and A is a $N \times N$ real matrix.

If we find a solution, how do we know if it is the only one? If we find a zero maximum pivot element, Gaussian elimination fails. How do we know if there is really no solution?

These are crucial questions if we intend linear systems to be the foundation upon which we build our other algorithms.

To answer these questions, we will look at the problem of solving $A\underline{x} = \underline{b}$ in a more abstract language.

By \mathbf{R}^N we mean the space of all real N -dimensional vectors.

What do we mean by “space”?

First, we know that we can “collect” all possible vectors $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix}$ where v_1, \dots, v_N are

all real scalars, to form the set of all N -dimensional vectors. **(1.3.1-2)**

By the term “space”, we make explicit not that the collection of all possible vectors that are real and N-dimensional, \mathbf{R}^N , have the following special properties:

> if $\underline{v} \in \mathbf{R}^N$ and $\underline{w} \in \mathbf{R}^N$ then $\underline{v} + \underline{w} \in \mathbf{R}^N$ (closure under addition) **(1.3.1-3)**

> if $\underline{v} \in \mathbf{R}^N$ and c is a real scalar, $c\underline{v} \in \mathbf{R}^N$ (closure under multiplication by a scalar)
(1.3.1-4)

> for any $\underline{u}, \underline{v}, \underline{w}, \in \mathbf{R}^N$, and for any real scalars c_1, c_2 , we have

$$\begin{array}{ll}
 \underline{u} + (\underline{v} + \underline{w}) = (\underline{u} + \underline{v}) + \underline{w} & c(\underline{v} + \underline{u}) = c\underline{v} + c\underline{u} \\
 \underline{u} + \underline{v} = \underline{v} + \underline{u} & (c_1 + c_2)\underline{v} = c_1\underline{v} + c_2\underline{v} \quad \mathbf{(1.3.1-5)} \\
 \underline{v} + \underline{0} = \underline{v} & (c_1 c_2)\underline{v} = c_1(c_2\underline{v}) \\
 \underline{v} + (-\underline{v}) = \underline{0} & 1\underline{v} = \underline{v}
 \end{array}$$

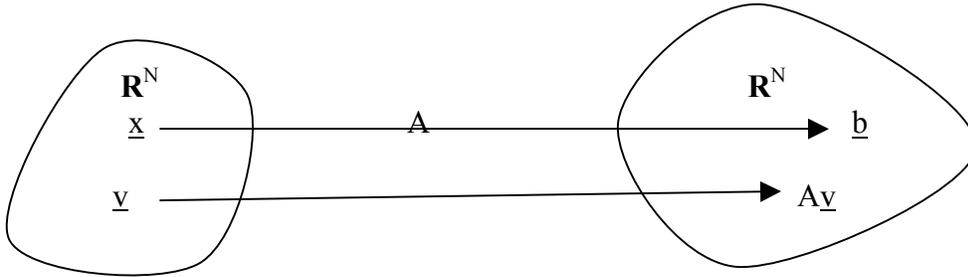
Since these properties hold, we call \mathbf{R}^N a vector space. In our later discussion of functional analysis, we will see that such a concept of a vector space provides a useful, general viewpoint for looking at other problems.

We are concerned now with real vectors, but we note that by merely relaxing the definitions in **(1.3.1-3)**, **(1.3.1-4)**, **(1.3.1-5)** to include all complex scalars and complex N-dimensional vectors, we have the space of all complex N-dimensional vectors \mathbf{C}^N .

This interpretation of A as a linear transformation provides the framework in which we can examine questions of existence and uniqueness of solutions to $A\underline{x} = \underline{b}$.

We interpret the process of solving $A\underline{x} = \underline{b}$ in the following manner.

Given the linear transformation A that maps $\underline{y} \in \mathbf{R}^N$ to a vector $A\underline{y} \in \mathbf{R}^N$; how do we find the vector (or vectors) $\underline{x} \in \mathbf{R}^N$, such that A maps \underline{x} into a particular vector \underline{b} , i.e. $A\underline{x} = \underline{b}$.



Given \underline{b} , how do we find \underline{x} that maps into it?

This conceptual point of view will be of great assistance in understanding when there is or is not a unique solution.