1.3.2 Multiplication of Matrices/Matrix Transpose

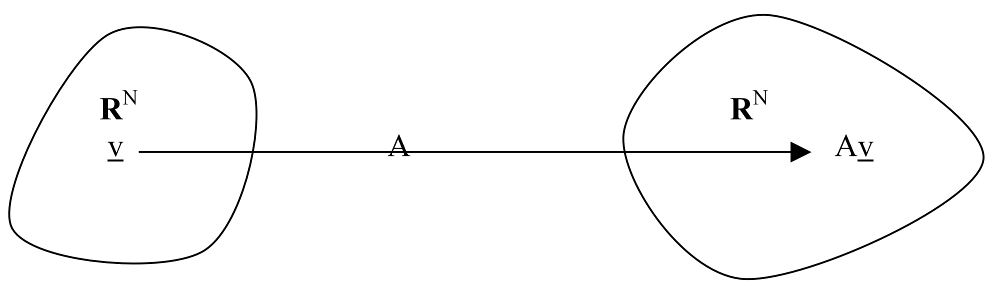
In section 1.3.1, we considered only square matrices, as these are of interest in solving linear problems $A\underline{x} = \underline{b}$.

The interpretation of a matrix as a linear transformation can be extended to non-square matrix. If we consider a M x N real matrix A, then A maps every vector $\underline{v} \in \mathbf{R}^N$ into a vector (now of dimensions m, not N) $A\underline{v} \in \mathbf{R}^N$, according to the rule

$$A\underline{v} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ : & : & : & \\ a_{m1} & a_{m2} & \dots & a_{mN} \end{bmatrix} \begin{bmatrix} v_1 \\ v_{12} \\ : \\ v_N \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \dots + a_{1N}v_N \\ a_{21}v_1 + a_{22}v_2 + \dots + a_{2N}v_N \\ : \\ a_{m1}v_1 + a_{m2}v_2 + \dots + a_{mN}v_N \end{bmatrix} \qquad \textbf{(1.3.2-1)}$$

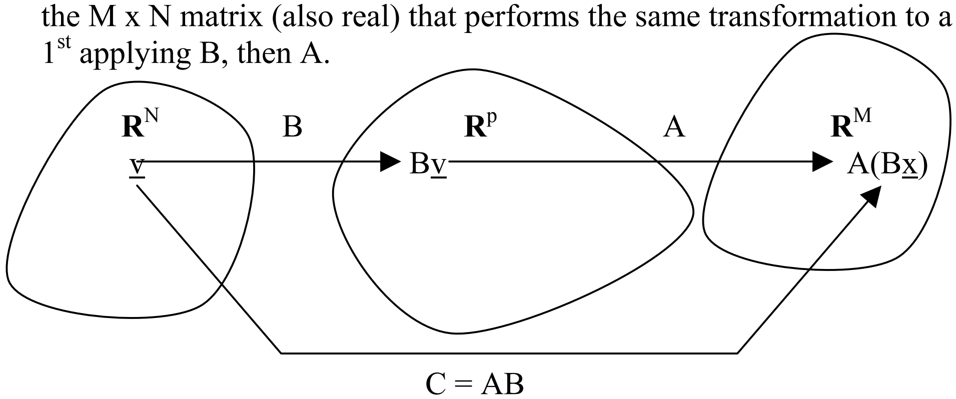
Note that the product $A\underline{v}$ is defined only if the number of columns of A equals the dimensions (# of components) of $\underline{v}$.

We give the M x N matrix A, with all a_{ij} real, the following pictorial interpretation:



The interpretation of non-square matrices as linear transformations provides the following rule for multiplying two real matrices:

Let A be a M x P real matrix, and let B be a P x N real matrix. We define C = AB to be the M x N matrix (also real) that performs the same transformation to a vector $\underline{v} \in \mathbf{R}^N$ as 1st applying B, then A.



First, to $\underline{v} \in \mathbf{R}^N$ we apply B,

$$B\underline{v} = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1N} \\ b_{21} & b_{22} & \dots & b_{2N} \\ : & : & & : \\ b_{p1} & b_{p2} & \dots & b_{pN} \end{bmatrix} \begin{bmatrix} v_1 \\ v_{12} \\ : \\ v_N \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{N} b_{1j} v_j \\ \sum_{j=1}^{N} b_{2j} v_j \\ : \\ \sum_{j=1}^{N} b_{pj} v_j \end{bmatrix} \qquad \textbf{(1.3.2-2)}$$

We then apply A to $B\underline{v} \in \mathbf{R}^p$,

$$A(B\underline{v}) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1p} \\ a_{21} & a_{22} & \dots & a_{2p} \\ : & : & & : \\ a_{m1} & a_{m2} & \dots & a_{mp} \end{bmatrix} \begin{bmatrix} \sum_{j=1}^{N} b_{1j} v_j \\ \sum_{j=1}^{N} b_{2j} v_j \\ : \\ \sum_{j=1}^{N} b_{pj} v_j \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{p} a_{1k} \sum_{j=1}^{N} b_{kj} v_j \\ \sum_{k=1}^{p} a_{2k} \sum_{j=1}^{N} b_{kj} v_j \\ : \\ \sum_{k=1}^{p} a_{mk} \sum_{j=1}^{N} b_{kj} v_j \end{bmatrix} \qquad \textbf{(1.3.2-3)}$$

If we compare to

$$C\underline{v} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1N} \\ c_{21} & c_{22} & \dots & c_{2N} \\ : & : & & : \\ c_{m1} & c_{m2} & \dots & c_{mN} \end{bmatrix} \begin{bmatrix} v_1 \\ v_{12} \\ : \\ v_N \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{N} c_{1j} v_j \\ \sum_{j=1}^{N} c_{2j} v_j \\ : \\ \sum_{j=1}^{N} c_{mj} v_j \end{bmatrix} \qquad \textbf{(1.3.2-4)}$$

We see that rearranging **(1.3.2-2)** yields

$$A(B\underline{v}) = \begin{bmatrix} \sum_{j=1}^{N}\left(\sum_{k=1}^{p} a_{1k} b_{kj}\right) v_j \\ \sum_{j=1}^{N}\left(\sum_{k=1}^{p} a_{2k} b_{kj}\right) v_j \\ \vdots \\ \sum_{j=1}^{N}\left(\sum_{k=1}^{p} a_{mk} b_{kj}\right) v_j \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{N} c_{1j} v_j \\ \sum_{j=1}^{N} c_{2j} v_j \\ \vdots \\ \sum_{j=1}^{N} c_{mj} v_j \end{bmatrix} \qquad \textbf{(1.3.2-5)}$$

The (i,j) element of the matrix C = AB is therefore

$$C_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} \qquad \textbf{(1.3.2-6)}$$

We compute this element by summing the product of elements A along row #I from left → right with those elements of B in column #j from top → bottom.

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ & \rightarrow & & \\ a_{i1} & a_{i2} & \cdots & a_{ip} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1N} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2N} \\ \vdots & \vdots & & \downarrow\vdots & & \\ b_{p1} & b_{p2} & \cdots & b_{pj} & \cdots & b_{pN} \end{bmatrix}$$

$$= \begin{bmatrix} \vdots \\ \vdots \\ \ldots\ldots c_{ij} \ldots\ldots \\ \vdots \\ \vdots \end{bmatrix} \text{row \#i}$$

column # j
(1.3.2-7)

column # j

row # i

We note that the product of two matrices A and B, C = AB, is defined only if the number of columns of A equals the number of rows of B.

Note also that in general AB $\neq$ BA **(1.3.2-8).** We define the <u>commutator</u> of A and B as

$$[A,B] \equiv AB - BA \qquad \textbf{(1.3.2-9)}$$

Note that we can interpret our rule for multiplying a vector $\underline{v} \in \mathbf{R}^N$ by an M x N matrix A by considering $\underline{v}$ to be a matrix of dimension N x 1, i.e. a column vector.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mN} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{N} a_{1k} v_k \\ \sum_{k=1}^{N} a_{2k} v_k \\ \vdots \\ \sum_{k=1}^{N} a_{mk} v_k \end{bmatrix} \qquad \textbf{(1.3.2-10)}$$

This is the convention that we will use. We can also write $\underline{v}$ as a row vector by taking the transpose,

$$\underline{v}^T = [v_1 \quad v_2 \quad \dots \quad v_N] \qquad \textbf{(1.3.2-11)}$$

We see that $\underline{v}^T$ is a 1 x N matrix.
The dot product $\underline{v} \bullet \underline{w}$ can therefore be written for $\underline{v}, \underline{w} \in \mathbf{R}^N$

$$\underline{v} \bullet \underline{w} = \underline{v}^T \underline{w} = [v_1 \quad v_2 \quad \dots \quad v_N] \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix} = v_1 w_1 + v_2 w_2 + \dots + v_N w_N \qquad \textbf{(1.3.2-12)}$$

We define a matrix transpose operation on a real matrix A of M rows and N columns as

$$A^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1N} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2N} \\ \vdots & \vdots & \vdots & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mN} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ a_{13} & a_{23} & \dots & a_{m3} \\ \vdots & \vdots & & \\ a_{1N} & a_{2N} & \dots & a_{mN} \end{bmatrix} \qquad \textbf{(1.3.2-13)}$$

If A is an M x N matrix, A^T is N x M and $(A^T)_{ij} = a_{ji}$ **(1.3.2-14)**