1.3.3 Basis sets and Gram-Schmidt Orthogonalization

Before we address the question of existence and uniqueness, we must establish one more tool for working with vectors – basis sets.

$$\text{Let } \underline{v} \in \mathbf{R}^N, \text{ with } \underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_3 \end{bmatrix} \qquad \textbf{(1.3.3-1)}$$

We can obviously define the set of N unit vectors

$$\underline{e}^{[1]} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \underline{e}^{[2]} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \ldots \quad \underline{e}^{[N]} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \qquad \textbf{(1.3.3-2)}$$

so that we can write $\underline{v}$ as

$$\underline{v} = v_1\underline{e}^{[1]} + v_2\underline{e}^{[2]} + \ldots + v_N\underline{e}^{[N]} \qquad \textbf{(1.3.3-3)}$$

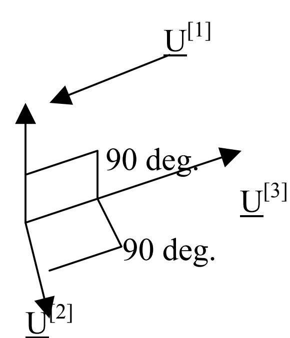
As <u>any</u> $\underline{v} \in \mathbf{R}^N$ can be written in this manner, the set of vectors $\{\underline{e}^{[1]}, \underline{e}^{[2]}, \ldots \underline{e}^{[N]}\}$ are said to form a <u>basis</u> for the vector space $\mathbf{R}^N$.

The same function can be performed by any set of mutually orthogonal vectors, i.e. a set of vectors $\{\underline{U}^{[1]}, \underline{U}^{[2]}, \ldots, \underline{U}^{[N]}\}$ such that

$$\underline{U}^{[j]} \bullet \underline{U}^{[k]} = 0 \quad \text{if } j \neq k \; \textbf{(1.3.3-4)}$$

This means that each $\underline{U}^{[j]}$ is mutually orthogonal to all of the other vectors. We can then write <u>any</u> $\underline{v} \in \mathbf{R}^N$ as

$$\underline{v} = v_1'\underline{e}^{[1]} + v_2'\underline{e}^{[2]} + \ldots + v_N'\underline{e}^{[N]} \qquad \textbf{(1.3.3-5)}$$



Where we use a prime to denote that

$$v_j' \neq v_j \qquad \textbf{(1.3.3-6)}$$

when comparing the expansions **(1.3.3-3)** and **(1.3.3-5)**

Orthogonal basis sets are very easy to use since the coefficients of a vector $\underline{v} \in \mathbf{R}^N$ in the expansion are easily determined.

We take the dot product of **(1.3.3-5)** with any basis vector $\underline{U}^{[k]}$, $k \in [1,N]$,

$$\underline{v} \bullet \underline{U}^{[k]} = v_1'(\underline{U}^{[1]} \bullet \underline{U}^{[k]}) + \ldots + v_k'(\underline{U}^{[k]} \bullet \underline{U}^{[k]}) + \ldots + v_N'(\underline{U}^{[N]} \bullet \underline{U}^{[k]}) \qquad \textbf{(1.3.3-6)}$$

Because

$$\underline{U}^{[j]} \bullet \underline{U}^{[k]} = (\underline{U}^{[k]} \bullet \underline{U}^{[k]})\delta_{jk} = \left|\underline{U}^{[k]}\right|^2 \delta_{jk} \qquad \textbf{(1.3.3-7)}$$

with

$$\delta_{jk} = \begin{cases} 1, j = k \\ 0, j \neq k \end{cases} \qquad \textbf{(1.3.3-8)}$$

then **(1.3.3-6)** becomes

$$\underline{v} \bullet \underline{U}^{[k]} = v_k' \left|\underline{U}^{[k]}\right|^2 \Rightarrow v_k' = \frac{\underline{v} \bullet \underline{U}^{[k]}}{\left|\underline{U}^{[k]}\right|^2} \qquad \textbf{(1.3.3-9)}$$

In the special case that all basis vectors are <u>normalized</u>, i.e. $\left|\underline{U}^{[k]}\right| = 1$ for all $k \in [1,N]$, we have an <u>orthonormal</u> basis set, and the coefficients of $\underline{v} \in \mathbf{R}^N$ are simply the dot products with each basis set vector.

Exmaple 1.3.3-1

Consider the orthogonal basis for $\mathbf{R}^3$

$$\underline{U}^{[1]} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \underline{U}^{[2]} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \underline{U}^{[3]} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \textbf{(1.3.3-10)}$$

for any $\underline{v} \in \mathbf{R}^3$, $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ what are the coefficients of the expansion

$$\underline{v} = v_1'\underline{U}^{[1]} + v_2'\underline{U}^{[2]} + v_3'\underline{U}^{[3]} \qquad \textbf{(1.3.3-11)}$$

First, we check the basis set for orthogonality

$$\underline{U}^{[2]} \bullet \underline{U}^{[2]} = [\,1 \quad 1 \quad 0] \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = (1)(1) + (1)(-1) + (0)(0) = 0$$

$$\underline{U}^{[1]} \bullet \underline{U}^{[3]} = [\,1 \quad 1 \quad 0] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = (1)(0) + (1)(0) + (0)(1) = 0 \qquad \textbf{(1.3.3-12)}$$

$$\underline{U}^{[2]} \bullet \underline{U}^{[3]} = [1 \quad -1 \quad 0] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = (1)(0) + (-1)(0) + (0)(1) = 0$$

We also have

$$\left|\underline{U}^{[1]}\right|^2 = [1 \quad 1 \quad 0] \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 2$$

$$\left|\underline{U}^{[2]}\right|^2 = [1 \quad -1 \quad 0] \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 2$$

$$\left|\underline{U}^{[3]}\right|^2 = [0 \quad 0 \quad 1] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1$$

(1.3.3-13)

So the coefficients of $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ are

$$v_1' = \frac{\underline{v} \bullet \underline{U}^{[1]}}{\left|\underline{U}^{[1]}\right|^2} = \frac{1}{2}[v_1 \quad v_2 \quad v_3] \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2}(v_1 + v_2)$$

$$v_2' = \frac{\underline{v} \bullet \underline{U}^{[2]}}{\left|\underline{U}^{[2]}\right|^2} = \frac{1}{2}[v_1 \quad v_2 \quad v_3] \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{2}(v_1 - v_2)$$

$$v_3' = \frac{\underline{v} \bullet \underline{U}^{[3]}}{\left|\underline{U}^{[3]}\right|^2} = \frac{1}{1}[v_1 \quad v_2 \quad v_3] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = v_3$$

Although orthogonal basis sets are very convenient to use, a set of N vectors $B = \{\underline{b}^{[1]}, \underline{b}^{[2]}, \ldots, \underline{b}^{[N]}\}$ need not be mutually orthogonal to be used as a basis – they need merely be linearly independent.

Let us consider a set of $M \le N$ vectors $\underline{b}^{[1]}, \underline{b}^{[2]}, \ldots, \underline{b}^{[M]} \in \mathbf{R}^N$. This set of M vectors is said to be linearly independent if

$$c_1\underline{b}^{[1]} + c_2\underline{b}^{[2]} + \ldots + c_M\underline{b}^{[M]} = 0 \quad \text{implies } c_1=c_2=\ldots=c_M=0 \qquad \textbf{(1.3.3-16)}$$

This means that no $\underline{b}^{[j]}$, $j \in [1,M]$ can be written as a linear combination of the other M-1 basis vectors.

For example, the set of 3 vectors for $\mathbf{R}^3$

$$\underline{b}^{[1]} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \quad \underline{b}^{[2]} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \underline{b}^{[3]} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \textbf{(1.3.3-17)}$$

is not linearly independent because we can write $\underline{b}^{[3]}$ as a linear combination of $\underline{b}^{[1]}$ and $\underline{b}^{[2]}$,

$$\underline{b}^{[1]} - \underline{b}^{[2]} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \underline{b}^{[3]} \qquad \textbf{(1.3.3-18)}$$

Here, a vector $\underline{v} \in \mathbf{R}^N$ is said to be a linear combination of the vectors $\underline{b}^{[1]}, \ldots, \underline{b}^{[M]} \in \mathbf{R}^N$ if it can be written as

$$\underline{v} = v_1'\underline{b}^{[1]} + v_2'\underline{b}^{[2]} + \ldots + v_M'\underline{b}^{[M]} \qquad \textbf{(1.3.3-19)}$$

We see that the 3 vectors of **(1.3.3-17)** do not form a basis for $\mathbf{R}^3$ since we cannot express any vector $\underline{v} \in \mathbf{R}^3$ with $v_3 \neq 0$ as a linear combination of $\{\underline{b}^{[1]}, \underline{b}^{[2]}, \underline{b}^{[3]}\}$ since

$$\underline{v} = v_1' \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + v_2' \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v_3' \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2v_1' - v_2' + v_3' \\ v_2' - v_3' \\ 0 \end{bmatrix} \qquad \textbf{(1.3.3-20)}$$

We see however that if we instead had the set of 3 linearly independent vectors

$$\underline{b}^{[1]} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \quad \underline{b}^{[2]} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \underline{b}^{[3]} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \qquad \textbf{(1.3.3-21)}$$

then we could write any $\underline{v} \in \mathbf{R}^3$ as

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1' \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + v_2' \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v_3' \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2v_1' + v_2' \\ v_2' \\ 2v_3' \end{bmatrix} \qquad \textbf{(1.3.3-22)}$$

(1.3.3-22) defines a set of 3 simultaneous linear equations

$$\begin{aligned} 2v_1' + v_2' &= v_1 \\ v_2' &= v_2 \\ 2v_3' &= v_3 \qquad \textbf{(1.3.3-23)} \end{aligned}$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

that we must solve for v_1', v_2', v_3',

$$v_1' = \frac{v_3}{2}, \qquad v_2' = v_2, \qquad v_1' = \frac{(v_1 - v_2')}{2} \qquad \textbf{(1.3.3-24)}$$

We therefore make the following statement:

Any set B of N linearly independent vectors $\underline{b}^{[1]}, \underline{b}^{[2]}, \ldots, \underline{b}^{[N]} \in \mathbf{R}^N$ can be used as a basis for $\mathbf{R}^N$.

We can pick any M subset of the linearly independent basis B, and define the span of this subset $\{\underline{b}^{[1]}, \underline{b}^{[2]}, \ldots, \underline{b}^{[M]}\} \subset$ B as the space of all possible vectors $\underline{v} \in \mathbf{R}^N$ that can be written as

$$\underline{v} = c_1 b^{[1]} + c_2 \underline{b}^{[2]} + \ldots + c_M \underline{b}^{[M]} \quad \textbf{(1.3.3-25)}$$

For the basis set **(1.3.3-21)**, we choose $\underline{b}^{[1]} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ and $\underline{b}^{[3]} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$.**(1.3.3-26)**

Then, span $\{\underline{b}^{[1]}, \underline{b}^{[3]}\}$ is the set of all vectors $\underline{v} \in \mathbf{R}^3$ that can be written as

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = c_1 b^{[1]} + c_3 b^{[3]} = c_1 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2c_1 \\ 0 \\ 2c_3 \end{bmatrix} \quad \textbf{(1.3.3-27)}$$

Therefore, for this case it is easy to see that $\underline{v} \in$ span $\{\underline{b}^{[1]}, \underline{b}^{[3]}\}$, if and only if ("iff") $v_2 = 0$.

Note that if $\underline{v} \in \text{span}\{\underline{b}^{[1]}, \underline{b}^{[3]}\}$ and $\underline{w} \in \text{span}\{\underline{b}^{[1]}, \underline{b}^{[3]}\}$, then automatically $\underline{v} + \underline{w} \in \text{span}\{\underline{b}^{[1]}, \underline{b}^{[3]}\}$.

We see then that $\text{span}\{\underline{b}^{[1]}, \underline{b}^{[3]}\}$ itself satisfies all the properties of a vector space identified in section 1.3.1.

Since $\text{span}\{\underline{b}^{[1]}, \underline{b}^{[3]}\} \subset \mathbf{R}^3$ (i.e. it is a subset of $\mathbf{R}^3$), we call $\text{span}\{\underline{b}^{[1]}, \underline{b}^{[3]}\}$ a subspace of $\mathbf{R}^3$.

This concept of basis sets also lets us formally identify the meaning of dimension – this will be useful in the establishment of criteria for existence/uniqueness of solutions.

Let us consider a vector space V that satisfies all the properties of a vector space identified in section 1.3.1.
We say that the dimension of V is N if every set of N+1 vectors $\underline{v}^{[1]}, \underline{v}^{[2]}, \ldots, \underline{v}^{[N+1]} \in V$ is linearly independent and if there exists some set of N linearly independent vectors $\underline{b}^{[1]}, \ldots, \underline{b}^{[N]} \in V$ that forms a basis for V. We say then that dim(V) = N. **(1.3.3-28)**

While linearly independent basis sets are completely valid, they are more difficult to use than orthogonal basis sets because one must solve a set of N linear algebraic equations to find the coefficients of the expansion

$$\underline{v} = v_1' \underline{b}^{[1]} + v_2' \underline{b}^{[2]} + \ldots + v_N' \underline{b}^{[N]} \qquad \textbf{(1.3.3-29)}$$

$$\begin{bmatrix} b_1^{[1]} & b_1^{[2]} & \ldots & b_1^{[N]} \\ b_2^{[1]} & b_2^{[2]} & \ldots & b_2^{[N]} \\ : & : & & : \\ b_N^{[1]} & b_N^{[2]} & \ldots & b_N^{[N]} \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \\ : \\ v_N' \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ : \\ v_N \end{bmatrix} \longleftarrow \text{O(N}^3\text{) effort to solve for all } v_j\text{'s} \qquad \textbf{(1.3.3-30)}$$

This requires more effort for an orthogonal basis $\{\underline{U}^{[1]}, \ldots, \underline{U}^{[N]}\}$ as

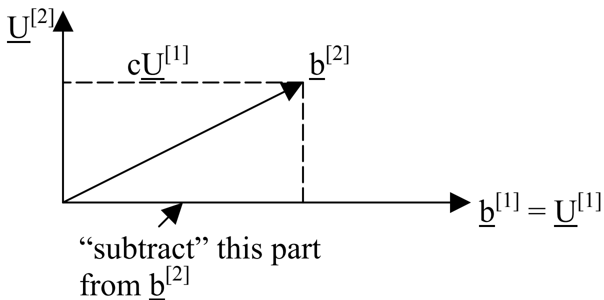
$$v_j' = \frac{\underline{v} \bullet \underline{U}^{[j]}}{\underline{U}^{[j]} \bullet \underline{U}^{[j]}} \longleftarrow \text{O(N}^2\text{) effort to find all } v_j\text{'s}$$

(1.3.3-9, repeated)

This provides an impetus to perform Gramm-Schmidt orthogonalization. We start with a linearly independent basis set $\{\underline{b}^{[1]}, \underline{b}^{[2]}, \ldots, \underline{b}^{[N]}\}$ for $\mathbf{R}^N$. From this set, we construct an orthogonal basis set $\{\underline{U}^{[1]}, \underline{U}^{[2]}, \ldots, \underline{U}^{[N]}\}$ through the following procedure:

1. First, set $\underline{U}^{[1]} = \underline{b}^{[1]}$ **(1.3.3-31)**

2. Next, we construct $\underline{U}^{[2]}$ such that $\underline{U}^{[2]} \bullet \underline{U}^{[1]} = 0$. Since $\underline{U}^{[1]} = \underline{b}^{[1]}$, and $\underline{b}^{[2]}$ and $\underline{b}^{[1]}$ are linearly independent, we can form an orthogonal vector $\underline{U}^{[2]}$ from $\underline{b}^{[2]}$ by the following procedure:



We write $\underline{U}^{[2]} = \underline{b}^{[2]} + c\underline{U}^{[1]}$ **(1.3.3-32)**

Then, taking the dot product with $\underline{U}^{[1]}$,

$$\underline{U}^{[2]} \bullet \underline{U}^{[1]} = 0 = \underline{b}^{[2]} \bullet \underline{U}^{[1]} + c\underline{U}^{[1]} \bullet \underline{U}^{[1]} \qquad \textbf{(1.3.3-33)}$$

Therefore

$$c = \frac{-\underline{b}^{[2]} \bullet \underline{U}^{[1]}}{\left|\underline{U}^{[1]}\right|^2} \qquad \textbf{(1.3.3-34)}$$

And our 2nd vector in the orthogonal basis is

$$\underline{U}^{[2]} = \underline{b}^{[2]} - \left[\frac{\underline{b}^{[2]} \bullet \underline{U}^{[1]}}{\left|\underline{U}^{[1]}\right|^2}\right]\underline{U}^{[1]} \qquad \textbf{(1.3.3-35)}$$

3. We now form $\underline{U}^{[3]}$ in a similar manner.
Since $\underline{U}^{[2]}$ is a linear combination of $\underline{b}^{[1]}$ and $\underline{b}^{[2]}$, we can add a component from $\underline{b}^{[3]}$ direction to form $\underline{U}^{[3]}$,

$$\underline{U}^{[3]} = \underline{b}^{[3]} + c_2\underline{U}^{[2]} + c_1\underline{U}^{[1]} \qquad \textbf{(1.3.3-36)}$$

First, we want $\underline{U}^{[3]} \bullet \underline{U}^{[1]} = 0 = \underline{b}^{[3]} \bullet \underline{U}^{[1]} + c_2 \underbrace{\underline{U}^{[2]} \bullet \underline{U}^{[1]}}_{=0} + c_1 \underline{U}^{[1]} \bullet \underline{U}^{[1]}$ **(1.3.3-37)**

so

$$c_1 = \frac{-\underline{b}^{[3]} \bullet \underline{U}^{[1]}}{\left|\underline{U}^{[1]}\right|^2} \qquad \textbf{(1.3.3-38)}$$

A similar condition that $\underline{U}^{[3]} \bullet \underline{U}^{[2]} = 0$ yields

$$c_2 = \frac{-\underline{b}^{[3]} \bullet \underline{U}^{[2]}}{\left|\underline{U}^{[2]}\right|^2} \qquad \textbf{(1.3.3-39)}$$

so that the 3rd member of the orthogonal basis set is

$$\underline{U}^{[3]} = \underline{b}^{[3]} - \left[\frac{\underline{b}^{[3]} \bullet \underline{U}^{[2]}}{\left|\underline{U}^{[2]}\right|^2}\right]\underline{U}^{[2]} - \left[\frac{\underline{b}^{[3]} \bullet \underline{U}^{[1]}}{\left|\underline{U}^{[1]}\right|^2}\right]\underline{U}^{[1]} \qquad \textbf{(1.3.3-40)}$$

4. Continue for $\underline{U}^{[j]}$, j = 4, 5, …, N where

$$\underline{U}^{[j]} = \underline{b}^{[j]} - \sum_{k=1}^{j-1}\left[\frac{\underline{b}^{[j]} \bullet \underline{U}^{[k]}}{\left|\underline{U}^{[k]}\right|^2}\right]\underline{U}^{k} \qquad \textbf{(1.3.3-41)}$$

5. Normalize vectors if desired (we can do this also during construction of orthogonal basis set)

$$\underline{U}^{[j]} \leftarrow \frac{\underline{U}^{[j]}}{\left|\underline{U}^{[j]}\right|} \qquad \textbf{(1.3.3-42)}$$

As an example, let us use this method to generate an orthogonal basis for $\mathbf{R}^3$ such that the 1st member of the basis set is

$$\underline{U}^{[1]} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \textbf{(1.3.3-43)}$$

First, we write a linearly independent basis that is not, in general, orthogonal. For example, we could choose

$$\underline{b}^{[1]} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \underline{b}^{[2]} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \underline{b}^{[3]} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \textbf{(1.3.3-44)}$$

We now perform Gram-Schmidt orthogonalization,

1. $\underline{U}^{[1]} = \underline{b}^{[1]} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \textbf{(1.3.3-45)}$
2. We next set

$$\underline{U}^{[2]} = \underline{b}^{[2]} - \left[\frac{\underline{b}^{[2]} \bullet \underline{U}^{[1]}}{\left|\underline{U}^{[1]}\right|^2} \right] \underline{U}^{[1]} \qquad \textbf{(1.3.3-35, repeated)}$$

$$\left|\underline{U}^{[1]}\right|^2 = [1 \quad 1 \quad 0] \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 2 \qquad \textbf{(1.3.3-46)}$$

$$\underline{b}^{[2]} \bullet \underline{U}^{[1]} = [1 \quad 0 \quad 0] \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 \qquad \textbf{(1.3.3-47)}$$

so

$$\underline{U}^{[2]} = \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\0\end{bmatrix} \quad \textbf{(1.3.3-48)}$$

$$\text{Note } \underline{U}^{[2]} \bullet \underline{U}^{[1]} = [1/2 \quad -1/2 \quad 0]\begin{bmatrix}1\\1\\0\end{bmatrix} = ½ - ½ = 0 \quad \textbf{(1.3.3-49)}$$

We now calculate

$$\underline{U}^{[3]} = \underline{b}^{[3]} - \left[\frac{\underline{b}^{[3]} \bullet \underline{U}^{[2]}}{\left|\underline{U}^{[2]}\right|^2}\right]\underline{U}^{[2]} - \left[\frac{\underline{b}^{[3]} \bullet \underline{U}^{[1]}}{\left|\underline{U}^{[1]}\right|^2}\right]\underline{U}^{[1]}$$

(1.3.3-41, repeated)

$$\left|\underline{U}^{[2]}\right|^2 = [1/2 \quad -1/2 \quad 0]\begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\0\end{bmatrix} = \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \quad \textbf{(1.3.3-50)}$$

$$\underline{b}^{[3]} \bullet \underline{U}^{[2]} = [0 \quad 0 \quad 1]\begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\0\end{bmatrix} = 0 \quad \textbf{(1.3.3-51)}$$

$$\underline{b}^{[3]} \bullet \underline{U}^{[1]} = [0 \quad 0 \quad 1]\begin{bmatrix}1\\1\\0\end{bmatrix} = 0 \quad \textbf{(1.3.3-52)}$$

We therefore have merely $\underline{U}^{[3]} = \underline{b}^{[3]} = \begin{bmatrix}0\\0\\1\end{bmatrix}$ **(1.3.3-53)**

Our orthogonal basis set is therefore

$$\underline{U}^{[1]} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \underline{U}^{[2]} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} \qquad \underline{U}^{[3]} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \textbf{(1.3.3-54)}$$