1.3.4 Null Space (kernel) and Existence/Uniqueness of Solutions

We now have the tools necessary to consider the existence and uniqueness of solutions to the linear system of equations

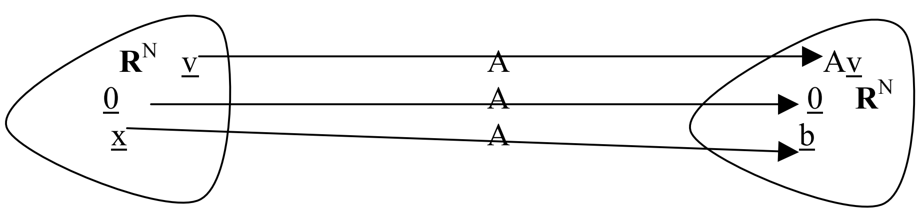
$A\underline{x} = \underline{b}$ **(1.3.4-1)**

Where $\underline{x}, \underline{b} \in \mathbf{R}^N$ and A is a N x N real matrix.

As described in section 1.3.1, we interpret A as a linear transformation that maps each $\underline{v} \in \mathbf{R}^N$ into some $A\underline{v} \in \mathbf{R}^N$ according to the rule

$$A\underline{v} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ : & : & & : \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{bmatrix} \begin{bmatrix} v_1 \\ v_{12} \\ : \\ v_N \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \dots + a_{1N}v_N \\ a_{21}v_1 + a_{22}v_2 + \dots + a_{2N}v_N \\ : \\ a_{N1}v_1 + a_{2N}v_2 + \dots + a_{NN}v_N \end{bmatrix} \quad \textbf{(1.3.4-2)}$$

Pictorially, we view the problem of solving $A\underline{x} = \underline{b}$ as finding the (or one of many?) vector(s) $\underline{x} \in \mathbf{R}^N$ that maps into a specific $\underline{b}$ under A.



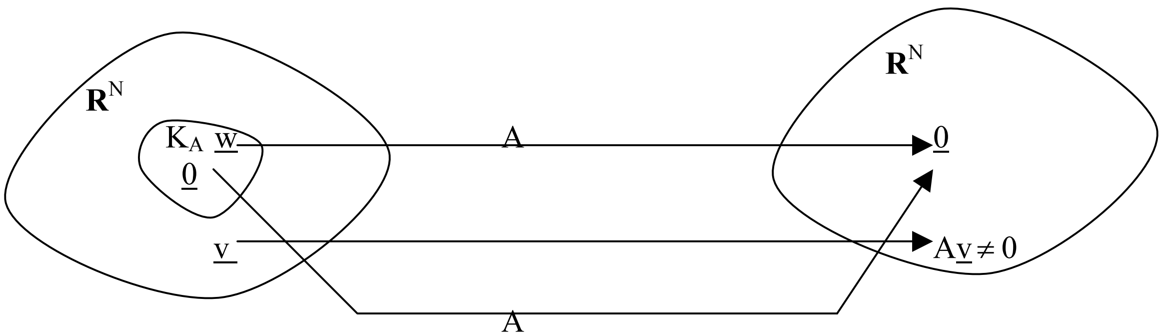
Here we have shown that for any real matrix A, the rule for forming $A\underline{v}$ **(1.3.4-3)** guarantees that

$A\underline{0}=\underline{0}$ **(1.3.4-4)**

Where $\underline{0}$ is the <u>null vector</u>, $\underline{0} = \begin{bmatrix} 0 \\ 0 \\ : \\ 0 \end{bmatrix}$ **(1.3.4-5)**

We always have one vector, $\underline{0}$, that maps into $\underline{0}$ under A. Crucial to the question of existence and uniqueness of solutions is the existence of any other vectors $\underline{w} \neq \underline{0}$ that also map into $\underline{0}$ under A.

We define the null space (or kernel) of a real matrix A to be the set of all vectors $\underline{w} \in \mathbf{R}^N$ such that $A\underline{w} = \underline{0}$. Pictorially, we view the kernel of A, denoted K_A, as



We use the concept of the kernel (null space) to prove the following theorems on existence/uniqueness of solutions to $A\underline{x} = \underline{b}$.

Theorem 1.3.4.1

Let $\underline{x} \in \mathbf{R}^N$ be a solution to the linear system $A\underline{x} = \underline{b}$, where $\underline{b} \in \mathbf{R}^N$, A is an N x N real matrix. If the kernel of A contains only the null vector, i.e. $K_A = \underline{0}$, then this solution is unique (no other solutions exist).

Proof:

Let $\underline{x}$ satisfy $A\underline{x} = \underline{b}$. Let $\underline{y}$ be some vector in $\mathbf{R}^N$ that also satisfies the system of equations $A\underline{y} = \underline{b}$.

If we define $\underline{v} = \underline{y} - \underline{x}$, we can write this 2nd solution as

$$\underline{y} = \underline{x} + \underline{v} \qquad \textbf{(1.3.4-6)}$$

Then,

$$A\underline{y} = A(\underline{x} + \underline{v}) = A\underline{x} + A\underline{v} \qquad \textbf{(1.3.4-7)}$$

Since $\underline{x}$ is a solution, $A\,\underline{x} = \underline{b}$, and

$$A\underline{y} = \underline{b} + A\underline{v} \qquad \textbf{(1.3.4-8)}$$

If $\underline{y}$ is to be a solution as well, then $A\underline{y} = \underline{b}$. This can then be the case only if

$$A\underline{v} = \underline{0} \qquad \textbf{(1.3.4-9)}$$

Therefore, if $\underline{x}$ is a solution, every other solution must differ from $\underline{x}$ by a vector $\underline{v} \in K_A$.

Since we have stated that for our matrix A, the only vector in the kernel is the null vector $\underline{0}$, there are no other solutions $\underline{y} \neq \underline{x}$ to $A\underline{x} = \underline{b}$.

Q.E.D. = "Quod Erat Demonstrondum"
"That which was to have been proven"

We have proven a theorem on uniqueness. We must not prove a theorem on existence.

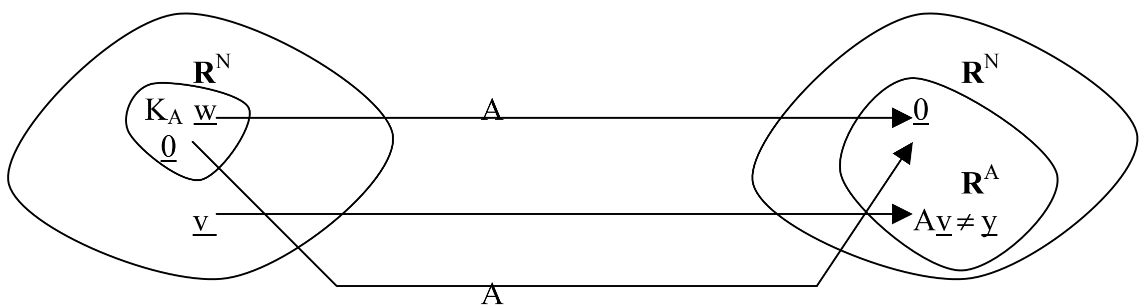
To do so, we define the range of A, denoted R_A, to be the subset of all vectors $\underline{y} \in \mathbf{R}^N$ such that there exists some $\underline{v} \in \mathbf{R}^N$ with $A\underline{v} = \underline{y}$.

There exists — some vector $\underline{v}$

$$R_A \equiv \{\underline{y} \in R^N \mid \exists \underline{v} \in R^N \text{ with } A\underline{v} = \underline{y}\} \qquad \textbf{(1.3.4-10)}$$

Every $\underline{y} \in \mathbf{R}^N$ — under the condition that

Pictorially, we view the range as



No vectors map into the port of $\mathbf{R}^N$ outside of the range.

Theorem 1.3.4.2:

Let A be a real N x N matrix with kernel $K_A \subset \mathbf{R}^N$ and Range $R_A \subset \mathbf{R}^N$. Then

(I) the dimensions of the kernel and of the range satisfy the "dimension theorem"

$$\dim(K_A) + \dim(R_A) = N \qquad \textbf{(1.3.4-11)}$$

(II) If the kernel contains only the null vector $\underline{0}$, $\dim(K_A) = 0$. As the range therefore has dimension N, $R_A = \mathbf{R}^N$, and for every $\underline{b} \in \mathbf{R}^N$, there exists some $\underline{x} \in \mathbf{R}^N$ with $A\underline{x} = \underline{b}$ (existence).

Proof:

(I) Let us use an orthonormal basis $\{\underline{U}^{[1],}\underline{U}^{[2]}, \ldots, \underline{U}^{[M]}, \underline{U}^{[M+1]}, \ldots, \underline{U}^{[N]}\}$ For $\mathbf{R}^N$ such that the 1st M vectors form a basis for the kernel K_A.

Since the kernel satisfies all the properties of a vector space itself, we can construct the M basis vectors for K_A, for example by Gram-Schmidt orthogonalization. Once we have identified these M basis vectors, we can continue with Gram-Schmidt orthogonalization to finish the basis set.

We can therefore write any $\underline{w} \in K_A$ as

$\underline{W} = c_1\underline{U}^{[1]} + c_2\underline{U}^{[2]} + \ldots + c_{M\underline{U}}{}^{[M]}$ **(1.3.4-12)**

And the dimension of the kernel is obviously M,

$\dim(K_A) = M$ **(1.3.4-13)**

We now write any arbitrary vector $\underline{v} \in \mathbf{R}^N$ as an expansion in the basis,

$\underline{v} = v_1'\underline{U}^{[1]} + v_1'\underline{U}^{[1]} + \ldots + v_M'\underline{U}^{[M]} + v_{M+1}'\underline{U}^{[M+1]} + \ldots + v_N'\underline{U}^{[N]}$ **(1.3.4-14)**

Then, taking the product with A,

$A\underline{v} = A\underbrace{\left(v_1'\underline{U}^{[1]} + v_1'\underline{U}^{[1]} + \ldots + v_M'\underline{U}^{[M]}\right)}_{=0} + v_{M+1}'A\underline{U}^{[M+1]} + \ldots + v_N'A\underline{U}^{[N]}$ **(1.3.4-15)**

We therefore see that any vector $A\underline{v} \subset \mathbf{R}_A$ can be written as a linear combination of the N – M vectors $\{A\underline{U}^{[M+1]}, \ldots, A\underline{U}^{[N]}\}$.

Therefore $\dim(R_A) = N - M$ and $\dim(K_A) + \dim(R_A) = N$

(II) Follows directly

Taken jointly, theorems **1.3.4.1** and **1.3.4.2** demonstrate that if $K_A = \underline{0}$, i.e. only the null vector maps into the null vector under A, then $A\underline{x} = \underline{b}$ has a unique solution for all $\underline{b}$.

What happens if the kernel of A is not empty, i.e. there exists some $\underline{w} \neq \underline{0}$? Let us consider a specific example.

Look at a system with

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \textbf{(1.3.4-16)}$$

Then for any $\underline{v} \in \mathbf{R}^3$

$$A\underline{v} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ v_3 \end{bmatrix} \qquad \textbf{(1.3.4-17)}$$

Writing

$$\underline{v} = v_1\underline{e}^{[1]} + v_2\underline{e}^{[2]} + v_3\underline{e}^{[3]}, \qquad \textbf{(1.3.4-18)}$$

$$A\underline{v} = v_1 A\underline{e}^{[1]} + v_2 A\underline{e}^{[2]} + v_3 A\, \underline{e}^{[3]} \qquad \textbf{(1.3.4-19)}$$

With

$$A\underline{e}^{[1]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \underline{0}$$

$$A\underline{e}^{[2]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \underline{0}$$

$$A\underline{e}^{[3]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underline{e}^{[3]}$$

(1.3.4-20)

Therefore

$$A\underline{v} = v_1\underline{0} + v_2\underline{0} + v_3\underline{e}^{[3]} = v_3\underline{e}^{[3]} \qquad \textbf{(1.3.4-21)}$$

This information is "lost" when mapped by A

We therefore see that for this A, any vector that is a linear combination of $\underline{e}^{[1]}$ and $\underline{e}^{[2]}$ is part of the kernel,

$$\underline{w} = w_1\underline{e}^{[1]} + w_2\underline{e}^{[2]} \in K_A \quad \textbf{(1.3.4-22)}$$

we then can say that $K_A = \text{span}\{\underline{e}^{[1]},\underline{e}^{[2]}\}$, and so $\dim(K_A) = z$. **(1.3.4-23)**

Also since for any $\underline{v} \in \mathbf{R}^3$, $A\underline{v} = \begin{bmatrix} 0 \\ 0 \\ v_3 \end{bmatrix} = v_3\underline{e}^{[3]}$; therefore $R_A = \text{span}\{\underline{e}^{[3]}\}$, $\dim(R_A) = 1$

(1.3.4-24)

As expected from the dimension theorem, $\dim(K_A) + \dim(R_A) = 3$ **(1.3.4-25)**

Now, does $A\underline{x} = \underline{b}$ have a solution?

- if $\underline{b} \in R_A$, i.e. $\underline{b} = \begin{bmatrix} 0 \\ 0 \\ b_3 \end{bmatrix}$ **(1.3.4-26) ,** then yes, there is a solution.

We easily see that a solution is

$$\underline{x} = \begin{bmatrix} 0 \\ 0 \\ b_3 \end{bmatrix} \textbf{(1.3.4-27)}, \; A\underline{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ b_3 \end{bmatrix} = \underline{b} \quad \textbf{(1.3.4-28)}$$

There are however an infinite number of solutions, since any vector $\underline{x} + w_1\underline{e}^{[1]} + w_2\underline{e}^{[2]}$ is also a solution as

$$\begin{aligned} A(\underline{x} + w_1\underline{e}^{[1]} + w_2\underline{e}^{[2]}) &= A\underline{x} + w_1A\underline{e}^{[1]} + w_2A\underline{e}^{[2]} \\ &= A\underline{x} + w_1\underline{0} + w_2\underline{0R_A} \quad \textbf{(1.3.4-29)} \\ &= \underline{b} \end{aligned}$$

- if $b \notin R_A$, i.e. $\underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ with either $b_1 \neq 0$ or $b_2 \neq 0$, then $A\underline{x} = \underline{b}$ has <u>no</u> solution.

We see therefore that we have the following three possibilities regarding the existence and uniqueness of solutions to the linear system $A\underline{x} = \underline{b}$, A N x N real matrix, $\underline{b} \in \mathbf{R}^N$.

Case I

The kernel of A is empty, i.e. $K_A = \underline{0}$. Then, $R_A = \mathbf{R}^N$ and for all $\underline{b} \in \mathbf{R}^N$ there exists a unique solution $\underline{x}$.

Case II

There exists $\underline{w} \neq 0$ for which $A\underline{w} = \underline{0}$. Let $\dim(K_A) = M$, and $\{\underline{U}^{[1]}, \underline{U}^{[2]}, \ldots, \underline{U}^{[M]}\}$ forms an orthonormal basis K_A,

$\underline{W} = c_1\underline{U}^{[1]} + c_2\underline{U}^{[2]} + \ldots + c_M\underline{U}^{[M]} \in K_A,\ A\underline{w} = \underline{0}$ **(1.3.4-30)**

If then $\underline{b} \bullet \underline{U}^{[1]} = \underline{b} \bullet \underline{U}^{[2]} = \ldots = \underline{b} \bullet \underline{U}^{[M]} = 0$, then $\underline{b} \in R_A$ and solutions exist, but there are an infinite number. If $A\underline{x} = \underline{b}$, then $A(\underline{x} + c_1\underline{U}^{[1]} + \ldots + c_M\underline{U}^{[M]}) = \underline{b}$ **(1.3.4-31)** as well.

Case III

Again $\dim(K_A) = M$, $M \geq 1$ and $\{\underline{U}^{[1]}, \ldots \underline{U}^{[M]}\}$ forms an orthonormal basis for K_A.

Now, for at least on $\underline{U}^{[j]}$, j = 1, 2, …, M, $\underline{b} \bullet \underline{U}^{[j]} \neq 0$. Therefore $\underline{b} \notin R_A$ and the system $A\underline{x} = \underline{b}$ has no solution.

While these rules provide insight into existence and uniqueness, to employ them we need:

1. A method to determine if $K_A = 0$ from the coefficients of A
2. A method to identify basis vectors for K_A

Point (1) is the subject of the next section. (2) is discussed in context of eigenvalues.