1.3.5 The Determinant Of A Square Matrix

In section 1.3.4 we have seen that the condition of existence and uniqueness for solutions to A $\underline{x} = \underline{b}$ involves whether $K_A = \underline{0}$, i.e. only $\underline{w} = \underline{0}$ has the property that $A\underline{w} = \underline{0}$.

To use this result, we need a method by which we can examine the elements of A to determine if $K_A = \underline{0}$.

For N = 1, this is simple. For the single equation

$$Ax = b \quad \textbf{(1.3.5-1)}$$

If $a \neq 0$, we have a single (unique) solution $x = \frac{b}{a}$. If a = 0, then if b = 0, there exists an infinite number of solutions. If $b \neq 0$, there is no solution.

For N > 1, we want a similar rule. Given an N x N real matrix A, we want a rule to calculate a scalar called the determinant, det(A), such that

$$\det(A) = \begin{cases} 0, \text{ then } A\underline{x} = \underline{b} \text{ has no unique soltuion} \\ c, c \neq 0, \text{ then } A\underline{x} = \underline{b} \text{ has a unique solution} \end{cases} \quad \textbf{(1.3.5-2)}$$

Since this determinant is to be used to determine whether a system $A\underline{x} = \underline{b}$ will have a unique solution, we can identify some characteristics that a suitable functional form of det(A) must possess.

Characteristic #1:

If we multiply any equation in our system, say the jth $a_{j1}x_1 + a_{j2}x_2 + \dots + a_{jN}x_N = b_j$ **(1.3.5-3)** by a scalar $c \neq 0$, we obtain an equation

$ca_{j1}x_1 + ca_{j2}x_2 + \dots + ca_{jN}x_N = cb_j$ **(1.3.5-4)**
As this new equation is completely equivalent to the first one, the determinants of the following 2 matrices should either both be zero or both be non-zero.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ \vdots & \vdots & & \vdots \\ a_{jN} & a_{j2} & \dots & a_{jN} \\ \vdots & \vdots & & \vdots \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ \vdots & \vdots & & \vdots \\ ca_{jN} & ca_{j2} & \dots & ca_{jN} \\ \vdots & \vdots & & \vdots \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{bmatrix} \qquad \textbf{(1.3.5-5)}$$

Moreover, if $c = 0$, then even if $\det(A) \neq 0$, the determinant of the 2^{nd} matrix in **(1.3.5-5)** should be zero.

We note that we can satisfy these requirements if our determinant function has the property that the determinant of the 2^{nd} matrix is $c * \det(A)$.

Characteristic #2.

The existence of a solution to $A\underline{x} = \underline{b}$ does not depend upon the order in which we write the equations. Therefore, we must be able to exchange any 2 rows in a matrix without affecting whether the determinant is zero or non-zero.

One way to satisfy this is if A' is the matrix obtained from A by interchanging any 2 rows, then our determinant should satisfy $\det(A') = \pm \det(A)$. **(1.3.5-6)**

Characteristic #3:

We can write the following 3 equations

$x + y + z = 4$
$2x + y + 3z = 7$
$3x + y + 6z = 2$ **(1.3.5-7)**

in matrix form with the labels

$x_1 = x \quad x_2 = y \quad x_3 = z$ **(1.3.5-8)**

to yield the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 6 \end{bmatrix}$$

We could just as well label the unknowns by

$x_1 = x \quad x_2 = z \quad x_3 = y$ **(1.3.5-9)**

In which case we obtain a matrix

$$A' = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 3 & 6 & 1 \end{bmatrix} \quad \textbf{(1.3.5-10)}$$

Obviously, such an interchange of columns does nothing to affect the existence and uniqueness of solutions. Therefore either det(A) and det(A') are both zero, or det(A) and det(A') are both non-zero.

One way to satisfy this is to make $\det(A') = \pm \det(A)$. **(1.3.5-11)**

Characteristic #4:

We can select any 2 equations, say # i and #j,

$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{iN}x_N = b_i$
$a_{j1}x_1 + a_{j2}x_2 + \ldots + a_{jN}x_N = b_j$ **(1.3.5-11)**

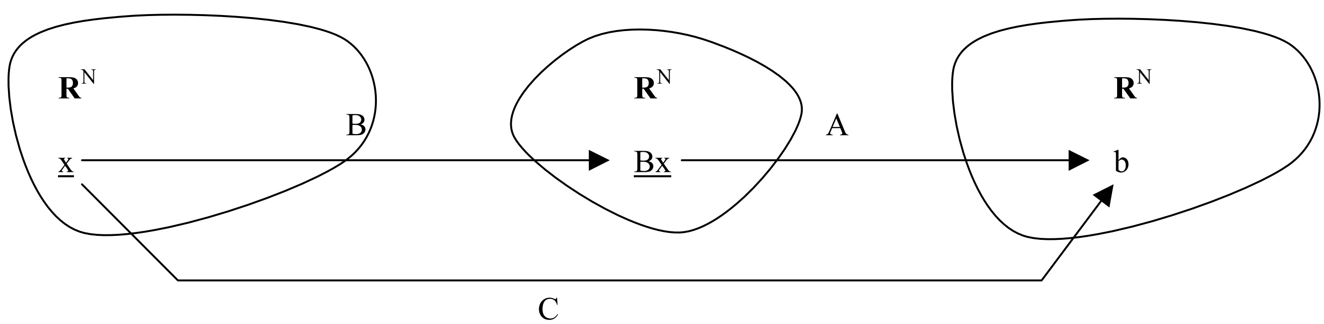
and replace them by the following 2, with $c \neq 0$

$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{iN}x_N = b_i$
$(ca_{i1} + a_{j1})x_1 + (ca_{i2} + a_{j2})x_2 + \ldots + (ca_{iN} + a_{jN})x_N = (cb_i + b_j)$ **(1.3.5-12)**

If A is the original matrix of the system, and A' is the new matrix obtained after making this replacement, than either det(A) and det(A') are both zero or they are both non-zero.

Characteristic #5:

If C = AB, the viewing $C\underline{x} = \underline{b}$ as



We see that $\det(C) \neq 0$ if and only if both $\det(A) \neq 0$ (so a unique $B\underline{x}$ exists) and if $\det(B) \neq 0$.

One way to ensure this is if det(C) = det(A) x det(B) **(1.3.5-13)**

Characteristic #6:

If any 2 rows of A are identical, the equations that they represent are dependent. We therefore do not have a unique solution, and must have det(A) = 0.

Similarly if all elements of a given row are zero, we have the equation $0 = b_j$, which is inconsistent if $b_j \neq 0$. Therefore, we must have det(A) = 0.

Characteristic #7:

If any 2 rows of A are equal, say columns #i and #j, then for all $M \in [1,N]$ $a_{Mi} = a_{Mj}$. We can therefore write each equation as

$$a_{M1}x_1 + a_{M2}x_2 + \ldots + a_{Mi}x_i + \ldots + a_{Mj}x_j + \ldots + a_{MN}x_N$$
$$= a_{M1}x_1 + a_{M2}x_2 + \ldots + a_{Mi}(x_i + x_j) + \ldots + a_{MN}x_N \quad \textbf{(1.3.5-14)}$$

Since x_i and x_j only appear together in this system of equations as the sum $x_i + x_j$, we could make the following change for any c that would not affect $A\underline{x}$,

$$x_i \leftarrow x_i + c \qquad x_j \leftarrow x_j - c \quad \textbf{(1.3.5-15)}$$

Therefore, we must have det(A) = 0.

Similarly, if any column of A contains all zeros, det(A) = 0.

We now have identified a number of properties that any functional form for det(A) must have to be a proper measure of existence and uniqueness for $A\underline{x} = \underline{b}$.

We now propose a functional form for the determinant, and show that it does satisfy these characteristics.

We define the determinant of the N x N matrix A as

$$\det(A) = \sum_{i_1=1}^{N}\sum_{i_2=1}^{N}....\sum_{i_N=1}^{N} E_{i_1 i_2 \ldots i_N} a_{i_1,1} a_{i_2,2} \ldots a_{i_N,N} \qquad \textbf{(1.3.5-16)}$$

Where

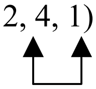
$$E_{i_1 i_2 \ldots i_N} \begin{cases} 0, \text{if any two of } \{i_1, i_2 ..., i_N\} \text{ are equal} \\ +1, \text{if } (i_1, i_2 ..., i_N) \text{ is an even parity permutation} \\ -1, \text{if } (i_1, i_2 ..., i_N) \text{ is an odd parity permutation} \end{cases} \qquad \textbf{(1.3.5-17)}$$

By “even parity permutation” we mean the following. Since $E_{i_1 i_2 \ldots i_N}$ = 0 if any two of the set $\{i_1, i_2 ..., i_N\}$, we know that the ordered set $(i_1, i_2 ..., i_N)$, if $E_{i_1 i_2 \ldots i_N}$ is to be non-zero, must be related to the ordered set (1, 2, 3, …, N) by some shuffling of the order.

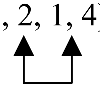
For example, consider $i_1 = 3$, $i_2 = 2$, $i_3 = 4$, $i = 1$, so
$(i_1, i_2, i_3, i_4) = (3, 2, 4, 1)$ **(1.3.5-18)**

We want to perform a sequence of interchanges to put it in the order (1, 2, 3, 4).

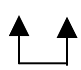
Interchange #1, (3, 2, 4, 1) → (3, 2, 1, 4) **(1.3.5-19)**



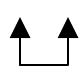
Interchange #2, (3, 2, 1, 4) → (3, 1, 2, 4)



Interchange #3, (3, 1, 2, 4) → (1, 3, 2, 4)



Interchange #4, (1, 3, 2, 4) → (1, 2, 3, 4) **(1.3.4-19)**



So we have put (3, 2, 4, 1) into order (1, 2, 3, 4) with four interchanges.

Note that we could do the same thing with only 2 interchanges:

(3, 2, 4, 1) → (1, 2, 4, 3) → (1, 2, 3, 4) **(1.3.5-20)**

or less efficiently, with six

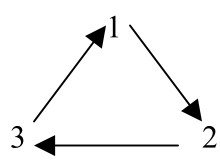
(3, 2, 4, 1) → (4, 2, 3, 1) → (2, 4, 3, 1) → (1, 4, 3, 2) → (1, 4, 2, 3) → (1, 4, 3, 2) → (1, 2, 3, 4) **(1.3.5-21)**

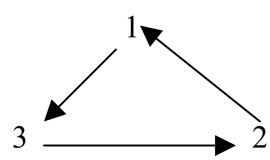
The number of interchanges by which (3, 2, 4, 1) is reordered into (1, 2, 3, 4) is therefore not unique; however, what is unique is that (3, 2, 4, 1) can only be reordered into (1, 2, 3, 4) in an <u>even</u> (0, 2, 4, 6) number of steps.

(3, 2, 4, 1) is therefore said to be an <u>even parity permutation</u> of (1,2,3,4).

If N = 3, we have the following parity assignments

<u>Even</u>	<u>Odd</u>
(1, 2, 3)	(3, 2, 1)
(2, 3, 1)	(2, 1, 3)
(3, 1, 2)	(1, 3, 2)





even = "clockwise order"   odd = "counter-clockwise order"

so $E_{123} = E_{231} = E_{312} = +1$, $E_{321} = E_{213} = E_{132} = -1$

while $E_{111} = E_{112} = E_{121} = E_{233} = 0$ **(1.3.5-22)**

For N = 2, $E_2 = +1$, $E_{21} = -1$ **(1.3.5-23)**

For a 2 x 2 matrix,

$$\det(A) = \begin{vmatrix} a_{11}a_{12} \\ a_{21}a_{22} \end{vmatrix} = E_{12}a_{11}a_{22} + E_{21}a_{21}a_2 = a_{11}a_2 - a_{21}a_{12} \qquad \textbf{(1.3.5-24)}$$

$i = 1$ $i_1 = 2$
$i_2 = 2$ $i_2 = 1$

For a 3 x 3 matrix A,

$$\det(A) = \begin{vmatrix} a_{11} a_{12} a_{13} \\ a_{21} a_{22} a_{23} \\ a_{31} a_{32} a_{33} \end{vmatrix} = \sum_{i_1=1}^{3}\sum_{i_2=1}^{3}\sum_{i_3=1}^{3} E_{i_1,i_2,i_3} a_{i_1,1} a_{i_2,2} a_{i_3,3} \qquad \textbf{(1.3.5-25)}$$

We use this formula to rearrange det(A) into a more recognizable form. First, split the summation over i_1 into $i_1=1$ and $i_1 \neq 1$.

$$\det(A) = \sum_{i_2=1}^{3}\sum_{i_3=1}^{3} E_{1,i_2,i_3} a_{11} a_{i_{21},2} a_{i_3,3} + \sum_{\substack{i_1=1,\\ i_1\neq 1}}^{3}\sum_{i_2=1}^{3}\sum_{i_3=1}^{3} E_{i_1,i_2,i_3} a_{i_1,1} a_{i_2,2} a_{i_3,3} \qquad \textbf{(1.3.5-26)}$$

Now if $i_1=1$, then $E_{1,i_2,i_3} = 0$ if $i_2 = 1$ or $i_3 = 0$, so the 1st term becomes det(A) =

$$a_{11} \sum_{i_2=1}^{3}\sum_{i_3=2}^{3} E_{1,i_2,i_3} a_{11} a_{i_2,2} a_{i_3,3} + \sum_{\substack{i_1=1,\\ i_1\neq 1}}^{3}\sum_{i_2=1}^{3}\sum_{i_3=1}^{3} E_{i_1,i_2,i_3} a_{i_1,1} a_{i_2,2} a_{i_3,3} \qquad \textbf{(1.3.5-27)}$$

We now split the summation over i_2,

$$\det(A) = a_{11} \sum_{i_2=2}^{3}\sum_{i_3=2}^{3} E_{1,i_2,i_3} a_{i_2,2} a_{i_3,3} + a_{12} \sum_{\substack{i_1=1,\\ i_1\neq 1}}^{3}\sum_{i_3=1}^{3} E_{i_1,1,i_3} a_{i_1,1} a_{i_3,3} + \sum_{\substack{i_1=1,\\ i_1\neq 1}}^{3}\sum_{\substack{i_2=1,\\ i_2\neq 1}}^{3}\sum_{i_3=1}^{3} E_{i_1,i_2,i_3} a_{i_1,1} a_{i_2,2} a_{i_3,3}$$

(1.3.5-28)

Then, we split the summation over i_3,

det(A) =

$$a_{11} \sum_{i_2=2}^{3}\sum_{i_3=2}^{3} E_{1,i_2,i_3} a_{i_2,2} a_{i_3,3} + a_{12} \sum_{\substack{i_1=1,\\ i_1\neq 1}}^{3}\sum_{i_3=1}^{3} E_{i_1,1,i_3} a_{i_1,1} a_{i_3,3} + a_{13} \sum_{\substack{i_1=1,\\ i_1\neq 1}}^{3}\sum_{\substack{i_2=1,\\ i_2\neq 1}}^{3} E_{i_1,i_2,1} a_{i_1,1} a_{i_2,2} + \sum_{\substack{i_1=1,\\ i_1\neq 1}}^{3}\sum_{\substack{i_2=1,\\ i_2\neq 1}}^{3}\sum_{\substack{i_3=1,\\ i_3\neq 1}}^{3} E_{i_1,i_2,i_3} a_{i_1,1} a_{i_2,2} a_{i_3,3}$$

(1.3.5-29)

Now for every term in the last summation of **(1.3.5-29)** $i_1 \neq 1$, $i_2 \neq 1$, $i_3 \neq 1$. This means that there must be some repeated index, e.g. $i_2=i_3$, in each term and so $E_{i_1,i_2,i_3} = 0$.

This last term is therefore zero and we have

$$\det(A) = a_{11}\sum_{i_2=2}^{3}\sum_{i_3=2}^{3} E_{1,i_2,i_3} a_{i_2,2} a_{i_3,3} + a_{12}\sum_{\substack{i_1=1,\\ i_1\neq 1}}^{3}\sum_{i_3=1}^{3} E_{i_1,1,i_3} a_{i_1,1} a_{i_3,3} + a_{13}\sum_{\substack{i_1=1,\\ i_1\neq 1}}^{3}\sum_{\substack{i_2=1,\\ i_2\neq 1}}^{3} E_{i_1,i_2,1} a_{i_1,1} a_{i_2,2} \quad \textbf{(1.3.5-30)}$$

Here we have added restriction $i_3 \neq 1$ on the summation in the 2nd term on the right since $i_2=1$ and $E_{i_1,1,i_3} = 0$ if $i_3 = 1$.

We now note that since 1 is the smallest number,

$$E_{1,i_2,i_3} \begin{cases} +1, \text{if } i_2 < i_3 \\ -1, \text{if } i_3 < i_2 \\ 0, \text{if } i_2 = i_3 \end{cases} \quad \textbf{(1.3.5-31)}$$

and so we can write $E_{1,i_2,i_3} = E_{i_2,i_3}$.

Next we look at $E_{i_1,1,i_3}$. By performing one interchange, we have
$(i_1, 1, i_3) \rightarrow (1, i_1, i_3)$.

So if, $(i_1, 1, i_3)$ is odd, $(1, i_1, i_3)$ is even

If$(i_1, 1, i_3)$ is even, $(1, i_1, i_3)$ is odd

In any event, $E_{i_1,1,i_3} = -E_{1,i_1,i_3} = -E_{i_1,i_3}$ **(1.3.5-32)**

Finally for (i_1, I, 1) we not that in 2 interchanges

(i_1, i_2, 1) → (1, i_2, i_1) → (1, i_1, i_2)

so that $E_{i_1,i_2,1} = E_{1,i_1,i_2} = E_{i_1,i_2}$ **(1.3.5-33)**

We therefore have

$$\det(A) = a_{11}\sum_{i_2=2}^{3}\sum_{i_3=2}^{3} E_{i_2,i_3} a_{i_2,2} a_{i_3,3} - a_{12}\sum_{\substack{i_1=1,\\ i_1\neq 1}}^{3}\sum_{\substack{i_3=1,\\ i_3\neq 1}}^{3} E_{i_1,i_3} a_{i_1,1} a_{i_3,3} + a_{13}\sum_{i_1=2}^{3}\sum_{i_2=2}^{3} E_{i_1,i_2} a_{i_1,1} a_{i_2,2} \quad \textbf{(1.3.5-34)}$$

Using the determinant formula for a 2 x 2 matrix **(1.3.5-24)**, we see that

$$\sum_{i_2=2}^{3}\sum_{i_3=2}^{3} E_{i_2,i_3} a_{i_2,2} a_{i_3,3} = a_{22}a_{33} - a_{32}a_{23} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \quad \textbf{(1.3.5-35)}$$

$$\sum_{i_1=2}^{3}\sum_{i_3=2}^{3} E_{i_1,i_3} a_{i_1,1} a_{i_3,3} = a_{21}a_{23} - a_{31}a_{33} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \quad \textbf{(1.3.5-36)}$$

$$\sum_{i_1=2}^{3}\sum_{i_2=2}^{3} E_{i_1,i_2} a_{i_1,1} a_{i_2,2} = a_{21}a_{32} - a_{31}a_{22} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \quad \textbf{(1.3.5-37)}$$

This yields the familiar formula for the determinant of a 3 x 3 matrix

$$\begin{vmatrix} a_{11}a_{12}a_{13} \\ a_{21}a_{22}a_{23} \\ a_{31}a_{32}a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \quad \textbf{(1.3.5-38)}$$

In the general formula **(1.3.5-16)** for det(A), we must have an expression that is define for N > 3m and that allows us to prove various properties of the determinant to show that it is valid measure for determining existence/uniqueness of solutions.

In general, we can determine the parity (even or odd) of a permutation $(i_1, i_2, \ldots, i_N)$ by the following method:

For each M = 1, 2, …., N, let α_M be the number of integers in the set $\{i_{M+1}, i_{M+2}, \ldots , i_N\}$ that are smaller then i_M.

The total number of inversion (pairwise interchanges) required to reorder $(i_1, i_2, \ldots, i_N)$ into (1, 2, …, N) using a particular straight-forward strategy is

$$V = \sum_{M=1}^{N-1} \alpha_M \qquad \textbf{(1.3.5-39)}$$

If v is even, $(i_1, i_2, \ldots, i_N)$ is even (note : 0 counts as even).
If v is odd, $(i_1, i_2, \ldots, i_N)$ is odd parity permutation.

This provides a well-defined rule to determining the value of
$E_{i_1, i_2, \ldots, i_N}$

Look at some examples:

(1, 2, 3, 4) : $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, v = 0 (even) $E_{1234} = +1$
(1, 3, 2, 4): $\alpha_1 = 0$, $\alpha_2 = 1$, $\alpha_3 = 0$, v = 1 (odd) $E_{1324} = -1$
(3, 4, 1, 2): $\alpha_1 = 2$, $\alpha_2 = 2$, $\alpha_3 = 0$, v = 4 (even) $E_{3412} = +1$

We now use the definition **(1.3.5-16)** of the determinant to prove several properties of the determinant.

Property I:

$$\det(A^T) = \det(A) \quad \textbf{(1.3.5-40)}$$

Proof:

The determinant of the transpose of A is $\det(A^T) = \sum_{i_1=1}^{N} \ldots \sum_{i_N=1}^{N} E_{i_1,\ldots i_N} a^T_{i_1,1} \ldots a^T_{i_N,N}$ **(1.3.5-41)**

Now, for every permutation (i_1, i_2, …, I_N), there exists another permutation (j_1, j_2, …, j_N) such that

$$a_{i_1,1} a_{i_2,2} \ldots a_{i_N,N} = a_{1,j_1} a_{2,j_2} \ldots a_{N,j_N} \quad \textbf{(1.3.5-42)}$$

order of 1st subscripts (i_1, i_2, …, i_N) (1, 2, 3…, N)

order of 2nd subscripts (1, 2, …, N) (j_1, j_2, …, j_N)

If we perform v pairwise exchanges to convert (i_1, i_2, …, i_N) → (1, 2, …, N),
Then in the same # of steps (1, 2, …, N) → (j_1, j_2, …, j_N).

Therefore, $E_{i_1,\ldots i_N} = E_{j_1,\ldots j_N}$ **(1.3.5-43)**

Using the definition of the transpose, $a^T_{ij} = a_{ji}$, so the determinant becomes

$$\det(A^T) = \sum_{j_1=1}^{N} \ldots \sum_{j_N=1}^{N} E_{j_1,\ldots j_N} a_{1,j_1} a_{2,j_2} \ldots a_{N,j_N} \quad \textbf{(1.3.5-44)}$$

Using **(1.3.5-42)** and **(1.3.5-43)**, we have

$$\det(A^T) = \sum_{i_1=1}^{N} \ldots \sum_{i_N=1}^{N} E_{i_1,\ldots i_N} a_{i_1,1} a_{i_2,2} \ldots a_{i_N,N} = \det(A) \quad \textbf{(1.3.5-45)} \quad \text{Q.E.D.}$$

Property II:

If every element in a row (column) of A is zero, then det(A) = 0.

Proof:

Let every element in column #M of A be zero. Then, in the formula for the determinant,

$$\det(A) = \sum_{i_1=1}^{N} \ldots \sum_{i_N=1}^{N} E_{i_1,\ldots i_N} a_{i_1,1} a_{i_2,2} \ldots a_{i_M,M} \ldots a_{i_N,N} \qquad \textbf{(1.3.5-46)}$$

We see that $a_{i_M,M} = 0$ for all i_M. As every term in the summation is therefore zero, det(A) = 0.

Let us now say that every element in row #M of a matrix B is zero. When we take the transpose, $b^T_{ij} = b_{ji}$, so every element in the mth column of B^T is zero. By the result above, $\det(B^T) = 0$. Using property I, **(1.3.5-45)**, we then have det(B) = 0.

Q.E.D.

Property III:

If every element in a row (column) of a matrix A is multiplied by a scalar c to form a matrix B, then det(B) = c*det(A).

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ : & : & & : \\ a_{M1} & a_{M2} & \dots & a_{MN} \\ : & : & & : \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{bmatrix} \quad B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ : & : & & : \\ ca_{M1} & ca_{M2} & \dots & ca_{MN} \\ : & : & & : \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{bmatrix} \qquad \textbf{(1.3.5-47)}$$

Proof:

We write the determinant for B, obtained from A by multiplying every element in row # M by a scalar c, as

$$\det(B) = \sum_{i_1=1}^{N} \dots \sum_{i_N=1}^{N} E_{i_1,\dots i_N} b_{i_1,1} b_{i_2,2} \dots b_{i_M,M} \dots b_{i_N,N} \qquad \textbf{(1.3.5-48)}$$

As $\det(B) = \det(B^T)$, we can also write the determinant as

$$\det(B) = \det(B^T) = \sum_{i_1=1}^{N} \dots \sum_{i_N=1}^{N} E_{i_1,\dots i_N} b_{1,i_1} b_{2,i_2} \dots b_{M,i_M} \dots b_{N,i_N} \qquad \textbf{(1.3.5-49)}$$

Substituting for b_{ij} in terms of a_{ij}, c we have

$$\det(B) = \sum_{i_1=1}^{N} \dots \sum_{i_N=1}^{N} E_{i_1,\dots i_N} a_{1,i_1} a_{2,i_2} \dots c a_{M,i_M} \dots a_{N,i_N}$$

$$= c \sum_{i_1=1}^{N} \dots \sum_{i_N=1}^{N} E_{i_1,\dots i_N} a_{1,i_1} a_{2,i_2} \dots a_{N,i_N}$$

$$= c^* \det(A^T) = c \det(A) \qquad \textbf{(1.3.5-50)}$$

From the rule $\det(A) = \det(A^T)$, it is clear that this formula holds also if we were to multiply every element in a column of A by the scalar c.

Q.E.D.

Property IV:

If 2 rows (columns) of A are interchanged to form a matrix B, then det(B) = -det(A).

Proof:

Let us interchange columns # r and s, r < s

$$A = \begin{bmatrix} a_{11} & \dots & a_{1r} & \dots & a_{1s} & \dots & a_{1N} \\ a_{21} & \dots & a_{2r} & \dots & a_{2s} & \dots & a_{2N} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{N1} & \dots & a_{Nr} & \dots & a_{Ns} & \dots & a_{NN} \end{bmatrix} \quad B = \begin{bmatrix} a_{11} & \dots & a_{1s} & \dots & a_{1r} & \dots & a_{1N} \\ a_{21} & \dots & a_{2s} & \dots & a_{2r} & \dots & a_{2N} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{N1} & \dots & a_{Ns} & \dots & a_{Nr} & \dots & a_{NN} \end{bmatrix} \qquad \textbf{(1.3.5-51)}$$

We write the determinant B as

$$\det(B) = \sum_{i_1=1}^{N} \dots \sum_{i_N=1}^{N} E_{i_1,\dots i_N} b_{i_1,1} b_{i_2,2} \dots b_{i_r,r} \dots b_{i_s,s} \dots b_{i_N,N}$$

$$= \sum_{i_1=1}^{N} \dots \sum_{i_N=1}^{N} E_{i_1,\dots i_N} a_{i_1,1} a_{i_2,2} \dots a_{i_r,s} \dots a_{i_s,r} \dots a_{i_N,N} \qquad \textbf{(1.3.5-52)}$$

where we have used $b_{i_r,r} = a_{i_r,s}$, $b_{i_s,s} = a_{i_s,r}$, according to the interchange of column # r and # s.

Now, if we use result for performing a pairwise interchange of i_r and i_s,

$$E_{i_1,\dots i_r,\dots,i_s,\dots,i_N} = - E_{i_1,\dots i_s,\dots,i_r,\dots,i_N} \qquad \textbf{(1.3.5-53)}$$

we have

$$\det(B) = - \sum_{i_1=1}^{N} \dots \sum_{i_N=1}^{N} E_{i_1,\dots i_s,\dots,i_r,\dots,i_N} a_{i_1,1} a_{i_2,2} \dots a_{i_r,s} \dots a_{i_s,r} \dots a_{i_N,N} \qquad \textbf{(1.3.5-54)}$$

We are now free to re-label the dummy indices $i_r \Leftrightarrow i_s$, and to switch the order in which we multiply the factors in each term to write

$$\det(B) = - \sum_{i_1=1}^{N} \dots \sum_{i_N=1}^{N} E_{i_1,\dots i_r,\dots,i_s,\dots,i_N} a_{i_1,1} a_{i_2,2} \dots a_{i_r,r} \dots a_{i_s,s} \dots a_{i_N,N} \qquad \det(B) = - \det(A) \qquad \textbf{(1.3.5-55)}$$

By using property $\det(A^T) = \det(A)$, we can demonstrate **(1.3.5-55)** holds when we switch 2 rows.   Q.E.D.

Property V:

If 2 rows (columns) of A are the same, det(A) = 0.

Proof:

Let B be the matrix that is obtained from A by interchanging the 2 rows (or columns) that are equal.
By property IV, det(B) = -det(A).
But, since B and A are identical, det(A) = det(B).
Therefore, we must have det(A) = 0. Q.E.D.

Property VI:

If $\underline{a}^{(M)}$ is the mth row vector of A, and we decompose this row vector into 2 parts, arbitrarily

$$\underline{A}^{(M)} = \underline{b}^{(M)} + \underline{d}^{(M)} \qquad \textbf{(1.3.5-56)}$$

And define matrices

$$A = \begin{bmatrix} \_\_\_\underline{a}^{(1)}\_\_\_ \\ \vdots \quad \vdots \\ \_\_\_\underline{a}^{(M)}\_\_\_ \\ \vdots \quad \vdots \\ \_\_\_\underline{a}^{(N)}\_\_\_ \end{bmatrix} \quad B = \begin{bmatrix} \_\_\_\underline{a}^{(1)}\_\_\_ \\ \vdots \quad \vdots \\ \_\_\_\underline{b}^{(M)}\_\_\_ \\ \vdots \quad \vdots \\ \_\_\_\underline{a}^{(N)}\_\_\_ \end{bmatrix} \quad D = \begin{bmatrix} \_\_\_\underline{a}^{(1)}\_\_\_ \\ \vdots \quad \vdots \\ \_\_\_\underline{d}^{(M)}\_\_\_ \\ \vdots \quad \vdots \\ \_\_\_\underline{a}^{(N)}\_\_\_ \end{bmatrix} \qquad \textbf{(1.3.5-57)}$$

Then det(A) = det(B) + det(D) **(1.3.5-58)**

Proof:

Write $\det(A) = \det(A^T) = \sum_{i_1=1}^{N}\dots\sum_{i_N=1}^{N} E_{i_1,\dots i_N} a_{1,i_1}\dots a_{M,i_M}\dots . a_{N,i_N}$ **(1.3.5-59)**

As $a_{M,i_M} = b_{M,i_M} + d_{M,i_M}$,

$$\det(A) = \sum_{i_1=1}^{N}\dots\sum_{i_N=1}^{N} E_{i_1,\dots i_N} a_{1,i_1}\dots(b_{M,i_M} + d_{M,i_M})\dots . a_{N,i_N}$$

$$= \sum_{i_1=1}^{N}\dots\sum_{i_N=1}^{N} E_{i_1,\dots i_N} a_{1,i_1}\dots b_{M,i_M}\dots . a_{N,i_N} + \sum_{i_1=1}^{N}\dots\sum_{i_N=1}^{N} E_{i_1,\dots i_N} a_{1,i_1}\dots d_{M,i_M}\dots d_{N,i_d} \qquad \textbf{(1.3.5-60)}$$

So, det(A) = det(B) + det (D) Q.E.D.

Property VII:

If a matrix B is obtained from A by adding c times one row (column) of A to another row (column) of A, det (B) = det(A).

Proof:

Let us define the following matrices in terms of their row vectors,

$$A = \begin{bmatrix} \_\_\_\underline{a}^{(1)}\_\_\_ \\ \vdots \quad \vdots \\ \_\_\_\underline{a}^{(j)}\_\_\_ \\ \vdots \quad \vdots \\ \_\_\_\underline{a}^{(k)}\_\_\_ \\ \vdots \quad \vdots \\ \_\_\_\underline{a}^{(N)}\_\_\_ \end{bmatrix} \quad B = \begin{bmatrix} \_\_\_\underline{a}^{(1)}\_\_\_ \\ \vdots \quad \vdots \\ \_\_\_\underline{a}^{(j)}\_\_\_ \\ \vdots \quad \vdots \\ \_\_\_\underline{a}^{(k)} + c\underline{a}^{(j)}\_\_\_ \\ \vdots \quad \vdots \\ \_\_\_\underline{a}^{(N)}\_\_\_ \end{bmatrix} \quad D = \begin{bmatrix} \_\_\_\underline{a}^{(1)}\_\_\_ \\ \vdots \quad \vdots \\ \_\_\_\underline{a}^{(j)}\_\_\_ \\ \vdots \quad \vdots \\ \_\_\_c\underline{a}^{(j)}\_\_\_ \\ \vdots \quad \vdots \\ \_\_\_\underline{a}^{(N)}\_\_\_ \end{bmatrix}$$

$$E = \begin{bmatrix} \_\_\_\underline{a}^{(1)}\_\_\_ \\ \vdots \quad \vdots \\ \_\_\_\underline{a}^{(j)}\_\_\_ \\ \vdots \quad \vdots \\ \_\_\_\underline{a}^{(j)}\_\_\_ \\ \vdots \quad \vdots \\ \_\_\_\underline{a}^{(N)}\_\_\_ \end{bmatrix} \qquad \textbf{(1.3.5-61)}$$

By property VI,

Det(B) = det(A) + det(D) **(1.3.5-62)**

By property III,

det(D) = c*det(E) **(1.3.5-63)**

So that

det(B) = det(A) + c*det(E) **(1.3.5-64)**

But, as 2 rows of E are identical, by property V, det(E) = 0.
Therefore

det(B) = det(A) **(1.3.5-65)**

Q.E.D.

Property VIII:

$$\det(AB) = \det(A) * \det(B) \quad \textbf{(1.3.5-66)}$$

We demonstrate this only for a 2 x 2 matrix,

$$\det(AB) = \sum_{i_1=1}^{2}\sum_{i_2=1}^{2} E_{i_1,i_2}\left[\sum\nolimits_{k_1=1}^{2} a_{1,k_1} b_{k_1,i_1}\right]\left[\sum_{k_2=1}^{2} a_{2,k_2} b_{k_2,i_2}\right]$$

$$= \sum_{i_1=1}^{2}\sum_{i_2=1}^{2} E_{i_1,i_2} \sum_{k_1=1}^{2}\sum_{k_2=1}^{2} a_{1,k_1} a_{2,k_2} b_{k_1,i_1} b_{k_2,i_2}$$

$$=E_{12}\sum_{k_1=1}^{2}\sum_{k_2=1}^{2} a_{1,k_1} a_{2,k_2} b_{k_1,1} b_{k_2,2} + E_{21}\sum_{k_1=1}^{2}\sum_{k_2=1}^{2} a_{1,k_1} a_{2,k_2} b_{k_1,2} b_{k_2,1}$$

$$= \sum_{k_1=1}^{2}\sum_{k_2=1}^{2} a_{1,k_1} a_{2,k_2}\left[E_{12} b_{k_1,1} b_{k_2,2} + E_{21} b_{k_1,2} b_{k_2,1}\right]$$

$$=\sum_{k_1=1}^{2}\sum_{k_2=1}^{2} a_{1,k_1} a_{2,k_2}\underbrace{\left[b_{k_1,1} b_{k_2,2} - b_{k_1,2} b_{k_2,1}\right]}_{=0 \text{ if } k_1 = k_2}$$

$$=\sum_{k_1=1}^{2}\sum_{k_2\neq k_1}^{2} a_{1,k_1} a_{2,k_2}\left[b_{k_1,1} b_{k_2,2} - b_{k_1,2} b_{k_2,1}\right]$$

$=a_{11}a_{22}\,[b_{11}b_{22} - b_{12}b_{21}] + a_{12}a_{21}\,[b_{21}b_{12} - b_{22}b_{11}]$

$= [a_{11}a_{22} - a_{12}a_{21}][b_{11}b_{22} - b_{12}b_{21}]$

$=\det(A) * \det(B)$

Property IX:

If A is an upper-triangular or lower-triangular matrix, then det(A) is equal to the product of the elements along the principal diagonal.

Proof:

Let us consider

$$L = \begin{bmatrix} L_{11} & & & \\ L_{211} & L_{22} & & \\ L_{N1} & L_{N2} & \dots & L_{NN} \end{bmatrix} \quad \textbf{(1.3.5-67)}$$

Then

$$\det(L) = \sum_{i_1=1}^{N} \dots \sum_{i_N=1}^{N} E_{i_1,\dots i_N} L_{i_1,1} L_{i_2,2} \dots . L_{i_N,N} \quad \textbf{(1.3.5-68)}$$

For every permutation (i_i, i_2, …, I_N) of (1, 2, …, N), we must have

$$i_1 + i_2 + \dots + i_N = 1 + 2 + \dots + N \quad \textbf{(1.3.5-69)}$$

So, in the expression above for det(L), if we have some L_M, I_M where I_M>M, then we must have some other I_r<r in the product. As $L_{ir,r} = 0$ for $I_r < r$, all terms with any off-diagonal elements of L are zero. The only term in det(L) that survives is $i_1 = 1$, $i_2 = 2$, …, $I_N = N$, $E_{i_1,\dots i_N} = E_{1,\dots i_N} = +1$,

So

$$\det(L) = L_{11} L_2 \dots L_{NN} \quad \textbf{(1.3.5-70)}$$

Similar logic shows that for an upper-triangular matrix

$$U = \begin{bmatrix} U_{11} & U_{11} & \dots & U_{11} \\ & U_{11} & \dots & U_{11} \\ & & & \vdots \\ & & & U_{11} \end{bmatrix} \quad \textbf{(1.3.5-71)}, \ \det(U) = U_{11} U_{22} \dots U_{NN} \quad \textbf{(1.3.5-72)}$$

Q.E.D.

We can now demonstrate that this functional form for det(A) satisfies all of the required characteristics that were identified on pages 1.3.5-2 and 1.3.5-5.

Characteristic #	Follows from property
1	III
2	IV
3	IV
4	VII
5	VIII
6	II, V
7	II, V

We therefore have in equation **(1.3.5-16)** a form for det(A) that we can use to judge existence/unqueness.

In practice, the most efficient way to compute det(A), or at least its magnitude, is to use property IX. Since row operations do not change values of the determinant (property VII), and exchanging 2 rows only changes the sign (property IV), then after N^3 FLOP's to perform Gaussian elimination with pivoting, we put A into an upper triangular form U such that

$$\det(A) = \pm U_{11} U_{22} \ldots U_{NN} \qquad \textbf{(1.3.5-73)}$$

This method is much faster than performing all of the summations necessary to evaluate **(1.3.5-16)** directly.