Non-Lecture

Review of Free Electromagnetic Field

Maxwell's Equations (SI):

(1) $\overline{\nabla} \cdot \overline{B} = 0$

(2) $\overline{\nabla} \cdot \overline{E} = \rho / \epsilon_0$

(3) $\overline{\nabla} \times \overline{E} = -\dfrac{\partial \overline{B}}{\partial t}$

(4) $\overline{\nabla} \times \overline{B} = \mu_0 \overline{J} + \epsilon_0 \mu_0 \dfrac{\partial \overline{E}}{\partial t}$

$\overline{E}$: electric field; $\overline{B}$: magnetic field; $\overline{J}$: current density; ρ: charge density; ϵ_0: electrical permittivity; μ_0: magnetic permittivity

We are interested in describing $\overline{E}$ and $\overline{B}$ in terms of a scalar and vector potential. This is required for our interaction Hamiltonian.

Generally: A vector field $\overline{F}$ assigns a vector to each point in space, and:

(5) $\overline{\nabla} \cdot \overline{F} = \dfrac{\partial F_x}{\partial x} + \dfrac{\partial F_y}{\partial y} + \dfrac{\partial F_z}{\partial z}$ is a scalar

For a scalar field ϕ

(6) $\nabla \phi = \dfrac{\partial \phi}{\partial x}\hat{x} + \dfrac{\partial \phi}{\partial y}\hat{y} + \dfrac{\partial \phi}{\partial z}\hat{z}$ is a vector

where $\hat{x}^2 + \hat{y}^2 + \hat{z}^2 = \hat{r}^2$ ← unit vector

Also:

(7) $\overline{\nabla} \times \overline{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$

Some useful identities from vector calculus are:

(8) $\overline{\nabla} \cdot \left(\overline{\nabla} \times \overline{F} \right) = 0$

(9) $\nabla \times \left(\nabla \phi \right) = 0$

(10) $$\nabla \times \left(\overline{\nabla} \times \overline{F}\right) = \overline{\nabla}\left(\overline{\nabla} \cdot \overline{F}\right) - \overline{\nabla}^2 \overline{F}$$

We now introduce a vector potential $\overline{A}(\bar{r},t)$ and a scalar potential $\varphi(\bar{r},t)$, which we will relate to $\overline{E}$ and $\overline{B}$

Since $\overline{\nabla} \cdot \overline{B} = 0$ and $\overline{\nabla}\left(\overline{\nabla} \times \overline{A}\right) = 0$:

(11) $$\overline{B} = \overline{\nabla} \times \overline{A}$$

Using (3), we have:

$$\overline{\nabla} \times \overline{E} = -\overline{\nabla} \times \frac{\partial \overline{A}}{\partial t}$$

or

(12) $$\overline{\nabla} \times \left[\overline{E} + \frac{\partial \overline{A}}{\partial t}\right] = 0$$

From (9), we see that a scalar product exists with:

(13) $$\overline{E} + \frac{\partial \overline{A}}{\partial t} = -\overline{\nabla}\varphi(\bar{r},t)$$

convention

or

(14) $$\overline{E} = \frac{\partial \overline{A}}{\partial t} - \nabla\varphi$$

So we see that the potentials $\overline{A}$ and φ determine the fields $\overline{B}$ and $\overline{E}$:

(15) $$\overline{B}(\bar{r},t) = \overline{\nabla} \times \overline{A}(\bar{r},t)$$

(16) $$\overline{E}(\bar{r},t) = -\overline{\nabla}\varphi(\bar{r},t) - \tfrac{\partial}{\partial t}\overline{A}(\bar{r},t)$$

We are interested in determining the wave equation for $\overline{A}$ and φ. Using (15) and differentiating (16) and substituting into (4):

(17) $$\overline{\nabla} \times \left(\overline{\nabla} \times \overline{A}\right) + \epsilon_0 \, \mu_0 \left(\frac{\partial^2 \overline{A}}{\partial t^2} + \overline{\nabla}\frac{\partial \varphi}{\partial t}\right) = \mu_0 \overline{J}$$

Using (10):

$$\left[-\overline{\nabla}^2\overline{A} + \in_0 \mu_0 \frac{\partial^2 \overline{A}}{\partial t^2}\right] + \overline{\nabla}\left(\overline{\nabla}\cdot\overline{A} + \in_0 \mu_0 \frac{\partial \varphi}{\partial t}\right) = \overline{\mu}_0 \,\overline{J} \tag{18}$$

From (14), we have:

$$\overline{\nabla}\cdot\overline{E} = -\frac{\partial\overline{\nabla}\cdot\overline{A}}{\partial t} - \overline{\nabla}^2\varphi$$

and using (2):

$$\frac{-\partial\overline{V}\cdot\overline{A}}{\partial t} - \overline{\nabla}^2\varphi = \rho / \in_0 \tag{19}$$

Notice from (15) and (16) that we only need to specify four field components (A_x, A_y, A_z, φ) to determine all six $\overline{E}$ and $\overline{B}$ components. But $\overline{E}$ and $\overline{B}$ do not uniquely determine $\overline{A}$ and φ. So, we can construct $\overline{A}$ and φ in any number of ways without changing $\overline{E}$ and $\overline{B}$. Notice that if we change $\overline{A}$ by adding $\overline{\nabla}\chi$ where χ is any function of $\overline{r}$ and t, this won't change $\overline{B}$ $\left(\nabla\times(\nabla\cdot B)=0\right)$. It will change E by $\left(-\frac{\partial}{\partial t}\overline{\nabla}\chi\right)$, but we can change φ to $\varphi' = \varphi - \frac{\partial\chi}{\partial t}$. Then $\overline{E}$ and $\overline{B}$ will both be unchanged. This property of changing representation (gauge) without changing $\overline{E}$ and $\overline{B}$ is gauge invariance. We can transform between gauges with:

$$\overline{A}'(\overline{r},t) = \overline{A}(\overline{r},t) + \overline{\nabla}\cdot\chi(\overline{r},t) \tag{20}$$

$$\varphi'(\overline{r},t) = \varphi(\overline{r},t) - \frac{\partial}{\partial t}\chi(\overline{r},t) \tag{21}$$

gauge transformation

Up to this point, A' and Q are undetermined. Let's choose a χ such that:

$$\overline{\nabla}\cdot\overline{A} + \in_0 \mu_0 \frac{\partial\varphi}{\partial t} = 0 \tag{22}$$

Lorentz condition

then from (17):

$$-\nabla^2\overline{A} + \in_0 \mu_0 \frac{\partial^2\overline{A}}{\partial t^2} = \mu_0 \overline{J} \tag{23}$$

The RHS can be set to zero for no currents.

From (19), we have:

(24) $$\in_0 \mu_0 \frac{\partial^2 \varphi}{\partial t^2} - \nabla^2 \varphi = \frac{\rho}{\in_0}$$

Eqns. (23) and (24) are wave equations for $\overline{A}$ and φ. Within the Lorentz gauge, we can still arbitrarily add another χ (it must only satisfy 22). If we substitute (20) and (21) into (24), we see:

(25) $$\nabla^2 \chi - \in_0 \mu_0 \frac{\partial^2 \chi}{\partial t^2} = 0$$

So we can make further choices/constraints on $\overline{A}$ and φ as long as it obeys (25).

For a field far from charges and currents, $J = 0$ and $\rho = 0$.

(26) $$-\overline{\nabla}^2 \overline{A} + \in_0 \mu_0 \frac{\partial^2 \overline{A}}{\partial t^2} = 0$$

(27) $$-\overline{\nabla}^2 \varphi + \in_0 \mu_0 \frac{\partial^2 \varphi}{\partial t^2} = 0$$

We now choose $\varphi = 0$ (Coulomb gauge), and from (22) we see:

(28) $$\overline{\nabla} \cdot \overline{A} = 0$$

So, the wave equation for our vector potential is:

(29) $$-\overline{\nabla}^2 \overline{A} + \in_0 \mu_0 \frac{\partial^2 \overline{A}}{\partial t^2} = 0$$

The solutions to this equation are plane waves.

(30) $$\overline{A} = \overline{A}_0 \sin\left(\omega t - \bar{k} \cdot \bar{r} + \alpha\right)$$

(31) $$= \overline{A}_0 \cos\left(\omega t - \bar{k} \cdot \bar{r} + \alpha'\right)$$

α: phase

$\bar{k}$ is the wave vector which points along the direction of propagation and has a magnitude:

(32) $$k^2 = \omega^2 \mu_0 \in_0 = \omega^2 / c^2$$

Since (28) $\overline{\nabla} \cdot \overline{A} = 0$

$$-\bar{k}\cdot\bar{A}_0\cos\left(\omega t-\bar{k}\cdot\bar{r}+\alpha\right)=0$$

(33) $\therefore\quad \bar{k}\cdot\bar{A}_0=0 \qquad\qquad \bar{k}\perp\bar{A}_0$

A_0 is the direction of the potential → polarization. From (15) and (16), we see that for $\varphi=0$:

$$\bar{E}=-\frac{\partial\bar{A}}{\partial t}=-\,\omega\bar{A}_0\cos\left(\omega t-\bar{k}\cdot\bar{r}+\alpha\right)$$

$$\bar{B}=\bar{\nabla}\times\bar{A}=-\left(\bar{k}\times\bar{A}_0\right)\cos\left(\omega t-\bar{k}\cdot\bar{r}+\alpha\right)$$

$\therefore\quad \bar{k}\perp\bar{E}\perp\bar{B}$