1.010 Uncertainty in Engineering
Fall 2008

Example Application 8
(hazard function)

OLD BETTER THAN NEW, NEW BETTER THAN OLD...

If a component or a system is subjected to a random environment, its reliability can be defined in terms of the random variable T = time to failure. In fact, the reliability of the system at any given time t is simply

$$\text{Reliability at } t = P[\text{no failure before } t] = P[T > t] = 1 - F_T(t) \tag{1}$$

where F_T is the cumulative distribution function (CDF) of T. An alternative characterization of component or system reliability is through the so-called hazard function h(t), which is defined such that h(t)dt is the probability that failure occurs in the time interval (t, t+dt), given that no failure occurred prior to t. First we show how h(t) is related to the CDF $F_T(t)$ and the PDF $f_T(t)$ of T and then comment on other properties of the function h(t).

• h(t) can be obtained from $F_T(t)$ and $f_T(t)$ as

$$h(t) = \frac{f_T(t)}{1 - F_T(t)} \tag{2}$$

In fact, using the definition of conditional probability,

$$
\begin{aligned}
h(t)dt &= P\big[\text{failure in } (t, t+dt) \big| \text{no failure prior to } t\big] \\
&= P\big[t < T \le t + dt \big| T > t\big] \\
&= \frac{P\big[(t < T \le t+dt) \cap (T > t)\big]}{P[T > t]} \\
&= \frac{P\big[(t < T \le t+dt)\big]}{P[T > t]} \\
&= \frac{f_T(t)dt}{1 - F_T(t)}
\end{aligned}
\tag{3}
$$

This produces Eq. 2.

- Conversely, $F_T(t)$ can be obtained from h(t) as

$$
\begin{aligned}
F_T(t) &= 1 - e^{-H(t)} \\
H(t) &= \int_0^t h(u)du
\end{aligned}
\tag{4}
$$

To prove Eq. 4, let G(t) = 1 - $F_T(t)$. Then G'(t) = dG(t)/dt = - $f_T(t)$ and, from Eq. 2, -h(t) = G'(t)/G(t). Integration of both sides from 0 to t gives

$$
-\int_0^t h(u)du = \int_0^t \frac{G'(u)}{G(u)}du = \ln G(t) - \ln G(0) \tag{5}
$$

Since $\int_0^t h(u)du = H(t)$ and G(0) = 1 (hence lnG(0) = 0), one concludes from Eq. 5 that lnG(t) = - H(t). Therefore $G(t) = e^{-H(t)}$, which implies $F_T(t) = 1 - e^{-H(t)}$ in Eq. 4.

Reliability of Old and New Items

An important special case of Eq. 2 is when T has exponential distribution, say with $F_T(t) = 1 - e^{-\lambda t}$. Then $f_T(t) = \lambda e^{-\lambda t}$ and, from Eq. 2, $h(t) = \lambda e^{-\lambda t}/e^{-\lambda t} = \lambda$ = constant. Therefore, a used item with exponential lifetime is "as good as new". This can be seen also through the distribution of the remaining lifetime. Suppose that the item has been properly working up to the present time t_o and let $T_o = T - t_o$ be the residual lifetime. Then

$$\begin{aligned} F_{T_o}(t) &= P[T_o \le t] \\ &= P[T \le t + t_o | T > t_o] \\ &= \frac{P[(T \le t + t_o) \cap (T > t_o)]}{P[T > t_o]} \\ &= \frac{e^{-\lambda t_o} - e^{-\lambda (t+t_o)}}{e^{-\lambda t_o}} \\ &= 1 - e^{-\lambda t} = F_T(t) \end{aligned} \tag{6}$$

Eq. 6 shows that the distribution of the remaining life is the same as the original distribution, i.e. the same as the distribution for a new item.

Items that are worse old than new have lifetime distributions with increasing hazard functions and items that are better old than new have decreasing hazard functions. Notice however that the hazard function needs not be monotonically increasing or decreasing; it may also fluctuate. For example, many new manufactured items go through a "break-in" period in which failure is relatively likely to occur due to manufacturing defects ("lemons"). For a period following this initial break-in phase, failure occurs much less frequently. However, after a while, the incidence of failures increases, due to factors like wear, aging and fatigue. For such items, the hazard function has a characteristic "bathtub" shape, as the one shown below (incidentally, such bathtub hazard functions are appropriate also for human life!).

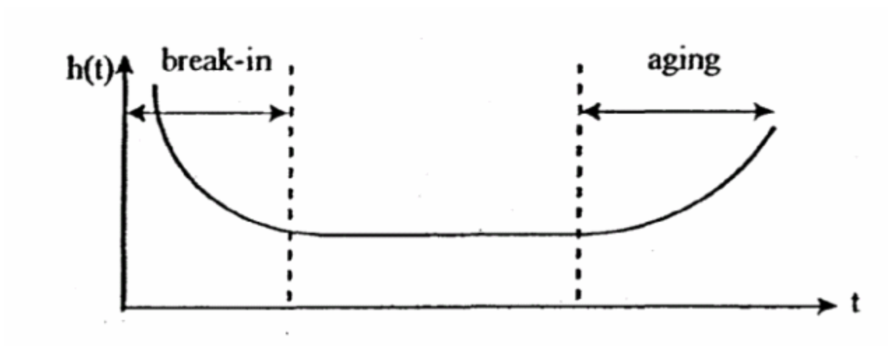

Figure 1: Schematic hazard function for items with break-in and aging.

Problem 8.1

(a) *You are about to buy a used car, which has accumulated 50,000 miles without major problems. You are worried that something bad may soon be happening to it. The dealer reassures you that the car "is very safe, because statistics show that its lifetime distribution is uniform between 0 and 150,000 miles." In his sale pitch, the dealer adds: " This means that the car has the same chance of dying on you in the next 50,000 miles as it did during the first 50,000 miles after rolling out of the factory. Therefore, it is as good as new." Using Eq. 2, calculate the hazard function $h(t)$ for the car when it was new (notice that here t has the units of miles). Compare $h(t)$ for $0 < t < 50{,}000$ miles and $50{,}000 < t < 100{,}000$ miles. Do you agree with the dealer that, for the next 50,000 miles, the car is "as good as new?" Explain your reasoning.*

(b) *Now suppose that the dealer tells you that, for $t > 50{,}000$ miles, "cars like this have a lifetime distribution with probability density function $f_T(t) = t^{-a}$, for some $a > 1$." Calculate and plot the hazard function $h(t)$ for $a = 2$ and $t > 50{,}000$ miles. Does this function increase or decrease with t? Is an older car of this type more or less reliable than a newer car?*