1.0 Appendix - Dealing with Uncertainty

1.1 Theory

In making measurements, in doing experiments to evaluate the behavior of a structure, or more simply to determine the value of a property, the instruments we use, whether unaided eyesight or laser interferometer, have a limited resolution. There will always be some uncertainty in the measured value which is on the order of the fineness of scale. When we make a measurement, we should record an estimate of the associated uncertainty. This is part and parcel of a value, as inseparable as the measured quantity's units.

We might indicate the uncertainty in the Data Sheet below, as in your Pasta report, in the third row as +/- ?? mm and the units of the column quantity in the second row.:

TABLE 1. Data Sheet Format

Spc #	**Length, L**	**Height, h**	**Span, s**	**h/L**	**R/L**	**$e|_{max}$**	**s/L**	
	mm	**mm**	**mm**				**measd.**	**graph**
	+/- ? mm	**+/- ? mm**	**+/- ?mm**	**+/-**	**+/-**	**+/-**	**+/-**	
1								

The question marks are to be filled in by the one(s) taking the measurement. Generally one estimates an uncertainty as one number, to be representative of a whole set of specimans subject to that measurement. For example, for the long strands, after measuring the lengths of several, make an estimate of uncertainty. That one number goes at the top of the column, replacing the questionmark. (You may want to record another estimate for the short specimans).

Ultimately we want to know the uncertainty in the quantity we seek to determine - the strain at fracture. The question then can be put: How do all the uncertainties of these individual, measurements combine and affect the uncertainty in the bending strain? Alternatively, and more in line with the lingo of the trade: How do these uncertainties *propagate* through to final result?

Consider a first step in moving from the measured values to the value for the ratio *h/L*. Given the measurements of *L* and *h* and associated estimates of uncertainties, call them u_L and u_h, what is the uncertainty in the ratio, *h/L*, or $u_{h/l}$?

We proceed as follows: For any group of specimans of like kind - e.g., plain spaghetti strands in their long, original state - we expect the measurements of length *L* and of the deflected height *h* to show only small variations in the neighbohood of some nominal values, L_n h_n. In this case, the value of the change in the quantity (*h/L*) can be aproximated, using a Taylor expansion, by

$$\Delta(h/L) = \left(\frac{\partial f}{\partial L}\right) \cdot \Delta L + \left(\frac{\partial f}{\partial h}\right) \cdot \Delta h$$

$$\Delta(h/L) = (-h/L^2) \cdot \Delta L + (1/L) \cdot \Delta h$$

Consider, for a moment, how one estimates the mean and variance of a *linear function* of two random variables.[1] If Y depends upon X_1 and X_2 according to

$$Y = a_1 \cdot X_1 + a_2 \cdot X_2$$

then the mean, or expected value of Y, computed according to

$$\bar{Y} = \frac{1}{n} \cdot \sum_{i=1}^{n} Y_i = E(Y)$$

is a linear function of the means of the variables X_1 and X_2. If we now interpret our small changes in *h* and *L* as random variables, of zero mean, i.e., in the notation of the equation immediately above: $\overline{\Delta(h/L)} = (-h/L^2) \cdot \overline{\Delta L} + (1/L) \cdot \overline{\Delta h}$

With $\overline{\Delta L} = 0$ and $\overline{\Delta h} = 0$ then the mean of Δ*(h/L) is* also zero. In this case , we must push further to obtain an uncertainty estimate for the latter. We will use the variance.

Returning to our general case, the variance of Y is given by:

$$Var(Y) = E(Y^2) - \bar{Y}^2$$

$$Var(Y) = a_1^2 \cdot Var(X_1) + a_2^2 \cdot Var(X_2) + 2a_1a_2 \cdot Cov(X_1, X_2)$$

In this, the covariance of X_1 and X_2 is computed according to

$$Cov(X_1, X_2) = E[(X_1 - \bar{X}_1) \cdot (X_2 - \bar{X}_2)]$$

Now if X_1 and X_2 are *independent random variables*, (as are the measurements *h* and *L*, *i.e.,* we would not expect our measurement of h to be influenced by our measurement of L,

1. Ang, A., Tang, Wilson, Probability Concepts in Engineering Planning and Design, Vol. I, John Wiley and Sons, New York, 1975, p191 ff.

assuming we are working with one set of specimans), then the covariance is zero and the last term vanishes from the expression for the variance.

We now make a crucial step: We associate (or define, if you like) the uncertainty in an experimental value with its variance. In this way we can write:

$$Var(\Delta(h/L)) = {a_1}^2 \cdot Var(\Delta L) + {a_2}^2 \cdot Var(\Delta h)$$

or

$$u_{h/L}^2 = (h/L^2)^2 \cdot u_L^2 + (1/L)^2 \cdot u_h^2$$

where we have equated the uncertainties to the variances and, from our expression for the linear sum, identified

$$a_1 = (-h/L^2) \qquad \text{and} \qquad a_2 = (1/L)$$

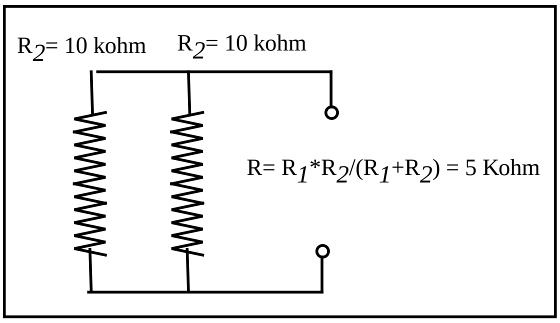
The above gives us a way to compute the uncertainty in the ratio *h/L* having (independent) estimates of the uncertainties of *h* and *L*, u_h and u_L

In general, if Y is a function of X_1, X_2,....X_n, and the X's are all independent, we can write:

$$u_Y^2 = \left(\frac{\partial Y}{\partial X_1}\right)^2 \cdot u_1^2 + \left(\frac{\partial Y}{\partial X_2}\right)^2 \cdot u_2^2 \text{........} + \left(\frac{\partial Y}{\partial X_n}\right)^2 \cdot u_n^2$$

1.2 Example: Two Resistors in Parallel

Taking an example from electrical circuits (we will be building some when we use strain gages), consider two resistors in parallel. Without too much loss in generality, we take them to be equal, say 10 kilo-ohms each.



An estimate of the uncertainty in the effective resistance *R* where

$$R = \frac{1}{R_1} + \frac{1}{R_2}$$

or

$$R = (R_1 \cdot R_2)/(R_1 + R_2)$$

is, after taking the partial derivatives, and noting they are squared:

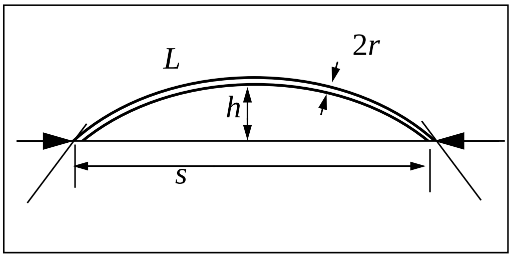
$$u^2_R = \left(\frac{R_2}{R_1 + R_2}\right)^4 \cdot u_2^2 \quad + \quad \left(\frac{R_1}{R_1 + R_2}\right)^4 \cdot u_2^2$$

If $R_1 = R_2$, this simplifies and can be written as; (check it out): $(u_R/R) = \frac{1}{\sqrt{2}} \cdot (u_1/R_1)$

This says that the uncertainty in the effective resistance, of the two in parallel *is less than* the uncertainty (expressed as a percentage) of either of the two taken alone! Can you explain this, perhaps counter-intuitive result?

1.3 Example: Propagation of Uncertainty with Pasta.

We consider, as an example, estimating the uncertainty in the bending strain at failure of the *thickest* pasta and only the case where the length is the *original length*. Our goal is to see how uncertainties in the measured values of *L, h*, and *r,* (actually 2r, the diameter), propagate through to produce an uncertainty in the bending strain, ε. We can then see if this brackets the results obtained from experiment on the 15 specimen. If so, then we have some confidence that our theory of failure is correct and that we have not overlooked some important factor - in the physical composition and behavior of the pasta and/or in our measurement technique that might explain a broader range of values of ε.



To determine the maximum bending strain at failure, we first make several measurements of the diameter, *d*, along a single strand to check its uniformity and across several different-strands. This produces, not just a nominal value for the diameter, e.g., 1.9 mm, but also a value for the associated uncertainty[1], e.g., 0.1mm.

1.3.1 Setting Uncertainties of Measured Values.

For each of the 15 strands tested, we measured the original length *L*, e.g., 240.0 mm, and estimated an associated uncertainty, e.g., 1mm. [Note: Strands of different length, *L*, of the same radius, will fail at different heights, *h*. But theory states that the radius of curvature should be *the same*,

1. The uncertainty in this instance may be taken as, not just the uncertainty due to the limited resolution of the calipers and our imperfect interpolation capabilities, but also the variability in diameter from specimen to specimen. In a more detailed experiment we might be required to pin down more precisely the characteristics of the distribution of the diameter (radius) considered as a random variable.

across all specimens. The uncertainty in this instance *should not* include the variation in original length. If, after analyzing the outcome of the experiments on the shorter strands of the same radius, we find our propagated uncertainty *does not bracket* the range of ε, we can conjecture that length *does matter* in some way and look for a physical or experimental reason why].

Applying the end load, we carefully noted the height, *h*, e.g., 40 mm, and span, *s*, e.g., 220 mm, just prior to fracture. After several trials, we made estimates of their associated uncertainties, e.g., +/- 2 mm and +/- 2 mm respectively.

Summarizing, letting the bold represent nominal values (later to be replaced by mean values), we write[1]

$$d = \mathbf{d} + u_d = 1.9 +/- 0.1 \text{ mm}$$

$$L = \mathbf{L} + u_L = 240 +/- 1.0 \text{ mm}$$

$$h = \mathbf{h} + u_h = 40 +/- 2.0 \text{ mm}$$

$$s = \mathbf{s} + u_s = 220 +/- 2.0 \text{ mm}$$

1.3.2 Propagation of Uncertainties.

As the simplest example of “propagation of uncertainty” consider how we obtain the uncertainty in the radius once having estimated the uncertainty associated with our measurement of the diameter. The radius is linearly related to the diameter; it’s just one-half the diameter. Our variance of the radius is then given in terms of the variance of the diameter by [2]

$$Var(r) = (1/2)^2\, Var(d)$$

or, since we interpret the variance as the square of the uncertainty, we have $u_r = (1/2)\, u_d$, and so

$$r = \mathbf{r} + u_r = 0.95 +/- 0.05 \text{ mm}.$$

This, at first sight, appears strange, i.e., we have a *smaller* uncertainty in the radius than the diameter. But note that the uncertainty relative to the nominal value is the same, for

$$u_r/\mathbf{r} = u_d/\mathbf{d} = .05/.95 \text{ (or } 1.0/1.95) = .053 = 5.3\%$$

Representing uncertainties as percentages of their nominal values will be pursued in what follows.

Recall the process for finding the strain: For each specimen, we compute the ratio of the height, *h*, at failure to the original length, *L*. With a value in hand for this ratio, we go to the graph and determine a value for the ratio of the radius of curvature at mid span, *R*, to

1. The nominal values are, to a certain extent, artificial; the estimates of uncertainty more real.
2. There is nothing approximate about this relationship in this instance since the relationship between diameter and radius is linear.

the original length. Knowing L, we can determine R. Knowing R and the radius, r, we can determine the strain.

Our propagation of uncertainty follows this same route. As a first step we determine an uncertainty to associate with the ratio *(h/L).*

We have from before:

$$u_{h/L} = \left[\left(\frac{\boldsymbol{h}}{\boldsymbol{L^2}}\right)^2 \cdot u_L^2 + \left(\frac{\boldsymbol{1}}{\boldsymbol{L}}\right)^2 \cdot u_h^2\right]^{1/2}$$

Dividing both sides by (***h/L***) we can write this in terms of uncertainties relative to the nominal values of their associated quantities.

$$\frac{u_{h/L}}{(\boldsymbol{h/L})} = \left[\left(\frac{u_L}{\boldsymbol{L}}\right)^2 + \left(\frac{u_h}{\boldsymbol{h}}\right)^2\right]^{1/2}$$

With

$u_L/\boldsymbol{L} = 1/240 = .0042$ *(note: less than 1%),*

and

$u_h/\mathbf{h} = 2/40 = .050$ (5.0%)

we obtain

$$\frac{u_{h/L}}{(\boldsymbol{h/L})} = 0.05\dot{0}2 \qquad (5\%)$$

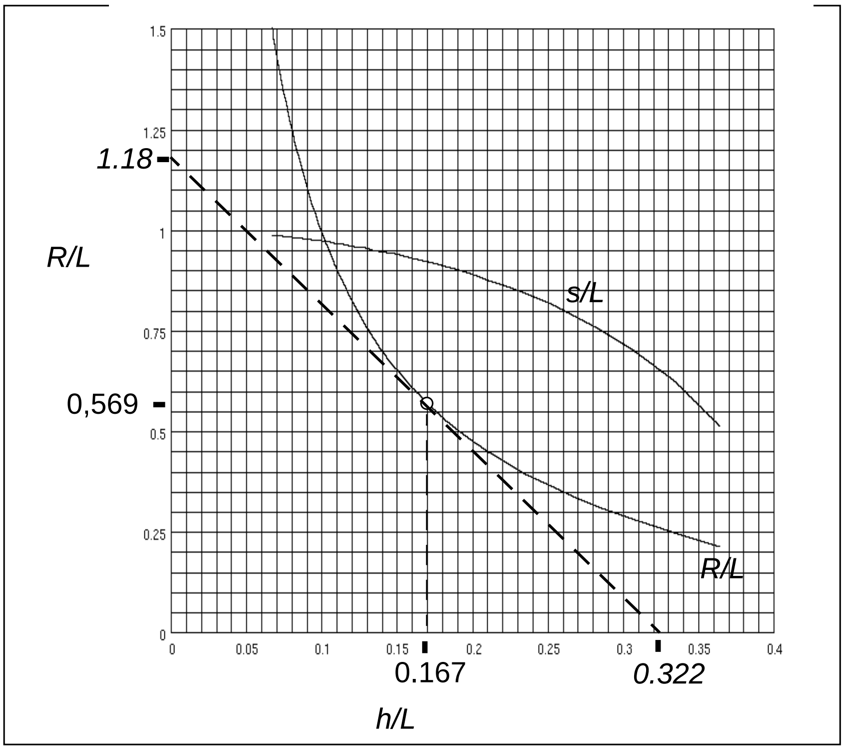
(Note how the uncertainty in h predominates relative to the uncertainty in L).

With the ratio of the nominal values (***h/L)*** = 40/240 = 0.167, the uncertainty of the ratio itself is

$u_{h/L} = 0.0084$

Next, we propagate this measure through to an uncertainty of R/L using the graph. To do so, think of the plot of R/L versus h/L as a one-to-one functional relationship, which it is, to which we can fit a linear approximation in the vicinity of ***h/L*** = 0.167 by graphical means.

This I have done, drawing a line tangent to the curve at this point.



We have, as a linear approximation,

$$R/L = 1.18 - (1.18/.322)^* (h/L) = 1.18 - 3.66(h/L)$$

Interpreting the square of the uncertainty as the variance and noting that

$$\frac{\partial}{\partial(h/L)}(R/L) = -3.66$$

we have

$$u_{R/L} = [(3.66)^2 \cdot (u_{h/L})^2]^{1/2} = 0.0307$$

and with

R/L = .568, $u_{R/L}$ / (***R/L)*** = .054 (5.4%)

I suggest another uncertainty associated with reading the graph. Think of the ordinate scale as a measuring stick. Its resolution is .05. There is some additional uncertainty in fixing *R/L* due to inaccurate readings of the scale, e.g., +/- 0.02. How to account for this?

One way is to think of the intercept of our linear approximation as being another independent random variable, *z*, (or if you like, it bounces up and down from one reading to another). Write

$R/L = z - 3.66(h/L)$ where $z = \mathbf{z} + u_z = 1.18 \pm 0.02$

Then our estimate of uncertainty in *R/L* becomes, according to our consideration of variance of the sum of two random variables,[1]

$$u_{R/L} = [1 \cdot u_z^2 + (3.66)^2 \cdot u_{h/L}^2]^{1/2} = 0.0367$$

So $\dfrac{u_{R/L}}{(\boldsymbol{R/L})} = 0.0645$ (6.5%)

To find the uncertainty in the strain, which is our goal, we need to take this process one step further. Writing

$\varepsilon = r/R = (r/L) / (R/L)$

we have for the uncertainty in the strain:

$$\frac{u_\varepsilon}{\varepsilon} = \left[\left(\frac{u_{r/L}}{\boldsymbol{r/L}}\right)^2 + \left(\frac{u_{R/L}}{\boldsymbol{R/L}}\right)^2\right]^{1/2}$$

so we need one additional bit of information, namely

$$\frac{u_{r/L}}{\boldsymbol{r/L}} = \left[\left(\frac{u_r}{\boldsymbol{r}}\right)^2 + \left(\frac{u_L}{\boldsymbol{L}}\right)^2\right]^{1/2}$$

which with $r = \boldsymbol{r} + u_r = 0.95 \pm .05$ mm and $L = \boldsymbol{L} + u_L = 240 \pm 1$mm gives

$$\frac{u_{r/L}}{\boldsymbol{r/L}} = 0.053 \ (5.3\%)$$

With this, and our already determined uncertainty for *R/L*, our final result of this propagation of uncertainties gives $u_e/\varepsilon = 0.084$ (8.4%)

which, with our nominal value of strain $\varepsilon = \mathbf{r/R} = 0.95/136.5 = 6.96$ E-03

which says that our values should lie within the range 6.4E-03 to 7.6 E-03.

1. Note we can not express this in terms of relative values so neatly.