Part III

ANALYSIS OF A MEMBER ELEMENT

10

Governing Equations for a Deformable Solid

10–1. GENERAL

The formulation of the governing equations for the behavior of a deformable solid involves the following three steps:

1. *Study of deformation.* We analyze the change in shape of a differential volume element due to displacement of the body. The quantities required to specify the deformation (change in shape) are conventionally called strains. This step leads to a set of equations relating the strains and derivatives of the displacement components at a point. Note that the analysis of strain is purely a *geometrical* problem.
2. *Study of forces.* We visualize the body to consist of a set of differential volume elements. The forces due to the interactions of adjacent volume elements are called *internal* forces. Also, the internal force per unit area acting on a differential area, say dA_j, is defined as the stress vector, $\bar{\sigma}_j$. In this step, we analyze the state of stress at a point, that is, we investigate how the stress vector varies with orientation of the area element. We also apply the conditions of static equilibrium to the volume elements. This leads to a set of differential equations (called stress equilibrium equations) which must be satisfied at each point in the interior of the body and a set of algebraic equations (called stress boundary conditions) which must be satisfied at each point on the surface of the body. Note that the study of forces is purely an *equilibrium* problem.
3. *Relate forces and displacements.* In this step, we first relate the stress and strain components at a point. The form of these equations depends on the material behavior (linear elastic, nonlinear elastic, inelastic, etc.). Substitution of the strain-displacement relations in the stress-strain relations leads to a set of equations relating the stress components and derivatives of the displacement components. We refer to this system as the stress-displacement relations.

The governing equations for a deformable solid consist of the stress equilibrium equations, stress-displacement relations, and the stress and displacement boundary conditions.

In this chapter, we develop the governing equations for a linearly elastic solid following the steps outlined above. We also extend the variational principles developed in Chapter 7 for an ideal truss to a three-dimensional solid.

In Chapter 11, we present St. Venant's theory of torsion-flexure of prismatic members and apply the theory to some simple cross sections. St. Venant's theory provides us with considerable insight as to the nature of the behavior and also as to how we can simplify the corresponding mathematical problem by introducing certain assumptions. The conventional engineering theory of prismatic members is developed in Chapter 12 and a more refined theory for thin walled prismatic members which includes the effect of warping of the cross section is discussed in Chapter 13. In Chapter 14, we develop the engineering theory for an arbitrary planar member. Finally, in Chapter 15, we present the engineering theory for an arbitrary space member.

10–2. SUMMATION CONVENTION; CARTESIAN TENSORS

Let **a** and **b** represent nth-order column matrices:

$$\mathbf{a} = \{a_1, a_2, \ldots, a_n\} \qquad \mathbf{b} = \{b_1, b_2, \ldots, b_n\} \tag{10–1}$$

Their scalar (inner) product is defined as

$$\mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{a} = a_1b_1 + a_2b_2 + \cdots + a_nb_n = \sum_{i=1}^{n} a_ib_i \tag{a}$$

To avoid having to write the summation sign, we introduce the convention that when an index is repeated in a term, it is understood the term is summed over the range of the index. According to this convention

$$\sum_{i=1}^{n} a_ib_i \equiv a_ib_i \qquad (i = 1, 2, \ldots, n) \tag{10–2}$$

and we write the scalar product as

$$\mathbf{a}^T\mathbf{b} = a_ib_i \tag{10–3}$$

The summation convention allows us to represent operations on multidimensional arrays in compact form. It is particularly convenient for formulation, i.e., establishing the governing equations. We illustrate its application below.

Example 10–1

1. Consider the product of a rectangular matrix, **a**, and a column vector, **x**.

$$\mathbf{c} = \mathbf{a}\mathbf{x} \qquad \mathbf{a} \text{ is } m \times n \tag{a}$$

The typical term is

$$c_i = \sum_{j=1}^{n} a_{ij}x_j \equiv a_{ij}x_j \tag{b}$$

2. Let **a**, **b** be square matrices, **x** a column vector, and f, g scalars defined by

$$f = \mathbf{x}^T\mathbf{a}\mathbf{x} \qquad g = \mathbf{x}^T\mathbf{b}\mathbf{x} \tag{c}$$

The matrix form of the product, fg, is

$$fg = (\mathbf{x}^T\mathbf{a}\mathbf{x})(\mathbf{x}^T\mathbf{b}\mathbf{x}) \tag{d}$$

One could expand (d) but it is more convenient to utilize (b) and write (c) as

$$f = a_{ij}x_ix_j \qquad g = b_{k\ell}x_kx_\ell \tag{e}$$

Then,

$$fg = a_{ij}b_{k\ell}x_ix_jx_kx_\ell = D_{ijk\ell}x_ix_jx_kx_\ell \tag{f}$$

3. We return to part 1. The inner product of **c** is a scalar, H,

$$H = \mathbf{c}^T\mathbf{c} = \mathbf{x}^T(\mathbf{a}^T\mathbf{a})\mathbf{x} \tag{g}$$

Using (b),

$$H = c_ic_i = a_{ik}a_{i\ell}x_kx_\ell \tag{h}$$

The outer product is a second-order array, **d**,

$$\mathbf{d} = \mathbf{c}\mathbf{c}^T = \mathbf{a}\mathbf{x}\mathbf{x}^T\mathbf{a}^T \tag{i}$$

and can be expressed as

$$d_{ij} = c_ic_j = a_{ik}a_{j\ell}x_kx_\ell = A_{ijk\ell}x_kx_\ell \tag{j}$$

According to the summation convention,

$$d_{ii} = d_{11} + d_{22} + \cdots = \text{trace of } \mathbf{d} \tag{k}$$

Then, we can write (h) as

$$H = d_{ii} = A_{iik\ell}x_kx_\ell \tag{l}$$

4. Let σ_{ij}, e_{ij} represent square second-order arrays. The inner product is defined as the sum of the products of corresponding elements:

$$\text{Inner product } (\sigma_{ij}, e_{ij}) = \sum_i \sum_j \sigma_{ij}e_{ij} = \sigma_{11}e_{11} + \sigma_{22}e_{22} + \cdots + \sigma_{12}e_{12} + \sigma_{21}e_{21} + \cdots \equiv \sigma_{ij}e_{ij} \tag{m}$$

In order to represent this product as a matrix product, we must convert σ_{ij}, e_{ij} over to one-dimensional arrays.

Let $b_1^{(1)}$, $b_2^{(1)}$, $b_3^{(1)}$ represent a one-dimensional set of elements associated with an orthogonal reference frame having directions $X_1^{(1)}$, $X_2^{(1)}$, $X_3^{(1)}$. If the corresponding set for a second reference frame is related to the first set by

$$\begin{aligned} b_j^{(2)} &= \alpha_{jk}\, b_k^{(1)} \\ \alpha_{jk} &= \underset{\substack{j,k=\\1,2,3}}{\cos}\,(X_j^{(2)}, X_k^{(1)}) \end{aligned} \tag{10–4}$$

we say that the elements of b comprise a first-order cartesian tensor. Noting (5–5), we can write (10–4) as

$$\mathbf{b}^{(2)} = \mathbf{R}^{12}\mathbf{b}^{(1)} \tag{10–5}$$

and it follows that the set of orthogonal components of a vector are a first-order cartesian tensor. We know that the magnitude of a vector is invariant. Then, the sum of the squares of the elements of a first-order tensor is invariant.

$$b_j^{(1)}b_j^{(1)} \equiv b_j^{(2)}b_j^{(2)} \tag{10–6}$$

A second-order cartesian tensor is defined as a set of doubly subscripted elements which transform according to

$$b_{jk}^{(2)} = \underset{\substack{j,k,m,n\\=1,2,3}}{\alpha_{jm}\alpha_{kn}}\, b_{mn}^{(1)} \tag{10–7}$$

An alternate form is

$$\mathbf{b}^{(2)} = \mathbf{R}^{12}\mathbf{b}^{(1)}(\mathbf{R}^{12})^T \tag{10–8}$$

The transformation (10–8) is orthogonal and the trace, sum of the principal second-order minors, and the determinant are invariant.†

$$\begin{aligned} \beta_1^{(2)} &= \beta_1^{(1)} \\ \beta_2^{(2)} &= \beta_2^{(1)} \\ \beta_3^{(2)} &= \beta_3^{(1)} \end{aligned} \tag{10–9}$$

where

$$\beta_1 = b_{jj}$$
$$\beta_3 = |\mathbf{b}|$$
$$\beta_2 = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{13} \\ b_{31} & b_{33} \end{vmatrix}$$

In the cases we encounter, **b** will be symmetrical.

10–3. ANALYSIS OF DEFORMATION; CARTESIAN STRAINS

Let P denote an arbitrary point in the undeformed state of a body and $\vec{r}$ the position vector for P with respect to 0, the origin of an orthogonal cartesian reference frame. The corresponding point and position vector in the deformed state are taken as P'; $\vec{\rho}$ and the movement from P to P' is represented by the displacement vector, $\vec{u}$. By definition,

$$\vec{\rho} = \vec{r} + \vec{u} \tag{10–10}$$

This notation is shown in Fig. 10–1.

† See Prob. 2–5.

Excluding rigid body motion, the displacement from the initial undeformed position will be small for a solid, and it is reasonable to take the initial cartesian coordinates (x_j) as the independent variables. This is known as the Lagrange

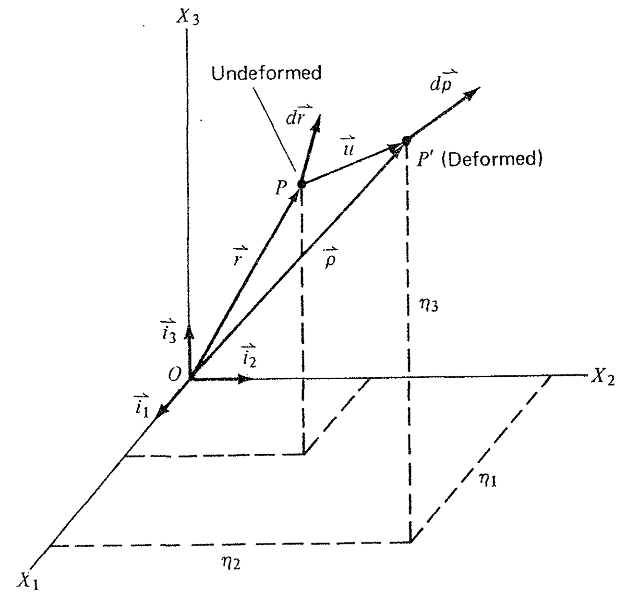


Fig. 10–1. Geometric notation.

approach. Also, to simplify the derivation, we work with cartesian components for $\vec{u}$. Then,

$$\begin{aligned} \vec{u} &= \vec{u}(x_j) = u_j\vec{\imath}_j \\ \vec{\rho} &= \vec{\rho}(x_j) \end{aligned} \tag{10–11}$$

We consider a differential line element at P represented by the vector $d\vec{r}$. (See Fig. 10–1). The initial length and direction cosines are ds and α_j. We are using the subscript notation for partial differentiation.

$$f_{,j} \equiv \frac{\partial}{\partial x_j} f$$
$$d\vec{r} = \vec{r}_{,j}\, dx_j = dx_j\vec{\imath}_j = ds(\alpha_j\vec{\imath}_j) \tag{10–12}$$

The corresponding line element in the deformed state is $d\vec{\rho}$. Since we are following the Lagrange approach, $\rho = \vec{\rho}(x_j)$, and we can write

$$d\vec{\rho} = \vec{\rho}_{,j}\, dx_j = ds(\alpha_j\vec{\rho}_{,j}) \tag{10–13}$$

The extensional strain, ε, is defined as the relative change in length with respect to the *initial* length.†

$$|d\bar{\rho}| = (1 + \varepsilon)\,|d\bar{r}| \tag{10–14}$$

Using the dot product, (10–14) becomes

$$(1 + \varepsilon)^2 = \frac{1}{ds^2}(d\bar{\rho} \cdot d\bar{\rho}) = \alpha_j \alpha_k \bar{\rho}_{,j} \cdot \bar{\rho}_{,k} \tag{a}$$

Finally, we write (a) as

$$\begin{aligned} \varepsilon(1 + \tfrac{1}{2}\varepsilon) &= \alpha_j \alpha_k e_{jk} \\ e_{jk} &= \tfrac{1}{2}(\bar{\rho}_{,j} \cdot \bar{\rho}_{,k} - \delta_{jk}) \end{aligned} \tag{10–15}$$

One can readily establish that (e_{jk}) is a second-order symmetrical cartesian tensor.‡

Taking the line element to be initially parallel to the X_i direction and letting ε_i represent the extensional strain, we see that

$$\begin{aligned} \varepsilon_i(1 + \tfrac{1}{2}\varepsilon_i) &= e_{ii} \qquad \text{(no sum)} \\ &= \tfrac{1}{2}(\rho_{,i} \cdot \bar{\rho}_{,i} - 1) \end{aligned} \tag{10–16}$$

To interpret the off-diagonal terms, e_{jk}, we consider 2 initially orthogonal line elements represented by $d\bar{r}'_1$, $d\bar{r}'_2$ (see Fig. 10–2) and having direction cosines

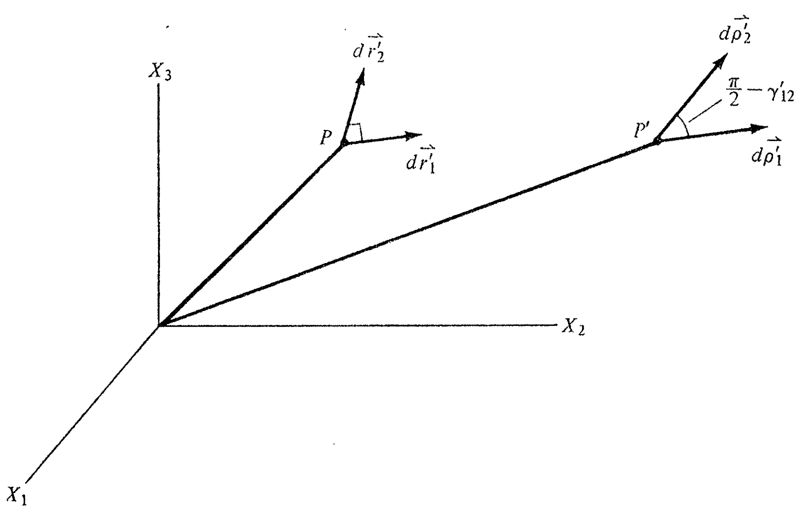


Fig. 10–2. Notation for shearing strain.

† This is the definition of Lagrangian strain. In the Eulerian approach, the cartesian coordinates (η_j) for the deformed state are taken as the independent variables,

$$u_j = u_j(\eta_k) \qquad x_j = x_j(\eta_k)$$

and the strain is defined as

$$|d\bar{r}| = (1 - \varepsilon)|d\bar{\rho}|$$

‡ See Prob. 10–4. It is known as Green's strain tensor. The elements, e_{jk}, are also called the components of finite strain. They relate the difference between the square of the initial and deformed lengths of the line element, i.e., an alternate definition of e_{jk} is

$$|d\bar{\rho}|^2 - ds^2 = 2e_{jk}\,dx_j\,dx_k$$

α_{1j}, α_{2j}. We define $\frac{\pi}{2} - \gamma'_{12}$ as the angle between the lines in the deformed state. The expression for γ'_{12}, which is called the shearing strain, follows by taking the dot product of the deformed vectors.

$$\cos\left(\frac{\pi}{2} - \gamma'_{12}\right) = \sin\gamma'_{12} = \frac{d\bar{\rho}'_1 \cdot d\bar{\rho}'_2}{|d\bar{\rho}'_1|\,|d\bar{\rho}'_2|} \tag{a}$$

Substituting for $d\bar{\rho}'_j$,

$$\begin{aligned} d\bar{\rho}'_j &= (\alpha_{jk}\bar{\rho}_{,k})ds'_j \\ |d\bar{\rho}'_j| &= (1 + \varepsilon'_j)ds'_j \end{aligned} \qquad \text{(sum on } k \text{ only)} \tag{b}$$

and noting that the lines are initially orthogonal,

$$\alpha_{j\ell}\alpha_{k\ell} = \delta_{jk} \tag{c}$$

(a) takes the form

$$(1 + \varepsilon'_1)(1 + \varepsilon'_2)\sin\gamma'_{12} = 2\alpha_{1j}\alpha_{2k}e_{jk} \tag{10–17}$$

Specializing (10–17) for lines parallel to X_i, X_j shows that e_{ij} is related to the shearing strain, γ_{ij}.

$$(1 + \varepsilon_i)(1 + \varepsilon_j)\sin\gamma_{ij} = 2e_{ij} = \bar{\rho}_{,i} \cdot \bar{\rho}_{,j} \tag{10–18}$$

Equations (10–15) and (10–17) are actually transformation laws for extensional and shearing strain. The state of strain is completely defined once the strain tensor is specified for a particular set of directions. To generalize these expressions, we consider two orthogonal frames defined by the unit vectors $\bar{i}_j$ and $\bar{i}'_j$ (see Fig. 10–3), take the initial frame parallel to the global frame $(\bar{i}_j = \bar{i}_j)$, and let $\alpha_{jk} = \bar{i}'_j \cdot \bar{i}_k$. With this notation:

$$\begin{aligned} e'_{ij} &= \alpha_{ik}\alpha_{j\ell}e_{k\ell} \\ \varepsilon'_i(1 + \tfrac{1}{2}\varepsilon'_i) &= e'_{ii} \\ (1 + \varepsilon'_i)(1 + \varepsilon'_j)\sin\gamma'_{ij} &= 2e'_{ij} \end{aligned} \quad \text{(no sum)} \tag{10–19}$$

The strain measures (ε, γ) are small with respect to unity for engineering materials such as metals and concrete. For example, $\varepsilon \approx 0(10^{-3})$ for steel. Therefore, it is quite reasonable (aside from the fact that it simplifies the expressions) to assume $\varepsilon, \gamma \ll 1$ in the strain expressions. The relations for "small" strain are:

$$\begin{aligned} \varepsilon_i &\approx e_{ii} \\ \gamma_{ij} &\approx 2e_{ij} \end{aligned} \tag{10–20}$$

It remains to expand e_{jk}. Now,

$$\bar{\rho} \doteq \bar{r} + \bar{u} \equiv (x_m + u_m)\bar{i}_m \tag{a}$$

Differentiating $\bar{\rho}$ with respect to x_j,

$$\bar{\rho}_{,j} = \frac{\partial\bar{\rho}}{\partial x_j} = (\delta_{mj} + u_{m,j})\bar{i}_m \tag{b}$$

and substituting into the definition of e_{jk} (Equation (10–15)) leads to

$$e_{jk} = \tfrac{1}{2}(u_{j,k} + u_{k,j}) + \tfrac{1}{2}u_{m,j}u_{m,k} \qquad \text{(sum on } m \text{ only)} \qquad (10\text{–}21)$$

In order to simplify (10–21), we must establish the geometrical significance of the various terms.

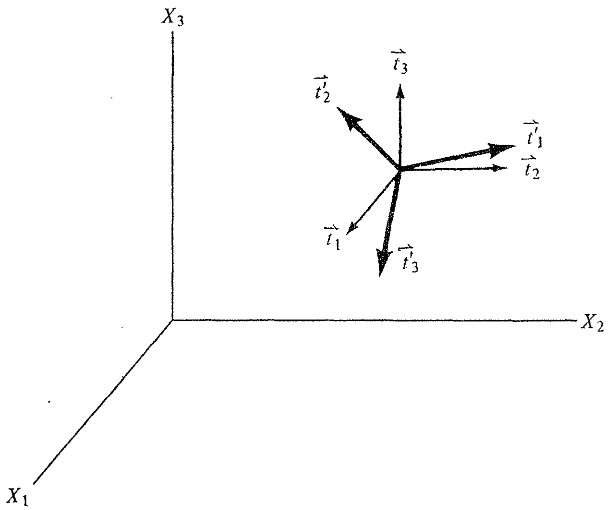


Fig. 10–3. Unit vectors defining transformation of orthogonal directions.

With this objective, we consider a line element initially parallel to the X_1 axis. Figure 10–4 shows the initial and deformed positions, and the angles θ_{12}, θ_{13} which define the rotation of the line toward the X_2, X_3 directions. The geometrical relations of interest to us are

$$\sin\theta_{13} = \frac{u_{3,1}}{1 + \varepsilon_1} \qquad \text{(a)}$$

$$\sin\theta_{12} = \frac{u_{2,1}}{(1 + \varepsilon_1)\cos\theta_{13}} \qquad \text{(b)}$$

$$(1 + \varepsilon_1)^2 = (1 + u_{1,1})^2 + u_{2,1}^2 + u_{3,1}^2 \qquad \text{(c)}$$

Also, by definition,

$$\varepsilon_1(1 + \tfrac{1}{2}\varepsilon_1) = e_{11} = u_{1,1} + \tfrac{1}{2}(u_{1,1}^2 + u_{2,1}^2 + u_{3,1}^2) \qquad \text{(d)}$$

We solve (a), (b) for $u_{2,1}$ and $u_{3,1}$,

$$\begin{aligned} u_{3,1} &= (1 + \varepsilon_1)\sin\theta_{13} \\ u_{2,1} &= (1 + \varepsilon_1)\sin\theta_{12}\cos\theta_{13} \end{aligned} \qquad (10\text{–}22)$$

and then solve (c) for $u_{1,1}$.

$$\begin{aligned} u_{1,1} &= (1 + \varepsilon_1)\{1 - A\}^{1/2} - 1 \\ A &= \sin^2\theta_{13} + \cos^2\theta_{13}\sin^2\theta_{12} \end{aligned} \qquad (10\text{–}23)$$

Applying the binomial expansion,

$$(1 - x)^{1/2} = 1 - \tfrac{1}{2}x(1 + \tfrac{1}{4}x + \cdots) \qquad (10\text{–}24)$$

to $(1 - A)^{1/2}$, we can write (10–23) as

$$u_{1,1} = \varepsilon_1\left\{1 - \frac{A}{2}\left(1 + \frac{1}{4}A + \cdots\right)\right\} - \frac{A}{2}\left(1 + \frac{1}{4}A + \cdots\right) \qquad (10\text{–}25)$$

In what follows, we assume *small* strain and express the derivatives and extensional strain (see Equation (d)) as

$$\begin{aligned} u_{3,1} &= 0(\theta_{13}) \qquad u_{2,1} = 0(\theta_{12}, \theta_{13}^2) \\ u_{1,1} &= 0(\varepsilon_1, \theta_{12}^2, \theta_{13}^2) \end{aligned} \qquad \text{(e)}$$

$$\varepsilon_1 \approx e_{11} = u_{1,1}(1 + 0(\varepsilon_1, \theta_{12}^2, \theta_{13}^2)) + 0(\theta_{12}^2, \theta_{13}^2) \qquad \text{(f)}$$

The various approximate theories are obtained by specializing (f).

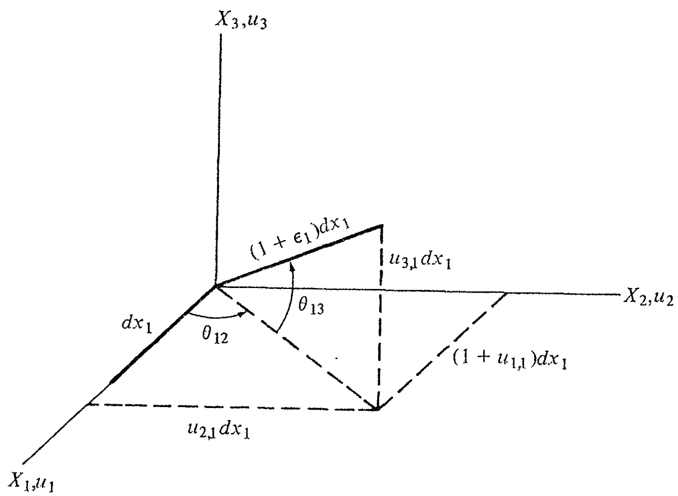


Fig. 10–4. Initial and deformed positions of a line element.

In the *linear geometric* case, the rotations are neglected with respect to strain. Formally, one sets $\theta_{12} = \theta_{13} = 0$ in (f) and the result is a linear relation between strain and displacement,

$$\varepsilon_1 \approx e_{11} \approx u_{1,1} \qquad \text{(g)}$$

Note that, according to this approximation, the *deformed* orientation coincides with the *initial* orientation. The general relations for the linear geometric case (small strain and infinitesimal rotation) are

$$\begin{aligned} \varepsilon_i &= e_{ii} = u_{i,i} \qquad \text{(no sum)} \\ \gamma_{ij} &= 2e_{ij} = u_{i,j} + u_{j,i} \end{aligned} \tag{10–26}$$

The next level of approximation is to consider θ^2 to be of the same order as strain.

$$\begin{aligned} \theta^2 &= 0(\varepsilon) \ll 1 \\ \sin\theta &\approx \theta \\ \cos\theta &\approx 1 \end{aligned} \tag{10–27}$$

We can neglect $u_{1,1}$ with respect to 1 in (f), but we must retain $u_{2,1}^2$ and $u_{3,1}^2$ since they are of $0(\theta^2)$.

$$\varepsilon_1 \approx e_{11} \approx u_{1,1} + \tfrac{1}{2}(u_{2,1}^2 + u_{3,1}^2) \tag{h}$$

The complete set of strain-displacement relations for small strain and *small-finite* rotation are listed below for reference.

$$\begin{aligned} \varepsilon_i &= e_{ii} = u_{i,i} + \tfrac{1}{2}(u_{j,i}^2 + u_{k,i}^2) \qquad \text{(no sum)} \\ \gamma_{ij} &= 2e_{ij} = u_{i,j} + u_{j,i} + u_{k,i}u_{k,j} \\ & i \neq j \neq k \end{aligned} \tag{10–28}$$

We utilize these expressions to develop a geometrically nonlinear formulation for a member in Chapter 18.

Lastly, if no restrictions are imposed on the magnitude of the rotations, one must use (10–21). The relations for *finite* rotation and small strain are

$$\begin{aligned} \varepsilon_i &= e_{ii} = u_{i,i} + \tfrac{1}{2}(u_{i,i}^2 + u_{j,i}^2 + u_{k,i}^2) \qquad \text{(no sum)} \\ \gamma_{ij} &= 2e_{ij} = u_{i,j}(1 + u_{i,i}) + u_{j,i}(1 + u_{j,j}) + u_{k,i}u_{k,j} \\ & i \neq j \neq k \end{aligned} \tag{10–29}$$

Note that the truss formulation presented in Chapter 6 allows for arbitrary magnitude of the rotations.

We have shown that linear strain-displacement relations are based on the following restrictions:

1. The strains are negligible with respect to unity, and
2. Products of the rotations are negligible with respect to the strains.

The first condition will always be satisfied for engineering materials such as metals, concrete, etc. Whether the second restriction is satisfied depends on the configuration of the body and the applied loading. If the body is massive in all three directions, the rotations are negligible with respect to the strains for an *arbitrary* loading. We have to include the nonlinear rotation terms in the strain displacement relations only if the body is thin (e.g., a thin plate or slender member) and the applied loading results in a significant change in the geometry. As an illustration, consider the simply supported member shown

in Fig. 10–5. We can neglect the change in geometry if only a transverse loading is applied (case 1). However, if both axial and transverse loads are applied (case 2), the change in geometry is no longer negligible and we must include the nonlinear rotation terms in the strain-displacement relations.

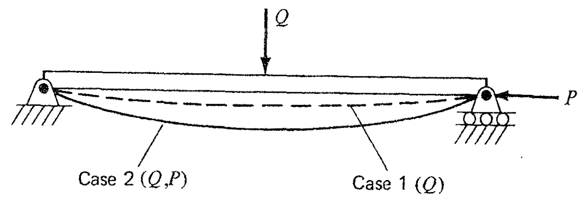


Fig. 10–5. Example of linear and geometrically nonlinear behavior.

To treat a geometrically nonlinear problem, we must work with the deformed geometry rather than the initial geometry. This can be defined by tracking the movement of a triad of line elements initially parallel to the global directions. We let $d\bar{r}_j$ be the initial set and $d\bar{\rho}_j$ the deformed set (see Fig. 10–6). By definition,

$$\begin{aligned} d\bar{r}_j &= dx_j\bar{\imath}_j \qquad \text{(no sum)} \\ d\bar{\rho}_j &= \bar{\rho}_{,j}\,dx_j \qquad \text{(no sum)} \\ |d\bar{\rho}_j| &= (1 + \varepsilon_j)dx_j \end{aligned} \tag{a}$$

The unit vector pointing in the direction of $d\bar{\rho}_j$ is denoted by $\bar{v}_j$. Using (a), we can write

$$\begin{aligned} \bar{v}_j &= \frac{1}{1 + \varepsilon_j}\bar{\rho}_{,j} \\ &\approx \bar{\rho}_{,j} \qquad \text{for small strain} \end{aligned} \tag{b}$$

Finally, we express $\bar{v}_j$ in terms of the unit vectors for the initial frame.

$$\begin{aligned} \bar{v}_j &= \beta_{jk}\bar{\imath}_k \\ \beta_{jk} &= \frac{1}{1 + \varepsilon_j}(\delta_{jk} + u_{k,j}) \\ &\approx \delta_{jk} + u_{k,j} \qquad \text{for small strain} \end{aligned} \tag{10–30}$$

We will utilize (10–30) in the next section to establish the stress equilibrium equations for the geometrically nonlinear case.

Equations (10–30) reduce to

$$v_j \approx \bar{\imath}_j \tag{10–31}$$

for the *geometrically linear* case and to

$$\begin{aligned} \bar{v}_j &\approx \bar{\imath}_j + \beta_{jk}\bar{\imath}_k + \beta_{j\ell}\bar{\imath}_\ell \qquad \text{(no sum)} \\ & j \neq k \neq \ell \end{aligned} \tag{10–32}$$

for the case of small strain and *small-finite* rotations.

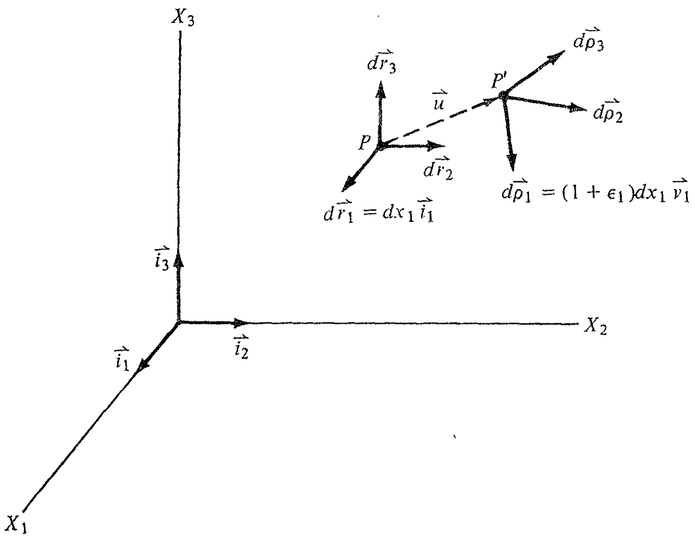


Fig. 10–6. Initial and deformed geometries.

10–4. ANALYSIS OF STRESS

The effects of the surroundings on a body such as contact pressure, gravitational attraction, etc., result in internal forces. In this section, we establish the *equilibrium* conditions for the internal forces in a body. This step is generally called the analysis of stress.

Consider a body subjected to some effect which results in internal forces. We pass a cutting plane through the deformed body and separate the two segments as shown in Fig. 10–7. We let m denote the *outward* normal direction for the internal face of body and refer to this face as the $+m$ face. In general, the subscript, m, is used for quantities associated with the $+m$ face. Now, we consider a differential area element ΔA_m, and let $\Delta \vec{F}_m$ be the *resultant* internal force vector acting on this element. The stress vector, $\vec{\sigma}_m$, is defined as

$$\vec{\sigma}_m = \lim_{\Delta A_m \to 0} \left(\frac{\Delta \vec{F}_m}{\Delta A_m}\right) \tag{10–33}$$

Note that $\vec{\sigma}_m$ has the units of force/area. Also, it depends on the *orientation* of the area element, i.e., on the direction of the outward normal. We do not allow for the possibility of the existence of a moment acting on a differential area element. One can include this effect by defining a couple-stress vector† in addition to a stress vector.

† See Ref. 6, p. 68.

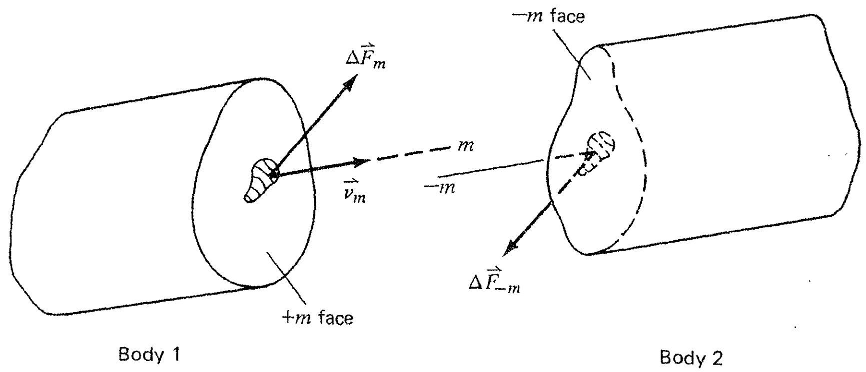
We consider next the corresponding area element in the $-m$ face. From Newton's law,

$$\Delta \vec{F}_{-m} = -\Delta \vec{F}_m \tag{a}$$

and it follows that

$$\vec{\sigma}_{-m} = -\vec{\sigma}_m \tag{10–34}$$

The stress vector has the same magnitude and line of action but it's sense is reversed.



Note: Deformed state

Fig. 10–7. Notation for internal force.

In order to analyze the state of stress at a point, say Q, we need an expression for the stress vector associated with an arbitrary plane through Q. With this objective, we consider the tetrahedron shown in Fig. 10–8. The orientation of the arbitrary plane is defined by q, the outward normal direction. The outward normals for the other three faces are parallel to the reference axes (X_j; $j = 1, 2, 3$). To simplify the notation, we use a subscript j for quantities associated with the X_j face, that is, the face whose outward normal points in the $+X_j$ direction. For example, we write

$$\begin{aligned} \vec{\sigma}_{X_j} &= \vec{\sigma}_j \\ \vec{\sigma}_{-X_j} &= \vec{\sigma}_{-j} = -\vec{\sigma}_j \\ \Delta A_{X_j} &= \Delta A_j \\ &\text{etc.} \end{aligned} \tag{10–35}$$

The force vectors acting at the centroids of the faces are shown in Fig. 10–8. The term $\Delta\vec{\sigma}_{()}$ represents the change in $\vec{\sigma}_{()}$ due to translation from Q to the centroid.

For equilibrium, the resultant force and moment vectors must vanish. In the limit (as $P \to Q$), the force system is concurrent and therefore we have to consider only the force equilibrium condition. From Fig. 10–8, we have

$$\vec{\sigma}_q + \Delta\vec{\sigma}_q = \frac{\Delta A_j}{\Delta A_q}(\vec{\sigma}_j + \Delta\vec{\sigma}_j) \qquad \text{(a)}$$

Now, ΔA_1 is the projection of ΔA_q on the X_2-X_3 plane. Noting that the projection of ΔA_q on a plane is equal to ΔA_q times the scalar product of $\vec{\imath}_q$ and the unit normal vector for the plane, and letting α_{qj} be the direction cosine for the q direction with respect to the X_j direction, we can write

$$\frac{\Delta A_j}{\Delta A_q} = \alpha_{qj} = \cos(q, X_j) = \vec{\imath}_q \cdot \vec{\imath}_j \qquad (10\text{–}36)$$

Finally, in the limit, Equation (a) reduces to

$$\vec{\sigma}_q = \alpha_{qj}\vec{\sigma}_j \qquad (10\text{–}37)$$

Once the stress vectors for three orthogonal planes at Q are known, we can determine the stress vector for an arbitrary plane through Q with (10–37).

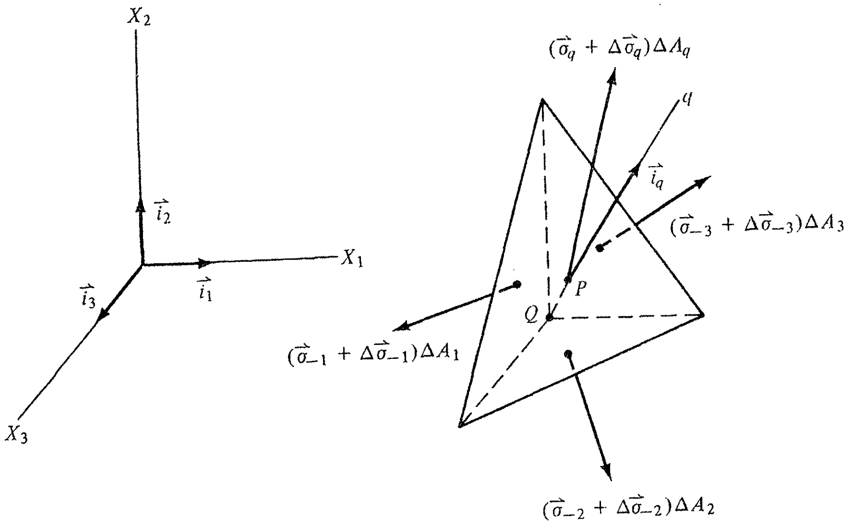


Fig. 10–8. Differential tetrahedral element.

Equation (10–37) is the transformation law for the *stress vector*. The component of $\vec{\sigma}_q$ in a particular direction is equal to the scalar product of $\vec{\sigma}_q$ and a unit vector pointing in the desired direction. Now, we express the stress vectors in terms of their components with respect to the coordinate axes X_j ($j = 1, 2, 3$).

$$\vec{\sigma}_j = \sigma_{jk}\vec{\imath}_k \qquad j = 1, 2, 3$$
$$\vec{\sigma}_q = \sigma_{qk}\vec{\imath}_k \qquad (10\text{–}38)$$

Note that the first subscript on a stress component always refers to the *face*, and the second to the *direction*. For example, σ_{12} acts on the X_1 face and points in the X_2 direction. The positive sense of the components for a negative face is *reversed* since $\vec{\sigma}_{-j} = -\vec{\sigma}_j$. The normal ($\sigma_{jj}$) and in-plane ($\sigma_{jk}$) components are generally called normal and shearing stresses. This notation is illustrated in Fig. 10–9.

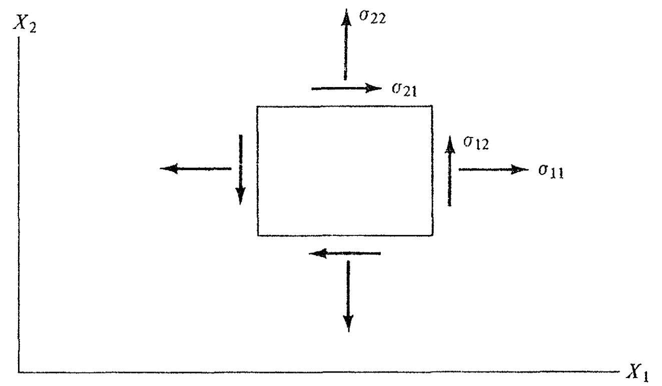


Fig. 10–9. Notation for stress components.

Substituting for the stress vectors in (10–37) results in

$$\sigma_{qk} = \alpha_{qj}\sigma_{jk} \qquad (10\text{–}39)$$

The component of $\vec{\sigma}_q$ with respect to an arbitrary direction, m, is determined from

$$\sigma_{qm} = \vec{\sigma}_q \cdot \vec{\imath}_m \qquad \text{(a)}$$

Letting

$$\vec{\imath}_m = \alpha_{mk}\vec{\imath}_k \qquad \text{(b)}$$

and noting (10–38), (a) expands to

$$\sigma_{qm} = \alpha_{qj}\alpha_{mk}\sigma_{jk} \qquad \text{(c)}$$

We generalize (c) for two orthogonal frames specified by the unit vectors $\vec{t}_j$, $\vec{t}'_j$ (see Fig. 10–3) where

$$\vec{t}_j = \vec{\imath}_j$$
$$\vec{t}'_j = \alpha_{jk}\vec{\imath}_k \qquad (10\text{–}40)$$

Defining σ'_{ij} as the component acting on the $\vec{t}'_i$ face in the $\vec{t}'_j$ direction and identifying $\vec{t}'_i$, $\vec{t}'_j$ with $\vec{\imath}_q$, $\vec{\imath}_m$, (c) takes the form

$$\sigma'_{ij} = \alpha_{ik}\alpha_{j\ell}\sigma_{k\ell} \qquad (10\text{–}41)$$

This result shows that the set (σ_{ij}) is a second-order cartesian tensor.

It remains to establish the equilibrium equations for a differential volume element. The equilibrium equations relate to the deformed state, i.e., we must consider a differential element on the *deformed* body. Since we have defined the stress components with respect to the global cartesian directions, it is natural to work with a rectangular parallelepiped having sides parallel to the global directions. This is shown in Fig. 10–10. Point 0 is at the centroid of the element.

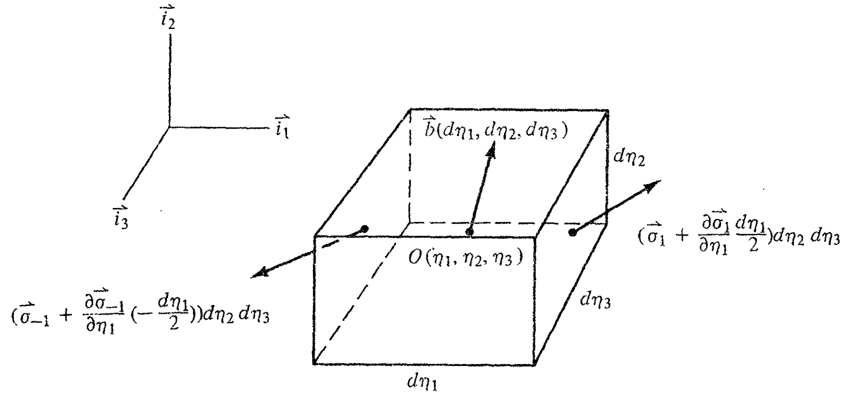


Fig. 10–10. Differential volume element in Eulerian representation.

The stress vectors are considered to be functions of the deformed† coordinates (η_i). We obtain the forces acting on the faces by expanding the stress vectors about 0 and retaining only the first two terms.‡ Letting $\vec{b}$ denote the external force per unit volume and enforcing the equilibrium conditions leads to

$$\sum \vec{F} = \vec{0} \Rightarrow \frac{\partial \vec{\sigma}_j}{\partial \eta_j} + \vec{b} = \vec{0} \qquad (10\text{–}42)$$

and

$$\sum \vec{M}_0 = \vec{0} \Rightarrow \vec{i}_j \times \vec{\sigma}_j = \vec{0} \qquad (10\text{–}43)$$

The scalar force equilibrium equations are obtained by expanding the vector equations using (10–38).

Force equilibrium $$\frac{\partial \sigma_{jk}}{\partial \eta_j} + b_k = 0 \qquad k = 1, 2, 3 \qquad (10\text{–}44)$$

Moment equilibrium $$\sigma_{jk} = \sigma_{kj} \qquad \begin{matrix} k \neq j \\ j, k = 1, 2, 3 \end{matrix} \qquad (10\text{–}45)$$

Moment equilibrium requires the shearing stress components to be symmetrical. Then, the stress tensor is symmetrical and there are only six independent stress measures for the three-dimensional case and three for the two-dimensional case.

† We are following the Eulerian approach here. Later we will shift back to the Lagrange approach.
‡ Second- and higher-order terms will vanish in the limit, i.e., when the element is shrunk to a point.

Equations (10–44) must be satisfied at each point in the interior of the body. Also, at the boundary, the stress components must equilibrate the applied surface forces.

We define $\vec{v}_n$ as the *outward* normal vector at a point on the *deformed* surface and write†

$$\vec{v}_n = \beta_{nj}\vec{i}_j \qquad (10\text{–}46)$$

The external force per unit deformed surface area is denoted by p_n.

$$\vec{p}_n = p_{nj}\vec{i}_j \qquad (10\text{–}47)$$

Applying (10–37) leads to the stress-boundary force-equilibrium relations:

$$\begin{gathered} \vec{p}_n = \beta_{nk}\vec{\sigma}_k \\ \Downarrow \\ p_{nj} = \beta_{nk}\sigma_{kj} \qquad j = 1, 2, 3 \end{gathered} \qquad (10\text{–}48)$$

When p_{nj} is prescribed, i.e., $p_{nj} = \bar{p}_{nj}$, (10–48) represent boundary conditions on the stress components. If u_j is prescribed, p_{nj} is a reaction.

Our derivation of strain-displacement relations employed the Lagrange approach, i.e., we considered the displacements (and strains) to be functions of the initial coordinates (x_i). The analysis of stress described above is based on the Eulerian approach, where the deformed coordinates are taken as the independent variables. This poses a problem since the strain and stress measures are referred to different volume elements. Figure 10–11 shows the initial and

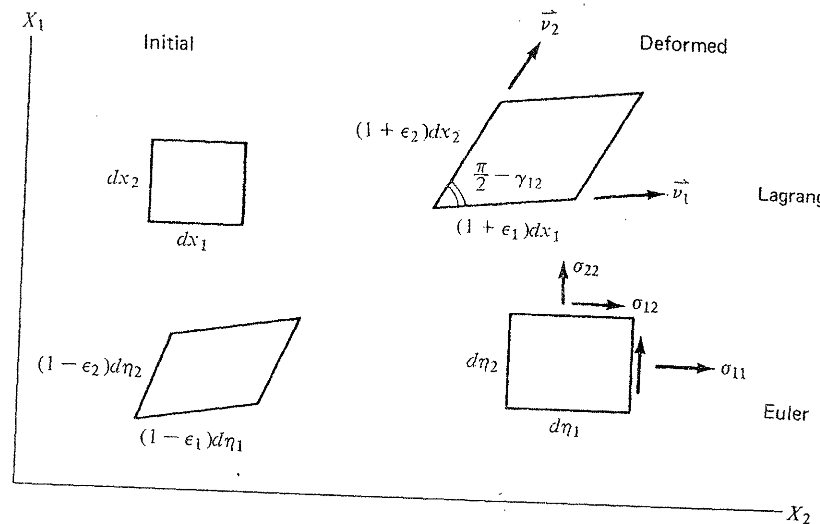


Fig. 10–11. Comparison of Eulerian and Lagrangian representations for a volume element.

†See Prob. 10–12.

deformed area elements corresponding to the two viewpoints. To be consistent with the Lagrange strains, we must work with a nonorthogonal parallelepiped whose sides are parallel to the deformed line elements in the analysis of stress. Conversely, to be consistent with the Eulerian stresses, we have to refer the strain measures to nonorthogonal directions in the initial state.

In the linear geometric case, we assume small strain and neglect the change in orientation due to rotation. The two approaches coalesce and we just have to replace η_j with x_j and β_{nk} with α_{nk} where α_{nk} is the direction cosine for the initial direction of the exterior normal. The *linear* equilibrium equations are:

$$\frac{\partial \sigma_{jk}}{\partial x_j} + b_k = 0 \qquad p_{nj} = \alpha_{nk}\sigma_{kj} \tag{10–49}$$

For the geometrically nonlinear case, we work with stress measures referred to the deformed directions (see Fig. 10–6) defined by the unit vectors, $\vec{v}_j$. We define $\vec{\sigma}_j^k$ as the stress vector per unit *initial* area acting on the face which initially is normal to the X_j direction, $\vec{b}^*$ as the force per unit *initial* volume, and $\vec{p}_n^*$ as the force per unit *initial* surface area. Figure 10–12 shows this notation for the two-dimensional case. The stress and force vectors are considered to be functions of the initial coordinates (x_i).

The equilibrium equations at an interior point are

$$\frac{\partial}{\partial x_j}\vec{\sigma}_j^k + \vec{b}^* = \vec{0} \qquad (1 + \varepsilon_j)\vec{v}_j \times \vec{\sigma}_j^k = \vec{0} \tag{10–50}$$

We express the body force and stress vectors as

$$\vec{b}^* = b_k^* \vec{i}_k \qquad \vec{\sigma}_j^k = \sigma_{ji}^k(1 + \varepsilon_i)\vec{v}_i \tag{10–51}$$

The set, σ_{ij}^k, is called the Kirchhoff stress tensor. Substituting for $\vec{v}_i$, using (10–30), results in the following scalar equations, which correspond to (10–44) and (10–45):

$$\frac{\partial}{\partial x_j}(\sigma_{j\ell}^k + \sigma_{ji}^k u_{\ell,i}) + b_\ell^* = 0 \qquad \ell = 1, 2, 3 \tag{10–52}$$

$$\sigma_{j\ell}^k = \sigma_{\ell j}^k \qquad j \neq \ell;\ j, \ell = 1, 2, 3 \tag{10–53}$$

The boundary equilibrium equations are obtained by expanding

$$\vec{p}_n^* = \alpha_{n\ell}\vec{\sigma}_\ell^k = p_{nj}^* \vec{i}_j \tag{10–54}$$

and have the form

$$p_{nj}^* = \alpha_{n\ell}(\sigma_{\ell j}^k + \sigma_{\ell i}^k u_{j,i}) \tag{10–55}$$

These equations apply for arbitrary strain and finite rotation. For small strain, we neglect the change in dimensions and shape of the volume element. This assumption is introduced by taking

$$b^* \approx b \qquad p^* \approx p \qquad \vec{\sigma}_j^k \approx \sigma_{ji}^k \vec{v}_i \tag{10–56}$$

Since the deformed unit vectors are orthogonal (to $\varepsilon \ll 1$), the Kirchhoff stresses σ_{ij}^k now comprise a second-order cartesian tensor and they transform according

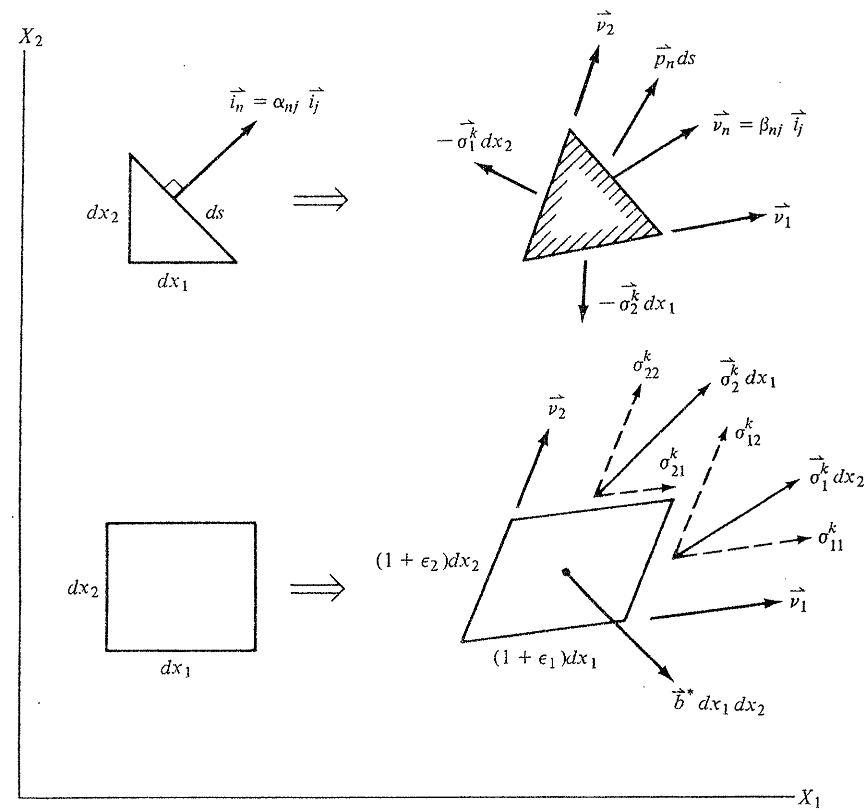


Fig. 10–12. Definition of stress components in Lagrangian representation.

to (10–41). The equations simplify somewhat if we assume *small-finite* rotation.† For infinitesimal rotation (linear geometry), $\sigma_{j\ell}^k \approx \sigma_{j\ell}$, $\vec{v}_j \approx \vec{i}_j$, and the equations reduce to (10–49), (10–50).

In what follows, we will work with the Kirchhoff stress components to keep the treatment general. However, we will assume small strain.

† See Prob. 10–16.

10–5. ELASTIC STRESS-STRAIN RELATIONS

A body is said to be elastic if it returns to its initial dimensions and shape when the applied forces are removed. The work done during the deformation process is independent of the order in which the body is deformed. We treat first an arbitrary elastic material and then specialize the results for a linearly elastic material.

Our starting point is the first law of thermodynamics:

$$\delta W = \delta V_T + \delta Q \tag{a}$$

where δW = first-order work done by the forces acting on the body
δV_T = first-order change in the total strain energy (also called internal energy)
δQ = first-order change in the total heat content.

When the deformation process is isothermal or adiabatic, $\delta Q = 0$, and (a) reduces to

$$\delta W = \delta V_T \tag{b}$$

Now, we apply (b) to a differential volume element in the deformed state (see, e.g., Fig. 10–12). We define V as the strain energy per unit initial volume. In general, V is a function of the deformation measures.

$$V = V(e_{ij}) = V(\varepsilon_1, \ldots, \gamma_{12}, \ldots) \tag{10–57}$$

The material is said to be hyperelastic (Green-type) when V is a continuous function. This requires

$$\frac{\partial^2 V}{\partial e_{k\ell}\,\partial e_{ij}} = \frac{\partial^2 V}{\partial e_{ij}\,\partial e_{k\ell}} \tag{10–58}$$

By definition,

$$\begin{aligned} \delta V_T &= \delta V(dx_1\, dx_2\, dx_3) \\ \delta V &= \frac{\partial V}{\partial e_{ij}}\,\delta e_{ij} \end{aligned} \tag{10–59}$$

where δe_{ij} is the first-order change† in e_{ij} due to an incremental displacement, $\Delta\vec{u}$. Also, one can show that the first order work done by the force vectors acting on the element is‡

$$\begin{aligned} \delta W &= ((\vec{\sigma}_1^k \cdot \Delta\vec{u})_{,1} + (\vec{\sigma}_2^k \cdot \Delta\vec{u})_{,2} + (\vec{\sigma}_3^k \cdot \Delta\vec{u})_{,3} + \vec{b} \cdot \Delta\vec{u})dx_1\, dx_2\, dx_3 \\ &= (\sigma_{ij}^k\, \delta e_{ij})dx_1\, dx_2\, dx_3 \end{aligned} \tag{10–60}$$

Equating δV_T and δW leads to the general form of the stress-strain relation for a Green-type material,

$$\sigma_{ij}^k = \frac{\partial V}{\partial e_{ij}} \tag{10–61}$$

† See Prob. 10–11.
‡ See Prob. 10–18. The forces are in equilibrium, ie, they satisfy (10–50).

This definition applies for arbitrary strain. Once V is specified, we can obtain expressions for the stresses in terms of the strains by differentiating V. Since V is continuous, the stress-strain relations must satisfy (10–58), which requires

$$\frac{\partial \sigma_{ij}^k}{\partial e_{mn}} = \frac{\partial \sigma_{mn}^k}{\partial e_{ij}} \tag{10–62}$$

In what follows, we restrict the discussion to small strain and a linearly elastic material, i.e., where the stress-strain relations are linear. We also shift from indicial notation to matrix notation, which is more convenient for this phase.

We list the stress and strain components in column matrices and drop the superscript k on the Kirchhoff stress components:

$$\begin{aligned} \boldsymbol{\sigma} &= \{\sigma_{11}^k, \sigma_{22}^k, \sigma_{33}^k, \sigma_{12}^k, \sigma_{23}^k, \sigma_{31}^k\} \\ \boldsymbol{\varepsilon} &= \{e_{11}, e_{22}, e_{33}, 2e_{12}, 2e_{23}, 2e_{31}\} \\ &= \{\varepsilon_1 \varepsilon_2 \varepsilon_3 \gamma_{12} \gamma_{23} \gamma_{31}\} \end{aligned} \tag{10–63}$$

With this notation,

$$\delta V = \sigma_{ij}^k\, \delta e_{ij} \Rightarrow \boldsymbol{\sigma}^T\, \delta\boldsymbol{\varepsilon} \tag{10–64}$$

The matrix transformation laws are†

$$\begin{aligned} \boldsymbol{\sigma}' &= \mathbf{T}_\sigma \boldsymbol{\sigma} \\ \boldsymbol{\varepsilon}' &= \mathbf{T}_\varepsilon \boldsymbol{\varepsilon} \end{aligned} \tag{10–65}$$

Since δV is invariant under a transformation of reference frames, the transformation matrices are related by

$$(\mathbf{T}_\sigma)^T \mathbf{T}_\varepsilon = \mathbf{I} \tag{10–66}$$

The total strain, $\boldsymbol{\varepsilon}$, is expressed as

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^0 + \mathbf{A}\boldsymbol{\sigma} \tag{10–67}$$

where $\boldsymbol{\varepsilon}^0$ contains the initial strains not associated with stress, e.g., strain due to a temperature increment, and $\mathbf{A}$ is called the material compliance matrix. We write the inverted relations as

$$\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^0) \tag{10–68}$$

where $\mathbf{D} = \mathbf{A}^{-1}$ is the material rigidity matrix. Equation (10–62) requires $\mathbf{D}$ (and $\mathbf{A}$) to be symmetrical. The elements of $\mathbf{A}$ are determined from material tests, and $\mathbf{D}$ is generated by inverting $\mathbf{A}$. Substituting for $\boldsymbol{\sigma}$ in (10–64), we obtain the form of the strain energy density for the linear case,

$$V = \tfrac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^0)^T \mathbf{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^0) \tag{10–69}$$

Since $V > 0$ for arbitrary $(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^0)$, $\mathbf{D}$ and $\mathbf{A}$ are positive definite matrices.

There are 21 material constants for a linearly elastic Green-type material. The number of independent constants is reduced if the material structure

† See Prob. 10–6, 10–13.

exhibits symmetry.† In what follows, we describe the transition from an anisotropic material to an isotropic material.

A material whose structure has three orthogonal axes of symmetry is called orthotropic. The structure of an orthotropic material appears identical after a 180° rotation about a symmetry axis. To determine the number of independent constants for this case, we suppose X_1, X_2, X_3 are axes of symmetry and consider a 180° rotation about X_2. We use a prime superscript to indicate the rotated axes. From Fig. 10–13,

$$\begin{aligned} X_1' &= -X_1 \\ X_3' &= -X_3 \\ X_2' &= X_2 \end{aligned} \tag{a}$$

The stress and deformation quantities are related by (we replace 1 by −1 and 3 by −3 in the shear terms)

$$\begin{array}{lll} \sigma_{ii}' = \sigma_{ii} & \varepsilon_i' = \varepsilon_i & i = 1, 2, 3 \\ \sigma_{12}' = -\sigma_{12} & \sigma_{23}' = -\sigma_{23} & \sigma_{13}' = \sigma_{13} \\ \gamma_{12}' = -\gamma_{12} & \gamma_{23}' = -\gamma_{23} & \gamma_{13}' = \gamma_{13} \end{array} \tag{b}$$

Now, the stress-strain relations must be identical in form. We expand $\boldsymbol{\varepsilon} = \mathbf{A}\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}' = \mathbf{A}\boldsymbol{\sigma}'$, and substitute for $\boldsymbol{\sigma}'$ using (b). Equating the expressions for ε_i

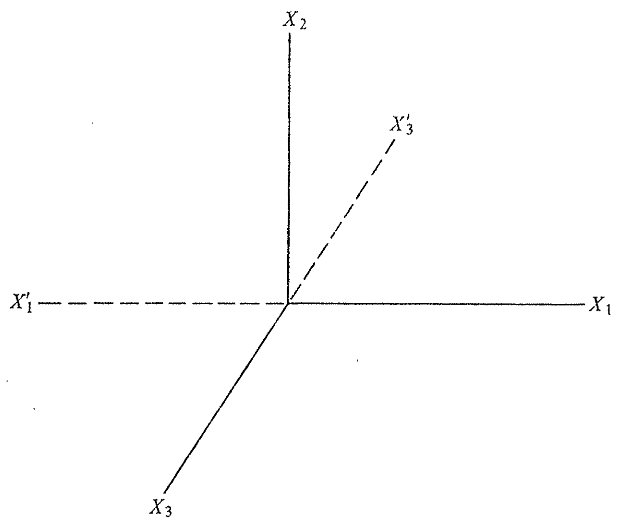


Fig. 10–13. Rotation of axes for symmetry with respect to the X_2-X_3 plane.

† A material whose structure has no symmetry is said to be anisotropic.

and ε_i' leads to the following relations between the elements of $\mathbf{A}$,

$$\begin{aligned} a_{14}\sigma_{12} + a_{15}\sigma_{23} &= -a_{14}\sigma_{12} - a_{15}\sigma_{23} \\ a_{24}\sigma_{12} + a_{25}\sigma_{23} &= -a_{24}\sigma_{12} - a_{25}\sigma_{23} \\ a_{34}\sigma_{12} + a_{35}\sigma_{23} &= -a_{34}\sigma_{12} - a_{35}\sigma_{23} \end{aligned} \tag{c}$$

For (c) to be satisfied, the coefficients must vanish identically. This requires

$$\begin{aligned} a_{14} &= a_{15} = 0 \\ a_{24} &= a_{25} = 0 \\ a_{34} &= a_{35} = 0 \end{aligned} \tag{d}$$

We consider next the expansions for γ_{ij}'. The symmetry conditions require $a_{46} = a_{56} = 0$. By rotating 180° about X_1, we find

$$a_{16} = a_{26} = a_{36} = a_{45} = 0 \tag{e}$$

A rotation about the X_3 axis will not result in any additional conditions. Finally, when the strains are referred to the *structural* symmetry axes, the stress-strain relations for an orthotropic material reduce to

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & & & \\ a_{12} & a_{22} & a_{23} & & \mathbf{0} & \\ a_{13} & a_{23} & a_{33} & & & \\ \hline & & & a_{44} & 0 & 0 \\ & \mathbf{0} & & 0 & a_{55} & 0 \\ & & & 0 & 0 & a_{66} \end{array}\right] \begin{Bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} \tag{10–70}$$

We see that $\mathbf{A}$ is quasi-diagonal and involves 9 independent constants. There is no interaction between extension and shear. Also, the shearing effect is uncoupled, i.e., σ_{12} leads only to γ_{12}.

An alternate form of the orthotropic stress-strain relations is

$$\begin{aligned} \varepsilon_1 &= \mu_1\,\Delta T + \frac{1}{E_1}\sigma_{11} - \frac{\nu_{21}}{E_2}\sigma_{22} - \frac{\nu_{31}}{E_3}\sigma_{33} \\ \varepsilon_2 &= \mu_2\,\Delta T + \frac{1}{E_2}\sigma_{22} - \frac{\nu_{12}}{E_1}\sigma_{11} - \frac{\nu_{32}}{E_3}\sigma_{33} \\ \varepsilon_3 &= \mu_3\,\Delta T + \frac{1}{E_3}\sigma_{33} - \frac{\nu_{13}}{E_1}\sigma_{11} - \frac{\nu_{23}}{E_2}\sigma_{22} \\ \gamma_{12} &= \frac{1}{G_{12}}\sigma_{12} \qquad \gamma_{23} = \frac{1}{G_{23}}\sigma_{23} \qquad \gamma_{31} = \frac{1}{G_{31}}\sigma_{31} \end{aligned} \tag{10–71}$$

where E_i are extensional moduli, G_{ij} are shear moduli, ν_{jk} are coupling coefficients, and ΔT is the temperature increment. The coupling terms are related by

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1} \qquad \frac{\nu_{31}}{E_3} = \frac{\nu_{13}}{E_1} \qquad \frac{\nu_{32}}{E_3} = \frac{\nu_{23}}{E_2} \tag{10–72}$$

It is relatively straightforward to invert these relations.† One should note that (10–71) apply only when X_i coincide with the material symmetry directions.‡

If the stress-strain relations are invariant for arbitrary directions in a plane, the material is said to be transversely orthotropic or isotropic with respect to the plane. We consider the case where the X_1 direction is the preferred direction, i.e., where the material is isotropic with respect to the X_2-X_3 plane. By definition, **A** is invariant when we transform from X_1-X_2-X_3 to X_1-X'_2-X'_3. This requires§

$$E_2 = E_3 \equiv E \qquad G_{12} = G_{31} \equiv G \qquad \frac{\nu_{21}}{E_2} = \frac{\nu_{31}}{E_3} \equiv \frac{\nu_1}{E}$$
$$\frac{\nu_{32}}{E_3} = \frac{\nu_{23}}{E_2} \equiv \frac{\nu}{E} \qquad \frac{1}{G_{23}} = \frac{2(1+\nu)}{E} \qquad \mu_2 = \mu_3 \equiv \mu \tag{10–73}$$

and the relations reduce to

$$\varepsilon_1 = \mu_1 \Delta T + \frac{1}{E_1}\sigma_{11} - \frac{\nu_1}{E}(\sigma_{22} + \sigma_{33})$$
$$\varepsilon_2 = \mu \Delta T + \frac{1}{E}(\sigma_{22} - \nu_1\sigma_{11} - \nu\sigma_{33}) \tag{10–74}$$
$$\varepsilon_3 = \mu \Delta T + \frac{1}{E}(\sigma_{33} - \nu_1\sigma_{11} - \nu\sigma_{22})$$
$$\gamma_{12} = \frac{1}{G_1}\sigma_{12} \qquad \gamma_{31} = \frac{1}{G_1}\sigma_{31} \qquad \gamma_{23} = \frac{2(1+\nu)}{E}\sigma_{23}$$

There are five independent constants (E, ν, E_1, ν_1, G_1).

Lastly, the material is called isotropic when the stress-strain relations are invariant for arbitrary directions, X'_1-X'_2-X'_3. For this case, $\mathbf{A} = \mathbf{A}'$ for arbitrary X'_1-X'_2-X'_3. The relations are obtained by specializing (10–74):

$$\varepsilon_i = \mu \Delta T + \frac{1}{E}(\sigma_{ii} - \nu(\sigma_{jj} + \sigma_{kk}))$$
$$\gamma_{ij} = \frac{2(1+\nu)}{E}\sigma_{ij} \tag{10–75}$$

Note that now there are only two independent constants (E, ν). The coupling coefficient, ν, is called Poisson's ratio.

The inverted form of (10–75) is written as

$$\sigma_{ii} = \sigma^0 + (\lambda + 2G)\varepsilon_i + \lambda(\varepsilon_j + \varepsilon_k)$$
$$\sigma_{ij} = G\gamma_{ij} \tag{10–76}$$
$$\sigma^0 = -(3\lambda + 2G)\mu \Delta T$$

† See Prob. 10–19 for the inverted form of (10–71).
‡ See Prob. 10–21.
§ See Prob. 10–22.

where λ, G are called Lamé constants and are related to E, ν by

$$G = \text{shear modulus} = \frac{E}{2(1+\nu)}$$
$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{10–77}$$

Since **D** must be positive definite, ν is restricted to $-1 < \nu < 1/2$. The limiting case where $\nu = +1/2$ is discussed in Problem 10–24.

10–6. PRINCIPLE OF VIRTUAL DISPLACEMENTS; PRINCIPLE OF STATIONARY POTENTIAL ENERGY; CLASSICAL STABILITY CRITERIA

Chapter 7 dealt with variational principles for an ideal truss. For completeness, we derive here the 3-dimensional form of the principle of virtual displacements, principle of stationary potential energy, and the classical stability criterion. The principle of virtual forces and stationary complementary energy are treated in the next section.

The principle of virtual displacements states that the first-order work done by the external forces (δW_E) is equal to the first order work done by the internal forces (δW_D) acting on the restraints for an arbitrary virtual displacement of the body from an equilibrium position.† In the continuous case, the external loading consists of body ($\bar{b}$) and surface ($\bar{p}$) loads and the internal forces are represented by the stress vectors.

We follow the Lagrange approach,‡ i.e., we work with Lagrange finite strain components (e_{jk}), Kirchhoff stresses ($\bar{\sigma}^k$), and external force measures per unit initial volume or area (b^*, $\bar{p}^*$). This is consistent with our derivation of the equilibrium equations. Let Δu denote the virtual displacement. The first-order external work is

$$\delta W_E = \iiint \bar{b}^* \cdot \Delta\bar{u}\, dx_1\, dx_2\, dx_3 + \iint \bar{p}^*_n \cdot \Delta\bar{u}\, d\Omega$$
$$= \iiint b^*_i\, \Delta u_i\, dx_1\, dx_2\, dx_3 + \iint p^*_{ni}\, \Delta u_i\, d\Omega \tag{10–78}$$

where Ω is the initial surface area. The total internal deformation work is obtained by summing the first-order work done by the stress vectors acting on a differential volume element.§

$$\delta W_D = \iiint \bar{\sigma}^k_j \cdot \Delta u_{,j}\, dx_1\, dx_2\, dx_3$$
$$= \iiint \sigma^k_{ij}\, \delta e_{ij}\, dx_1\, dx_2\, dx_3 \tag{10–79}$$

Equating (a) and (b), we obtain the 3-dimensional form of the principle of

† See Sec. 7–2.
‡ See Fig. 10–12.
§ See (10–60).

virtual displacements,

$$\delta W_D = \delta W_E$$
$$\Downarrow$$
$$\iiint \bar{\sigma}_j^k \cdot \Delta\bar{u}_{,j}\, dx_1\, dx_2\, dx_3 = \iiint \bar{b}^* \cdot \Delta\bar{u}\, dx_1\, dx_2\, dx_3 + \iint \bar{p}_n^* \cdot \Delta\bar{u}\, d\Omega \quad (10\text{–}80)$$
$$\Downarrow$$
$$\iiint \sigma_{ij}^k\, \delta e_{ij}\, dx_1\, dx_2\, dx_3 = \iiint b_i^*\, \Delta u_i\, dx_1\, dx_2\, dx_3 + \iint p_{ni}^*\, \Delta u_i\, d\Omega$$

Requiring (10–80) to be satisfied for arbitrary (continuous) $\Delta\bar{u}$ is equivalent to enforcing the equilibrium equations.

To show this, we work with the vector form and utilize the following *integration by parts* formula:†

$$\iiint \bar{f} \cdot \frac{\partial \bar{g}}{\partial x_j}\, dx_1\, dx_2\, dx_3 = \iint \bar{f} \cdot \bar{g}\alpha_{nj}\, d\Omega - \iiint \bar{g} \cdot \frac{\partial \bar{f}}{\partial x_j}\, dx_1\, dx_2\, dx_3 \quad (10\text{–}81)$$

where α_{nj} is the direction cosine for the initial *outward* normal (n) with respect to the X_j direction. Operating on the left-hand term and equating coefficients of $\Delta\bar{u}$ in the volume and surface integrals leads directly to (10–50) and (10–54).

The principle of virtual displacements applies for arbitrary loading (static or dynamic) and material behavior. When the behavior is elastic and the loading is independent of time, it can be interpreted as a variational principle for the displacements. The essential steps required for the truss formulation are described in Sec. 7–4. Their extension to a continuous body is straightforward.

When the behavior is elastic,

$$\sigma_{ij}^k = \frac{\partial V}{\partial e_{ij}} \quad \text{(a)}$$

Letting V_T denote the total strain energy, the left-hand side of (10–80) reduces to

$$\delta W_D = \iiint \sigma_{ij}^k\, \delta e_{ij}\, dx_1\, dx_2\, dx_3 \Rightarrow \iiint \delta V\, dx_1\, dx_2\, dx_3 = \delta V_T \quad \text{(b)}$$

We consider the surface area to consist of 2 zones as shown in Fig. 10–14.

$$\Omega = \Omega_d + \Omega_\sigma$$

where displacements are prescribed on Ω_d,

$$u_i = \bar{u}_i \qquad \text{on } \Omega_d \quad (10\text{–}82)$$

and surface force intensities are prescribed on Ω_σ,

$$p_{ni} = \bar{p}_{ni} \qquad \text{on } \Omega_\sigma$$

The displacement variation, Δu_i, is admissible if it is continuous and satisfies

$$\Delta u_i = 0 \qquad \text{on } \Omega_d \quad (10\text{–}83)$$

We also consider the surface and body forces to be independent of the displacements. With these definitions, the principle of virtual displacements is trans-

† See Prob. 10–25.

formed to

$$\begin{aligned} \delta\Pi_p &= 0 \qquad \text{for arbitrary admissible } \Delta u_i \\ \Pi_p &= V_T - \iiint \bar{b}_i^* u_i\, dx_1\, dx_2\, dx_3 - \iint_{\Omega_\sigma} \bar{p}_{ni}^*\, u_i\, d\Omega \end{aligned} \quad (10\text{–}84)$$

where Π_p is the total potential energy functional. According to (10–84), the displacements defining an equilibrium position correspond to a stationary value of the total potential energy functional. Note that this result applies for arbitrary strain and finite rotations. The only restrictions are elastic behavior and static loading.

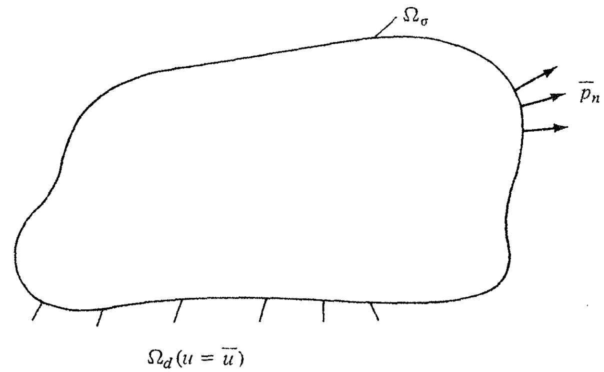


Fig. 10–14. Classification of boundary zones.

Example 10–2

Direct methods of variational calculus such as Rayleigh-Ritz, Galerkin, weighted residuals, and others are applied to Π_p to determine approximate solutions for the displacements. In the Rayleigh-Ritz method, one expresses the displacements in terms of unknown parameters, q, and prescribed functions, $\phi(x_1, x_2, x_3)$,

$$u_i = u_i^0 + \sum_{j=1}^{N} q_j^{(i)} \phi_j^{(i)} \quad \text{(a)}$$

where

$$\left.\begin{aligned} u_i^0 &= \bar{u}_i \\ \phi_j^{(i)} &= 0 \qquad \text{for } j = 1, 2, \ldots, N \end{aligned}\right\} \text{ on } \Omega_d \quad \text{(b)}$$

The displacement boundary conditions on Ω_d are called "essential" boundary conditions. Substituting for u_i transforms Π_p to a function of the q's. When the material is linearly elastic, V is a quadratic function of the strains. Then, V will involve up to fourth-degree terms for the geometrically nonlinear case. If the behavior is completely *linear*, Π_p reduces to:

$$\begin{aligned} \Pi_p &= \text{Const.} + \mathbf{q}^T\mathbf{Q} + \tfrac{1}{2}\mathbf{q}^T\mathbf{K}\mathbf{q} \\ \mathbf{q} &= \{q_1^{(1)} q_2^{(1)} \cdots q_N^{(1)} q_1^{(2)} \cdots q_N^{(3)}\}\ (3N \times 1) \\ &\mathbf{K} \text{ is symmetrical} \end{aligned} \quad \text{(c)}$$

Finally, requiring Π_p to be stationary for arbitrary $\delta\mathbf{q}$ leads (for linear behavior) to

$$\mathbf{Kq} = \mathbf{Q} \tag{d}$$

The strains are evaluated by operating on (a) and the stresses are determined from the stress-strain relations.

Polynomials and trigonometric functions are generally used to construct the spatial distribution functions. The mathematical basis for direct methods is treated in numerous texts (see Refs. 9, 10).

The "classical" stability criterion for a stable equilibrium position is†

$$\delta^2 W_D - \delta^2 W_E > 0 \qquad \text{for arbitrary } \Delta\bar{u}$$

where $\delta^2 W_E = \delta(\delta W_E)$ is the second-order work done by the external forces during the incremental displacement, $\Delta\bar{u}$, and $\delta^2 W_D = \delta(\delta W_D)$ is the second-order work done by the internal forces acting on the restraints during the incremental deformation resulting from $\Delta\bar{u}$. The form of the work terms for a continuous body are obtained by operating on (10–78) and (10–79):‡

$$\begin{aligned}
\delta^2 W_E &= \iiint \delta\bar{b}^* \cdot \Delta\bar{u}\, dx_1\, dx_2\, dx_3 + \iint \delta\bar{p}^* \cdot \Delta\bar{u}\, d\Omega \\
&= \iiint \delta b_i^*\, \Delta u_i\, dx_1\, dx_2\, dx_3 + \iint \delta p_i^*\, \Delta u_i\, d\Omega \\
\delta^2 W_D &= \iiint \delta\bar{\sigma}^k \cdot \Delta\bar{u}_{,j}\, dx_1\, dx_2\, dx_3 \\
&= \iiint (\delta\sigma_{ij}^k\, \delta e_{ij} + \sigma_{ij}^k\, \delta^2 e_{ij}) dx_1\, dx_2\, dx_3
\end{aligned} \tag{10–85}$$

If $\delta^2 W_D = \delta^2 W_E$ for a particular $\Delta\bar{u}$, the equilibrium position is neutral. The position is unstable if $\delta^2 W_D < \delta^2 W_E$. Note that $\delta\bar{b}$, $\delta\bar{p}$ are *null* vectors when the forces are prescribed.

For elastic behavior, the incremental deformation work is equal to the increment in strain energy ($\delta W_D = \delta V_T$), and (10–84) can be written as

$$\delta^2\Pi_p = \delta(\delta\Pi_p) > 0 \qquad \text{for arbitrary } \Delta\bar{u} \tag{10–86}$$

It follows that a stable equilibrium position corresponds to a relative minimum value of the total potential energy. Bifurcation (neutral equilibrium) occurs when $\delta^2\Pi_p = 0$ for some $\Delta\bar{u}$, say $\Delta\bar{u}_B$. If the loading is prescribed, $\delta^2\Pi_p = \delta^2 V_T$, and $\delta^2 V_T = 0$ at bifurcation.

The governing equations for bifurcation can be obtained by expanding $\delta^2 W_D = \delta^2 W_E$. This involves transforming the integrand of $\delta^2 W_D$ by applying (10–81). Since bifurcation corresponds to the existence of an alternate equilibrium position, it is more convenient to form the incremental equations directly. The equations for the case of linearly elastic material and prescribed external forces are listed below.

† See Sec. 7–6 for a derivation of the classical stability criterion.
‡ See Probs. 10–11, 10–18.

1. ***Equilibrium Equation in the Interior***

$$\frac{\partial}{\partial x_j}(\delta\sigma_{j\ell}^k + u_{\ell,i}\,\delta\sigma_{ji}^k + \sigma_{ji}^k\,\Delta u_{\ell,i}) = 0 \qquad \ell = 1, 2, 3$$

2. ***Stress-Boundary Force Equations on Ω_σ***

$$\alpha_{n\ell}(\delta\sigma_{\ell j}^k + u_{j,i}\,\delta\sigma_{\ell i}^k + \sigma_{\ell i}^k\,\Delta u_{j,i}) = 0 \qquad j = 1, 2, 3 \tag{10–87}$$

3. ***Stress-Strain Relations***

$$\delta\boldsymbol{\sigma} = \mathbf{D}\,\delta\boldsymbol{\varepsilon}$$

4. ***Strain-Displacement Relations***

$$\begin{aligned}
\delta e_{ij} &= \tfrac{1}{2}(\Delta u_{i,j} + \Delta u_{j,i} + u_{m,i}\,\Delta u_{m,j} + u_{m,j}\,\Delta_{m,i}) \\
\Delta u_i &= 0 \qquad \text{on } \Omega_d
\end{aligned}$$

10–7. PRINCIPLE OF VIRTUAL FORCES; PRINCIPLE OF STATIONARY COMPLEMENTARY ENERGY

Let u_i be the actual displacements in a body due to some loading and e_{ij} the geometrically linear strain measures corresponding to u_i. The strain and displacement measures are related by

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{a}$$

Also,

$$u_i = \bar{u} \qquad \text{on } \Omega_d \tag{b}$$

Once the strains are known, we can find the displacements by solving (a) and enforcing (b). The principle of virtual forces is basically a procedure for determining displacements without having to operate on (a). It applies only for *linear* geometry. We developed its form for an ideal truss in Sec. 7–3. We will follow the same approach here to establish the three-dimensional form.

The essential step involves selecting a statically permissible force system, i.e., a force system which satisfies the linear equilibrium equations. For the continuous case, the force system consists of stresses, $\Delta\sigma_{ij}$; surface forces, $\Delta\bar{p}_{ni}$, on Ω_σ; and reactions, Δp_{ni}, on Ω_d. Static permissibility requires

$$\begin{aligned}
\Delta\sigma_{ji,j} &= 0 \\
\Delta\bar{p}_{ni} &= \alpha_{nj}\,\Delta\sigma_{ji} \qquad \text{on } \Omega_\sigma \\
\Delta p_{ni} &= \alpha_{nj}\,\Delta\sigma_{ji} \qquad \text{on } \Omega_d
\end{aligned} \tag{10–88}$$

If we multiply e_{ij} by $\Delta\sigma_{ij}$, integrate over the volume using (10–81), and note the static relations, we obtain the following identity,†

$$\iiint e_{ij}\,\Delta\sigma_{ij}\, dx_1\, dx_2\, dx_3 = \iint_{\Omega_\sigma} u_i\,\Delta\bar{p}_{ni}\, d\Omega + \int_{\Omega_d} \bar{u}_i\,\Delta p_{ni}\, d\Omega \tag{10–89}$$

† See Prob. 10–26.

which is referred to as the principle of virtual forces (or stresses). This result is applicable for arbitrary material behavior. However, the geometry must be linear.

Suppose the translation at a point Q on Ω_σ in the direction defined by $\bar{\imath}_q$ is desired (see Fig. 10–15). Let d_Q be the displacement. We apply a unit force at Q in the $\bar{\imath}_q$ direction and generate a statically permissible stress field.

$$(1)\ \bar{\imath}_q \text{ at point } Q \Rightarrow \Delta\sigma_{ij}^{(q)} \quad \text{and} \quad \Delta p_{ni}^{(q)}$$

The integral on Ω_σ reduces to $(1)d_Q$, and it follows that

$$d_Q = \iiint e_{ij}\, \Delta\sigma_{ij}^{(q)}\, dx_1\, dx_2\, dx_3 - \iint_{\Omega_d} \bar{u}_i\, \Delta p_{ni}^{(q)}\, d\Omega \qquad (10\text{–}90)$$

A second application is in the force method, where one reduces the governing equations (stress equilibrium and stress displacement) to a set of equations

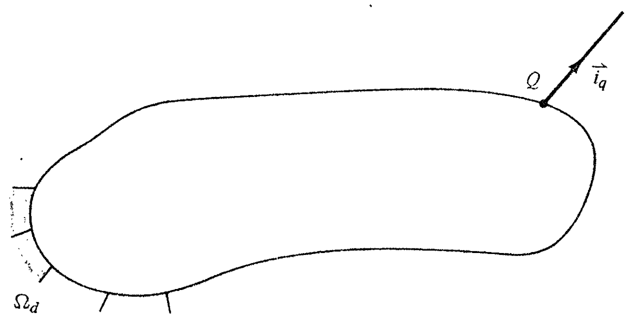


Fig. 10–15. Notation for determination of the translation at point Q.

involving only force unknowns. We start by expressing the stress field in terms of a prescribed distribution (σ^0) and a "corrective" field (σ^c),

$$\sigma_{ij} = \sigma_{ij}^0 + \sigma_{ij}^c \qquad (10\text{–}91)$$

where σ_{ij}^0 is a particular solution of the equilibrium equations which satisfies the boundary conditions on Ω_σ,

$$\begin{aligned} \sigma_{ji,j}^0 + b_i &= 0 \\ \alpha_{nj}\sigma_{ji}^0 &= \bar{p}_{ni} \qquad \text{on } \Omega_\sigma \end{aligned} \qquad (10\text{–}92)$$

and σ_{ij}^c satisfies

$$\begin{aligned} \sigma_{ji,j}^c &= 0 \\ \alpha_{nj}\sigma_{ji}^c &= 0 \qquad \text{on } \Omega_\sigma \\ \alpha_{nj}\sigma_{ji}^c &= p_{ni}^c \qquad \text{on } \Omega_d \end{aligned} \qquad (10\text{–}93)$$

Stress fields satisfying (10–93) are called *self-equilibrating* stress fields. For the ideal truss, σ^0 corresponds to the forces in the primary structure due to the prescribed loading and σ^c represents the contribution of the force redundants.

The governing equations for the force redundants were obtained by enforcing geometric compatibility, i.e., the bar elongations are constrained by the requirement that the deformed bar lengths *fit* in the assembled structure.

Geometric compatibility for a continuum requires the strains to lead to *continuous* displacements. One can establish the strain compatibility equations by operating on the strain- displacement relations. This approach is described in Prob. 10–10. One can also obtain these equations with the principle of virtual forces by taking a self-equilibrating force system. Letting $\Delta\sigma^c, \Delta p^c$ denote the virtual stress system, (10–89) reduces to

$$\iiint e_{ij}\, \Delta\sigma_{ij}^c\, dx_1\, dx_2\, dx_3 = \iint_{\Omega_d} \bar{u}_i\, \Delta p_{ni}^c\, d\Omega \qquad (10\text{–}94)$$

The compatibility equations are determined by expressing σ_{ij}^c in terms of stress functions and integrating the left-hand term by parts. We illustrate its application to the plane stress problem.

Example 10–3

If the stress components associated with the normal direction to a plane are zero, the stress state is called *planar*. We consider the case where $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$. The equilibrium equations and stress-boundary force relations reduce to

$$\begin{aligned} \sigma_{11,1} + \sigma_{21,2} + b_1 &= 0 \\ \sigma_{12,1} + \sigma_{22,2} + b_2 &= 0 \end{aligned} \qquad \text{(a)}$$

$$\begin{aligned} p_{n1} &= \alpha_{n1}\sigma_{11} + \alpha_{n2}\sigma_{21} \\ p_{n2} &= \alpha_{n1}\sigma_{12} + \alpha_{n2}\sigma_{22} \end{aligned} \qquad \text{(b)}$$

The stress field, σ_{ij}^c, must satisfy (a) with $b_1 = b_2 = 0$ and also $p_{n1} = p_{n2} = 0$ on Ω_σ. We can satisfy the equilibrium equations by expressing σ_{ij}^c in terms of a function, ψ, as follows:†

$$\begin{aligned} \sigma_{11}^c = \psi_{,22} \qquad \sigma_{22}^c &= \psi_{,11} \\ \sigma_{12}^c = \sigma_{21}^c &= -\psi_{,12} \end{aligned} \qquad \text{(c)}$$

The boundary forces corresponding to σ_{ij}^c are

$$p_{n1}^c = \frac{\partial}{\partial s}\psi_{,2} \qquad p_{n2}^c = -\frac{\partial}{\partial s}\psi_{,1} \qquad \text{(d)}$$

where s is the arc length on the boundary (sense is from $X_1 \to X_2$).

Substituting for σ^c, p^c in terms of ψ, (10–94) expands to

$$\iint (\varepsilon_1\, \Delta\psi_{,22} + \varepsilon_2\, \Delta\psi_{,11} - \gamma_{12}\, \Delta\psi_{,12})dx_1\, dx_2 - \int_{S_d} \left(\bar{u}_1 \frac{\partial}{\partial s}\Delta\psi_{,2} - \bar{u}_2 \frac{\partial}{\partial s}\Delta\psi_{,1}\right) dS = 0 \qquad \text{(e)}$$

There is no loss in generality by taking $\Delta\psi = 0$ on S. Then, integrating (e) by parts,

$$\iint (\varepsilon_{1,22} + \varepsilon_{2,11} - \gamma_{12,12})\Delta\psi\, dx_1\, dx_2 = 0 \qquad \text{(f)}$$

† See Prob. 10–14.

and requiring (f) to be satisfied for arbitrary $\Delta\psi$ results in the strain compatibility equation,

$$\varepsilon_{1,22} + \varepsilon_{2,11} - \gamma_{12,12} = 0 \tag{g}$$

which is actually a continuity requirement

$$u_{1,122} + u_{2,211} - (u_{1,212} + u_{2,112}) = 0 \tag{h}$$

We express (g) in terms of ψ by substituting for the strains in terms of the stresses.†

The principle of virtual forces is also employed to generate approximate solutions for the stresses. It is convenient to shift over to matrix notation for this discussion, and we write (10–94) as

$$\iiint \boldsymbol{\varepsilon}^T \Delta\boldsymbol{\sigma}^c \, dx_1 \, dx_2 \, dx_3 = \iint_{\Omega_d} \bar{\mathbf{u}}^T \Delta\mathbf{p}^c \, d\Omega$$

We express the stress matrix in terms of prescribed stress states and unknown parameters, a_i,

$$\begin{aligned} \boldsymbol{\sigma} &= \boldsymbol{\sigma}^0 + \boldsymbol{\sigma}^c \\ &= \boldsymbol{\sigma}^0 + a_1\boldsymbol{\phi}_1 + a_2\boldsymbol{\phi}_2 + \cdots + a_r\boldsymbol{\phi}_r \end{aligned} \tag{10–95}$$

where $\boldsymbol{\sigma}^0$ satisfies (10–92) and $\boldsymbol{\phi}_i$ $(i = 1, 2, \ldots, r)$ are self-equilibrating stress states, i.e., they satisfy the homogenous equilibrium equations and boundary conditions on Ω_σ. The corresponding surface forces are

$$\begin{aligned} &\mathbf{p} = \mathbf{p}^0 + a_1\boldsymbol{\theta}_1 + a_2\boldsymbol{\theta}_2 + \cdots + a_r\boldsymbol{\theta}_r \\ &\left.\begin{aligned} &\mathbf{p}^0 = \mathbf{p} \\ &\boldsymbol{\theta}_i = \mathbf{0} \qquad (i = 1, 2, \ldots, r) \end{aligned}\right\} \quad \text{on } \Omega_\sigma \end{aligned} \tag{10–96}$$

Taking virtual-force systems corresponding to Δa_i $(i = 1, 2, \ldots, r)$ results in r equations for the parameters.

$$\iiint \boldsymbol{\varepsilon}^T\boldsymbol{\phi}_i \, dx_1 \, dx_2 \, dx_3 = \iint_{\Omega_d} \bar{\mathbf{u}}^T\boldsymbol{\theta}_i \, d\Omega \qquad i = 1, 2, \ldots, r \tag{10–97}$$

In order to proceed, we need to introduce the material properties. When the material is linearly elastic,

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^0 + \mathbf{A}\boldsymbol{\sigma} = \boldsymbol{\varepsilon}^0 + \mathbf{A}\boldsymbol{\sigma}^0 + a_j\mathbf{A}\boldsymbol{\phi}_j \tag{a}$$

and the equations expand to

$$\begin{aligned} f_{ij}a_j &= d_i \qquad i, j = 1, 2, \ldots, r \\ f_{ij} = f_{ji} &= \iiint \boldsymbol{\phi}_i^T\mathbf{A}\boldsymbol{\phi}_j \, dx_1 \, dx_2 \, dx_3 \\ d_i &= \iint_{\Omega_d} \bar{\mathbf{u}}^T\boldsymbol{\theta}_i \, d\Omega - \iiint \boldsymbol{\phi}_i^T(\boldsymbol{\varepsilon}^0 + \mathbf{A}\boldsymbol{\sigma}^0) dx_1 \, dx_2 \, dx_3 \end{aligned} \tag{10–98}$$

One should note that (10–97) are *weighted* compatibility conditions. The true stresses must satisfy both equilibrium and compatibility throughout the

† See Prob. 10–27.

domain. We call $\boldsymbol{\sigma}^c$ the *corrective* stress field since it is required to correct the compatibility error due to $\boldsymbol{\sigma}^0$.

For completeness, we describe here how one establishes a variational principle for σ_{ij}^c. Our starting point is (10–94) restricted to *elastic* behavior. We define $V^* = V^*(\sigma_{ij})$ according to

$$\delta V^* = e_{ij} \, \Delta\sigma_{ij} = \boldsymbol{\varepsilon}^T \Delta\boldsymbol{\sigma} \tag{10–99}$$

and call V^* the *complementary energy density*. The form of V^* for a linearly elastic material is

$$V^* = \boldsymbol{\sigma}^T\boldsymbol{\varepsilon}^0 + \tfrac{1}{2}\boldsymbol{\sigma}^T\mathbf{A}\boldsymbol{\sigma} \tag{10–100}$$

By definition, V^* complements V, i.e.,

$$V + V^* = \sigma_{ij}e_{ij} \tag{10–101}$$

Then, letting

$$V_T^* = \iiint V^* \, dx_1 \, dx_2 \, dx_3 \tag{10–102}$$

we can write (10–94) as

$$\begin{aligned} &\delta\Pi_c = 0 \qquad \text{for arbitrary } \Delta\sigma_{ij}^c \\ &\Pi_c = V_T^* - \iint_{\Omega_d} \bar{u}_i p_{ni} \, d\Omega = \Pi_c(\sigma_{ij}^c) \end{aligned} \tag{10–103}$$

This form is called the principle of stationary complementary energy and shows that the *true* stresses correspond to a stationary value of Π_c.

Since p_{ni} is linear in σ_{ij}, the second variation of Π_c reduces to

$$\delta^2\Pi_c = \delta^2 V_T^* = \iiint \delta e_{ij} \, \Delta\sigma_{ij}^c \, dx_1 \, dx_2 \, dx_3 \tag{10–104}$$

We shift over to matrix notation and express $\delta\boldsymbol{\varepsilon}$ as

$$\delta\boldsymbol{\varepsilon} = \mathbf{A}_t \, \Delta\boldsymbol{\sigma}^c \tag{10–105}$$

where $\mathbf{A}_t$ represents the tangent compliance matrix. Now, $\mathbf{A}_t$ must be positive definite in order for the material to be stable.† Then, $\delta^2\Pi_c > 0$ for arbitrary $\Delta\boldsymbol{\sigma}^c$ and we see that the solution actually corresponds to a *relative minimum value* of Π_c.

The approximate method described earlier can be applied to Π_c. Substituting for $\boldsymbol{\sigma}$ given by (10–95) converts Π_c to a function of the stress parameters $(a_1, a_2, \ldots, a_r)$. When the material is linearly elastic,

$$\Pi_c = \tfrac{1}{2}\mathbf{a}^T\mathbf{f}\mathbf{a} - \mathbf{a}^T\mathbf{d} + \text{const} \tag{10–106}$$

The equations for the stress parameters follow by requiring Π_c to be stationary for arbitrary Δa_i:

$$\begin{aligned} \delta\Pi_c = \Delta\mathbf{a}^T(\mathbf{f}\mathbf{a} - \mathbf{d}) &= 0 \\ &\Downarrow \\ \mathbf{f}\mathbf{a} &= \mathbf{d} \end{aligned} \tag{10–107}$$

† The classical stability criterion specialized for elastic material and linear geometry requires $\delta^2 V = \delta\boldsymbol{\varepsilon}^T\mathbf{D}_t \, \delta\boldsymbol{\varepsilon} > 0$ for arbitrary $\delta\boldsymbol{\varepsilon}$ which, in turn, requires $\mathbf{D}_t$ to be positive definite. Since $\mathbf{A}_t = \mathbf{D}_t^{-1}$, it follows that $\mathbf{A}_t$ must be positive definite for a stable material.

Operating on $\delta\Pi_c$,

$$\delta^2\Pi_c = \Delta\mathbf{a}^T\mathbf{f}\,\Delta\mathbf{a} \tag{10–108}$$

and noting that $\delta^2\Pi_c > 0$, we conclude that $\mathbf{f}$ is positive definite.

REFERENCES

1. CRANDALL, S. J., and N. C. DAHL: *An Introduction to the Mechanics of Solids*, McGraw-Hill, New York, 1959.
2. BISPLINGHOFF, R. L., MAR., J. W., and T. H. H. PIAN: *Statics of Deformable Solids*, Addison-Wesley, Reading, Mass., 1965.
3. WANG, C. T.: *Applied Elasticity*, McGraw-Hill, New York, 1953.
4. TIMOSHENKO, S. J., and J. N. GOODIER: *Theory of Elasticity*, 3d ed., McGraw-Hill, New York, 1970.
5. SOKOLNIKOFF, I. S.: *Mathematical Theory of Elasticity*, 2d ed., McGraw-Hill, New York, 1956.
6. FUNG, Y. C.: *Foundations of Solid Mechanics*, Prentice-Hall, 1965.
7. LEKHNITSKII, S. G.: *Theory of Elasticity of an Anisotropic Elastic Body*, Holden-Day, San Francisco, 1963.
8. WASHIZU, K.: *Variational Methods in Elasticity and Plasticity*, Pergamon Press, 1968.
9. HILDEBRAND, F. B.: *Methods of Applied Mathematics*, Prentice-Hall, 1965.
10. CRANDALL, S. J.: *Engineering Analysis*, McGraw-Hill, New York, 1956.

PROBLEMS

10–1. Write out the expanded form of the following products. Consider the repeated indices to range from 1 to 2.

(a) $a_{ijk}x_jx_k$
(b) $\frac{1}{2}\sigma_{ij}(u_{i,j} + u_{j,i})$ where $\sigma_{ij} = \sigma_{ji}$
(c) $(\delta_{mj} + u_{m,j})(\delta_{mk} + u_{m,k}) - \delta_{jk}$

10–2. Let f be a continuous function of x_1, x_2, x_3. Establish the transformation laws for $\partial f/\partial x_j$ and $\partial^2 f/\partial x_j\,\partial x_k$.

10–3. Establish the transformation law for $a_{ij}b_k$ where a_{ij}, b_k are cartesian tensors.

10–4. Prove that

$$e_{jk} = \tfrac{1}{2}(\vec{\rho}_{,j}\cdot\vec{\rho}_{,k} - \delta_{jk})$$

is a second-order cartesian tensor. *Hint*: Expand

$$\vec{\rho}_{,j'}\cdot\vec{\rho}_{,k'} = \frac{\partial\vec{\rho}}{\partial x'_j}\cdot\frac{\partial\vec{\rho}}{\partial x'_k}$$

10–5. Equations (10–19) are the strain transformation laws. Since e_{ij} is a symmetrical second-order cartesian tensor, there exists a particular set of directions, say X_j^p, for which e_{ij}^p is a diagonal array. What are the strain components for the X^p frame? Consider a rectangular parallelepiped having sides dX_j^p in the undeformed state. What is its deformed shape and relative change in volume, ε_v^p, with respect to its initial volume? Specialize the expression for ε_v^p for *small* strain. Then determine ε_v for the initial (X_j) directions and *small* strain. Finally, show that ε_v is invariant.

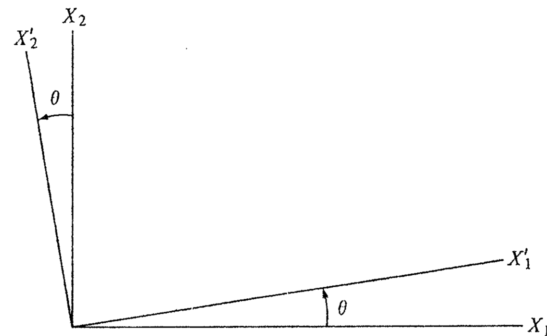
10–6.

(a) Specialize (10–19) for small strain and write out the expressions for ε'_i, γ'_{ij} in terms of $\varepsilon_1, \varepsilon_2, \ldots \gamma_{13}$.
(b) Let $\boldsymbol{\varepsilon} = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \gamma_{12}, \gamma_{23}, \gamma_{31}\}$. We can express the strain transformation (small strain) as

$$\boldsymbol{\varepsilon}' = \mathbf{T}_\varepsilon\boldsymbol{\varepsilon}$$

Develop the form of $\mathbf{T}_\varepsilon$ using the results of part a.
(c) Evaluate $\mathbf{T}_\varepsilon$ in terms of $\cos\theta$, $\sin\theta$ for the rotation shown below. Comment on the transformation law for the out-of-plane shear strains $\gamma'_{31}, \gamma'_{32}$.

Prob. 10–6



10–7. In the Eulerian approach, the cartesian coordinates (η_i) for the deformed state are taken to be the independent variables, i.e.,

$$u_j = u_j(\eta_k) \qquad x_j = x_j(\eta_k)$$

Almansi's strain tensor is defined as

$$|d\vec{\rho}|^2 - (ds)^2 = 2E_{jk}\,d\eta_j\,d\eta_k$$

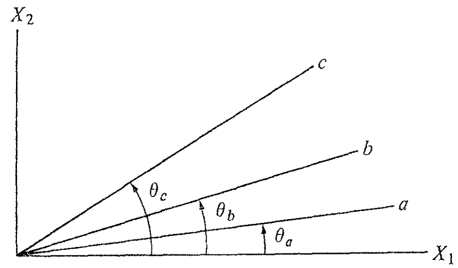
Determine the expression for E_{jk} in terms of the displacements. Compare the result with (10–21).

10–8. Consider the case of two-dimensional deformation in the X_1-X_2 plane ($\varepsilon_3 = \gamma_{13} = \gamma_{23} = 0$). Let $\varepsilon_a, \varepsilon_b, \varepsilon_c$ be the extensions in the a, b, c directions defined below and let $\boldsymbol{\varepsilon}_N = \{\varepsilon_a, \varepsilon_b, \varepsilon_c\}$. We can write

$$\boldsymbol{\varepsilon}_N = \mathbf{B}\boldsymbol{\varepsilon}$$
$$\boldsymbol{\varepsilon} = \mathbf{B}^{-1}\boldsymbol{\varepsilon}_N$$

(a) Determine the general form of $\mathbf{B}$.
(b) Determine $\mathbf{B}^{-1}$ for $\theta_a = 0, \theta_b = 45°, \theta_c = 90°$.
(c) Determine $\mathbf{B}^{-1}$ for $\theta_a = 0, \theta_b = 60°, \theta_c = 120°$.
(d) Extend (a) to the three-dimensional case. Consider six directions having direction cosines $\alpha_{j1}, \alpha_{j2}, \alpha_{j3}$ with respect to X_1, X_2, X_3. Can we select the six directions arbitrarily?

Prob. 10–8



10–9. For small strain, the volumetric strain is

$$\varepsilon_v = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = e_{11} + e_{22} + e_{33}$$

Rather than work with e_{ij}, one can express it as the sum of two tensors,

$$e_{ij} = e_{ij}^{(s)} + e_{ij}^{(d)}$$

where $e_{ij}^{(s)}$ is called the spherical strain tensor,

$$e_{ij}^{(s)} = \tfrac{1}{3}\delta_{ij}\varepsilon_v = \delta_{ij}\varepsilon_m$$

and $e_{ij}^{(d)}$ is the deviator strain tensor.

(a) Write out the expanded form for $e_{ij}^{(s)}$ and $e_{ij}^{(d)}$.
(b) Determine the first invariant of $e_{ij}^{(s)}$, $e_{ij}^{(d)}$ and compare with the invariant of e_{ij}.

10–10. This question concerns strain compatibility equations.

(a) Show that

$$\frac{\partial^2 e_{nk}}{\partial x_m \partial x_\ell} + \frac{\partial^2 e_{m\ell}}{\partial x_n \partial x_k} \equiv \frac{\partial^2 e_{n\ell}}{\partial x_m \partial x_k} + \frac{\partial^2 e_{mk}}{\partial x_n \partial x_\ell}$$

where

$$e_{nk} = e_{kn} = \frac{1}{2}\left(\frac{\partial u_n}{\partial x_k} + \frac{\partial u_k}{\partial x_n}\right)$$

and k, ℓ, m, n range from 1 to 3. This expression leads to six independent conditions, called *geometric compatibility relations*, on the strain measures.

(b) Show that for two-dimensional deformation in the X_1-X_2 plane ($\varepsilon_3 = \varepsilon_{13} = \varepsilon_{23} = 0$; this called *plane strain*) there is only *one* compatibility equation, and it has the following form:

$$\varepsilon_{1,22} + \varepsilon_{2,11} = \gamma_{12,12}$$

Is the following strain state permissible?

$$\varepsilon_1 = k(x_1^2 + x_2^2)$$
$$\varepsilon_2 = kx_2^2$$
$$\gamma_{12} = 2kx_1x_2$$
$$k = \text{constant}$$

10–11. Equation (10–21) defines the strain measures due to displacements, u_i. To analyze geometrically nonlinear behavior, one can employ an incremental formulation. Let Δu_i represent the displacement increment and Δe_{jk} the incremental strain. We write

$$\Delta e_{jk} = \delta e_{jk} + \tfrac{1}{2}\delta^2 e_{jk}$$

where δe_{jk} contains linear terms (Δu_i) and $\delta^2 e_{jk}$ involves quadratic terms. The δ-symbol denotes the first-order change in a functional and is called the variational operator (see Ref. 8). We refer to δe as the first variation of e. Determine the expressions for δe, $\delta^2 e$.

10–12. Let $\bar{i}_n$ be the unit vector defining the initial orientation of the differential line element $d\bar{r}_n$ at a point.

$$d\bar{r}_n = ds\bar{i}_n \qquad \bar{i}_n = \alpha_{nj}\bar{\imath}_j$$

The unit vector defining the orientation in the deformed state is $\bar{v}_n$.

$$d\bar{\rho}_n = (1 + \varepsilon)ds\bar{v}_n \qquad \bar{v}_n = \beta_{nj}\bar{\imath}_j$$

Determine the general expression for β_{nj}. Then specialize it for small strain.

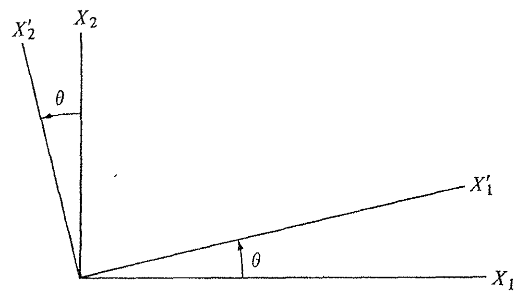
10–13. The several parts of this question concerns stress transformation.

(a) Starting with (10–41), write out the expressions for σ'_{ii}, σ'_{ij} in terms of $\sigma_{11}, \sigma_{22}, \ldots, \sigma_{13}$.
(b) Let $\boldsymbol{\sigma} = \{\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}\}$ = stress matrix. We express the stress transformation as a matrix product.

$$\boldsymbol{\sigma}' = \mathbf{T}_\sigma \boldsymbol{\sigma}$$

Develop the form of $\mathbf{T}_\sigma$ using the results of part *a*.
(c) Evaluate $\mathbf{T}_\sigma$ in terms of $\cos\theta$, $\sin\theta$ for the axes shown.

Prob. 10–13



(d) Plane stress refers to the case where $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$. We work with reduced stress and strain matrices,

$$\boldsymbol{\sigma} = \{\sigma_{11}, \sigma_{22}, \sigma_{12}\}$$
$$\boldsymbol{\varepsilon} = \{\varepsilon_1, \varepsilon_2, \gamma_{12}\}$$

and write the transformations in the same form as the three-dimensional case:

$$\boldsymbol{\sigma}' = \mathbf{T}_\sigma \boldsymbol{\sigma}$$
$$\boldsymbol{\varepsilon}' = \mathbf{T}_\varepsilon \boldsymbol{\varepsilon}$$

Evaluate $\mathbf{T}_\sigma$ from part *c* above and $\mathbf{T}_\varepsilon$ from Prob. 10–6. Verify that

$$\mathbf{T}_\sigma^T \mathbf{T}_\varepsilon = \mathbf{I}_3$$

10–14. This question develops a procedure for generating self-equilibrating stress fields.

(a) Expand the linear equilibrium equations, (10–49) and (10–50).

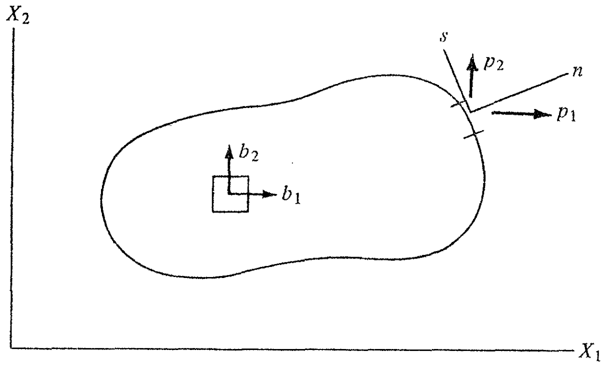
(b) Specialize the equilibrium equations for plane stress ($\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$).

(c) Suppose we express the two-dimensional stress components in terms of a function $\psi = \psi(x_1, x_2)$, as follows:

$$\sigma_{11} = \psi,_{22} - \int_{x_1} b_1 \, dx_1$$
$$\sigma_{22} = \psi,_{11} - \int_{x_2} b_2 \, dx_2$$
$$\sigma_{12} = \sigma_{21} = -\psi,_{12}$$

The notation for body and surface forces is defined in the following sketch.

Prob. 10–14



Verify that this definition satisfies the equilibrium equations in the interior. Show that the expressions for p_1 and p_2 in terms of derivatives with respect to x_1, x_2, and s are

$$p_1 = \frac{\partial}{\partial s}\psi,_2 - \alpha_{n1} \int_{x_1} b_1 \, dx_1$$

$$p_2 = -\frac{\partial}{\partial s}\psi,_1 - \alpha_{n2} \int_{x_2} b_2 \, dx_2$$

10–15. The mean stress, σ_m, is defined as

$$\sigma_m = \tfrac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$$

Rather than work with σ_{ij}, we can express it as the sum of two tensors,

$$\sigma_{ij} = \sigma_{ij}^{(s)} + \sigma_{ij}^{(d)}$$

where $\sigma_{ij}^{(s)}$ is called the spherical stress tensor,

$$\sigma_{ij}^{(s)} = \delta_{ij}\sigma_m$$

and $\sigma_{ij}^{(d)}$ is the deviator stress tensor.

(a) Write out the expanded forms for $\sigma_{ij}^{(s)}$ and $\sigma_{ij}^{(d)}$.

(b) Determine the first invariant of $\sigma_{ij}^{(s)}$, $\sigma_{ij}^{(d)}$.

10–16. Establish the stress-equilibrium equations for *small-finite* rotation and small strain.

10–17. Starting with (10–52), (10–55) specialized for small strain, establish the incremental equilibrium equations in terms of $\Delta\sigma^k$, Δu, Δb^*, and Δp^*. Group according to linear and quadratic terms. Specialize these equations for the case where the initial position is geometrically linear, i.e., where we can approximate β_{jk} with α_{jk} in the incremental equations.

10–18. Prove (10–60). *Hint*:

$$\delta e_{jk} = \tfrac{1}{2}(\bar{\rho}_{,j} \cdot \delta\bar{\rho}_{,k} + \bar{\rho}_{,k} \cdot \delta\bar{\rho}_{,j})$$
$$\delta\bar{\rho}_{,j} = \Delta\bar{u}_{,j}$$

10–19. Verify that the inverted form of (10–71) is

$$\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^0)$$

where

$$D_{11} = E_1/C_3 \qquad D_{12} = (C_2/C_1)D_{11} \qquad D_{13} = C_4 D_{11}$$
$$D_{22} = E_2/C_1 + (C_2/C_1)D_{12}$$
$$D_{23} = \nu_{32}E_2/C_1 + (C_2/C_1)D_{13}$$
$$D_{33} = E_3 + \nu_{31}D_{13} + \nu_{32}D_{23}$$

and

$$C_1 = 1 - \nu_{32}^2(E_2/E_3)$$
$$C_2 = \nu_{21} + \nu_{31}\nu_{32}(E_2/E_3)$$
$$C_3 = 1 - \frac{E_1}{E_3}\left\{\nu_{31}^2 + \frac{E_3}{E_2}\left(\frac{C_2}{C_1}\right)C_2\right\}$$
$$C_4 = \nu_{31} + \left(\frac{C_2}{C_1}\right)\nu_{32}$$

Specialize for plane strain ($\varepsilon_3 = \gamma_{13} = \gamma_{23} = 0$)

10–20. Consider 2 sets of orthogonal directions defined by the unit vectors $\bar{t}_j$ and $\bar{t}'_j$. The stress-strain relations for the two frames are

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^0 + \mathbf{A}\boldsymbol{\sigma}$$
$$\boldsymbol{\varepsilon}' = (\boldsymbol{\varepsilon}^0)' + \mathbf{A}'\boldsymbol{\sigma}'$$

Express $\mathbf{A}'$ in terms of $\mathbf{A}$ and $\mathbf{T}_\sigma$. Also determine $\mathbf{D}'$.

10–21. Consider the three-dimensional stress-strain relations defined by (10–71).

(a) Specialize for plane stress ($\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$).

(b) Let

$$\boldsymbol{\sigma} = \{\sigma_{11}, \sigma_{22}, \sigma_{12}\}$$
$$\boldsymbol{\varepsilon} = \{\varepsilon_1, \varepsilon_2, \gamma_{12}\}$$
$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^0 + \mathbf{A}\boldsymbol{\sigma}$$

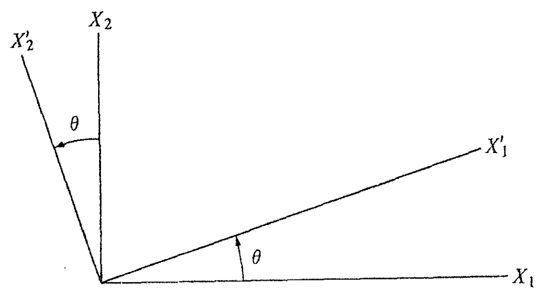
Verify that **D** has the following form:

$$\mathbf{D} = \frac{E_1}{1 - n\nu_{21}^2}\begin{bmatrix} 1 & \nu_{21} & 0 \\ \nu_{21} & \dfrac{1}{n} & 0 \\ 0 & 0 & \dfrac{G}{E_1}(1 - n\nu_{21}^2) \end{bmatrix}$$

where $n = \dfrac{E_1}{E_2}$.

(c) Assuming X_1-X_2 in the sketch are material symmetry directions, determine $\mathbf{D}'$ for the X_1'-X_2' frame. Use the results of Prob. 10–13, 10–20. What relations between the properties are required in order for $\mathbf{D}'$ to be identical to **D**?

Prob. 10–21



10–22. Verify (10–73). Start by requiring equal properties for the X_2 and X_3 directions. Then introduce a rotation about the X_1 axis and consider the expression for γ_{23}'. Isotropy in the X_2-X_3 plane requires

$$\gamma_{23}' = \frac{1}{G_{23}}\sigma_{23}'$$

10–23. Verify that the directions of principal stress and strain coincide for an isotropic material. Is this also true for an orthotropic material?

10–24. Equations (10–76) can be written as

$$\sigma_{ij} = \sigma^0\delta_{ij} + \lambda\varepsilon_v\,\delta_{ij} + 2Ge_{ij}$$

where ε_v is the volumetric strain. Using the notation introduced in Probs. 10–9 and 10–15—

(a) Show that

$$\sigma_m = K\varepsilon_v + \sigma^0$$

where K is the bulk modulus $= (E/3(1 - 2\nu))$. Discuss the case where $\nu = \frac{1}{2}$.

(b) Show that

$$\sigma_{ij}^{(d)} = 2Ge_{ij}^{(d)}$$

(c) Verify that the strain-energy density can be written as

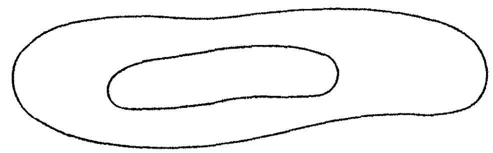
$$\begin{aligned} V &= \tfrac{1}{2}\sigma_{ij}(e_{ij} - \delta_{ij}e_{ii}^0) \\ &= \tfrac{1}{2}\sigma_m(\varepsilon_v - \varepsilon_v^0) + \tfrac{1}{2}\sigma_{ij}^{(d)}e_{ij}^{(d)} \\ &= V^{(s)} + V^{(d)} \end{aligned}$$

Determine $V^{(s)}$ and $V^{(d)}$ for the isotropic case.

(d) When $\nu = \frac{1}{2}$, $\varepsilon_v = \varepsilon_v^0$. We must work with 7 stress measures (σ_{ij}, σ_m) and the mean stress has to be determined from an equilibrium consideration. Summarize the governing equations for the incompressible case.

10–25. Prove (10–81) for the two-dimensional case. Is this formula restricted to a specific direction of integration on the boundary? Does it apply for a multi-connected region, such as shown in the figure below?

Prob. 10–25



10–26. Verify Equation (10–89).

10–27. Refer to Example 10–3. Express (g) in terms of ψ. Consider the material to be orthotropic.

10–28. Verify that the stationary requirement

$$\delta\Pi_R = 0 \qquad \text{for arbitrary } \Delta u_i, \Delta\sigma_{ij}^k, \Delta p_{ni}$$

where

$$\begin{aligned} \Pi_R &= \iiint(e_{ij}\sigma_{ij}^k - V^* - \bar{b}_i^* u_i)dx_1\,dx_2\,dx_3 \\ &\quad - \iint_{\Omega_\sigma} \bar{p}_{ni}^* u_i\,d\Omega - \iint_{\Omega_d} p_{ni}(u_i - \bar{u}_i)d\Omega \\ \sigma_{ij}^k &= \text{Kirchhoff stress} \\ e_{ij} &= \text{Lagrange strain} = \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{m,i}u_{m,j}) \\ V^* &= \text{complementary energy density (initial volume)} \\ \bar{b}^*, \bar{p}^* &= \text{prescribed force measures (initial dimensions)} \end{aligned}$$

leads to the complete set of governing equations for an elastic solid, i.e.,

1. stress equilibrium equations
2. stress-displacement relations
3. stress boundary conditions on Ω_σ
4. displacement boundary conditions on Ω_d
5. expressions for the reaction surface forces on Ω_d

This variational statement is called Reissner's principle (see Ref. 8).

(a) Transform Π_R to Π_p by requiring the stresses to satisfy the stress displacement relations. *Hint*: Note (10–101).

(b) Transform Π_R to $-\Pi_c$ by restricting the geometry to be linear ($\sigma^k = \sigma$ and $e_{ij} = (u_{i,j} + u_{j,i})/2$) and requiring the stresses to satisfy the stress equilibrium equations and stress boundary conditions on Ω_σ. *Hint*: Integrate $\sigma_{ij}e_{ij}$ by parts, using (10–81).

10–29. Interpret (10–90) as

$$d_Q = \frac{\partial}{\partial P_Q}\Pi_c$$

where P_Q is a force applied at Q in the direction of the displacement measure, d_Q.

11

St. Venant Theory of Torsion-Flexure of Prismatic Members

11–1. INTRODUCTION AND NOTATION

A body whose cross-sectional dimensions are small in comparison with its axial dimension is called a *member*. If the centroidal axis is straight and the shape and orientation of the normal cross section are constant,† the member is said to be *prismatic*. We define the member geometry with respect to a global reference frame (X_1, X_2, X_3), as shown in Fig. 11–1. The X_1 axis is taken to coincide with the centroidal axis and X_2, X_3 are taken as the principal inertia directions. We employ the following notation for the cross-sectional properties:

$$\begin{aligned} A &= \iint dx_2\, dx_3 = \iint dA \\ I_2 &= \iint (x_3)^2\, dA \\ I_3 &= \iint (x_2)^2\, dA \end{aligned} \tag{11–1}$$

Since X_2, X_3 pass through the centroid and are principal inertia directions, the centroidal coordinates and product of inertia vanish:

$$\begin{gathered} x_{2,c} = \frac{1}{A}\iint x_2\, dA = 0 \qquad x_{3,c} = \frac{1}{A}\iint x_3\, dA = 0 \\ I_{23} = \iint x_2 x_3\, dA = 0 \end{gathered} \tag{11–2}$$

One can work with an arbitrary orientation of the reference axes, but this will complicate the derivation.

St. Venant's theory of torsion-flexure is restricted to *linear* behavior. It is an *exact* linear formulation for a prismatic member subjected to a prescribed

† The case where the cross-sectional shape is constant but the orientation varies along the centroidal axis is treated in Chapter 15.