Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
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1.2 Kinematics of Fluid Motion -the Eulerian picture

Consider two neighboring stations (not two fluid particles) $\vec{x}$ and $\vec{x}'$ at the same instant t, where $\delta\vec{x} = \vec{x}' - \vec{x}$ is small. The fluid velocity at the two stations are related by

$$\vec{q}(\vec{x}', t) = \vec{q}(\vec{x}, t) + (\vec{x}' - \vec{x}) \cdot \nabla\vec{q}(\vec{x}, t) + O(\vec{x}' - \vec{x})^2 \tag{1.2.1}$$

Hence

$$\delta\vec{q}(\vec{x}, t) = \vec{q}(\vec{x}', t) - \vec{q}(\vec{x}, t) = \delta\vec{x} \cdot \nabla\vec{q}(\vec{x}, t) + O(\delta\vec{x})^2 \tag{1.2.2}$$

Let us introduce the index notation:

$$q_1 = u, \quad q_2 = v, \quad q_3 = w; \quad x_1 = x, \quad x_2 = y, \quad x_3 = z \tag{1.2.3}$$

and Einstein's convention: Repeated indices are summed over the range from 1 to 3, and the summation symbol is omitted but implied. For example,

$$\sum_{i=1}^{3} q_i q_i = q_i q_i = q_1^2 + q_2^2 + q_3^3 = \vec{q} \cdot \vec{q}$$

Thus we may write (1.2.2) as

$$\delta q_i = \delta x_j \frac{\partial q_i}{\partial x_j}, \quad i = 1, 2, 3. \tag{1.2.4}$$

Now

$$\frac{\partial q_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial q_i}{\partial x_j} - \frac{\partial q_j}{\partial x_i}\right) \tag{1.2.5}$$

Define the rate-of -strain tensor by

$$e_{ij} = \frac{1}{2}\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}\right) \tag{1.2.6}$$

and the vorticity tensor by

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial q_i}{\partial x_j} - \frac{\partial q_j}{\partial x_i}\right) \tag{1.2.7}$$

Note that

$$e_{ij} = e_{ji}, \quad \Omega_{ij} = -\Omega_{ji} \tag{1.2.8}$$

and (1.2.4) becomes

$$\delta q_i = \delta x_j e_{ij} + \delta x_j \Omega_{ij} \tag{1.2.9}$$

Let us examine the physics of these terms.

1.2.1 Rate-of-strain tensor

In matrix form, the rate-of -strain tensor is :

$$\begin{aligned}
\{e_{ij}\} &= \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial q_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial q_1}{\partial x_2}+\frac{\partial q_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial q_1}{\partial x_3}+\frac{\partial q_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial q_2}{\partial x_1}+\frac{\partial q_1}{\partial x_2}\right) & \frac{\partial q_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial q_2}{\partial x_3}+\frac{\partial q_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial q_3}{\partial x_1}+\frac{\partial q_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial q_3}{\partial x_2}+\frac{\partial q_2}{\partial x_3}\right) & \frac{\partial q_3}{\partial x_3} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} \end{pmatrix}
\end{aligned} \tag{1.2.10}$$

First, the diagonal terms. It is easy to see that $e_{11} = \partial u/\partial x$ is the rate of stretching per unit length in the direction of x, $e_{22} = \partial v/\partial y$ is the rate of stretching per unit length in the direction of y, and $e_{33} = \partial w/\partial z$ is the rate of stretching per unit length in the direction of z. They are the normal components of the rate of strain tensor.

Note that

$$e_{11} + e_{22} + e_{33} = e_{kk} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \vec{q} \tag{1.2.11}$$

is the rate of volume dilatation due to fluid motion. For a proof, let us consider a cube with sides $(x, x+\Delta x)$, $(y, y+\Delta y)$ and $(z, z+\Delta z)$. After δt, the side along x will lengthen from Δx to $\Delta x + \Delta x \frac{\partial u}{\partial x}\delta t = \Delta x\left(1+\frac{\partial u}{\partial x}\delta t\right)$. Similarly, the side along y will lengthen from Δy to $\Delta y\left(1+\frac{\partial v}{\partial y}\delta t\right)$, and the side along z lengthens from Δz to $\Delta z\left(1+\frac{\partial w}{\partial z}\delta t\right)$. Consequently the volume $V(t) = \Delta x\Delta y\Delta z$ will change to

$$\begin{aligned}
V(t+\delta t) &= \Delta x\left(1+\frac{\partial u}{\partial x}\delta t\right)\Delta y\left(1+\frac{\partial v}{\partial y}\delta t\right)\Delta z\left(1+\frac{\partial w}{\partial z}\delta t\right) \\
&= V(t)\left[1+\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\delta t + O(\delta t)^2\right]
\end{aligned}$$

Hence, the rate of volume dilatation is

$$\lim_{\delta t=0}\frac{1}{V}\frac{V(t+\delta t)-V(t)}{\delta t} = \frac{1}{V}\frac{dV}{dt} = \left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) = \nabla\cdot\vec{q} \tag{1.2.12}$$

Next, the off-diagonal terms. Referring to Figure 1.2.1, consider a plane flow in which $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial x}$ do not vanish. In the time interval δt the side Δx rotates counterclockwise for an angle $\delta\theta_1 = \frac{\Delta v\delta t}{\Delta x} = \frac{\partial v}{\partial x}\delta t$. The side Δy rotates counterclockwise for an angle $\delta\theta_2 = -\frac{\Delta u\delta t}{\Delta y} = -\frac{\partial u}{\partial y}\Delta t$. The total rate of angular deformation is

$$\frac{\delta\theta_1}{\delta t} - \frac{\delta\theta_2}{\delta t} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{1.2.13}$$

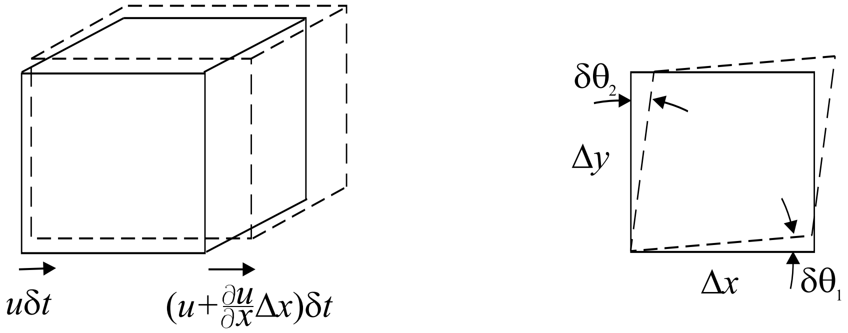

Figure 1.2.1: Rate of strain tensor components

Thus $e_{12} = e_{xy}$ is a rate of angular deformation, called the rate of shear strain. Other components e_{13} and e_{23} can be interpreted similarly.

1.2.2 Vorticity tensor

The matrix form of Ω_{ij} is

$$\begin{aligned}
\{\Omega_{ij}\} &= \begin{pmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{pmatrix} \\
&= \begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial q_1}{\partial x_2} - \frac{\partial q_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial q_1}{\partial x_3} - \frac{\partial q_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial q_2}{\partial x_1} - \frac{\partial q_1}{\partial x_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial q_2}{\partial x_3} - \frac{\partial q_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial q_3}{\partial x_1} - \frac{\partial q_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial q_3}{\partial x_2} - \frac{\partial q_2}{\partial x_3}\right) & 0 \end{pmatrix} \qquad (1.2.14) \\
&= \begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) & 0 & \frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) & 0 \end{pmatrix}
\end{aligned}$$

Because of the anti-symmetry, there are only three independent components, which can also be used to define the vorticity vector $\vec{\zeta}$:

$$\begin{aligned}
\vec{\zeta} &= \nabla \times \vec{q} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \\
&= \vec{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \vec{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \vec{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad (1.2.15)
\end{aligned}$$

Hence

$$\{\Omega_{ij}\} = \frac{1}{2}\begin{pmatrix} 0 & -\zeta_3 & \zeta_2 \\ \zeta_3 & 0 & -\zeta_1 \\ -\zeta_2 & \zeta_1 & 0 \end{pmatrix} \qquad (1.2.16)$$

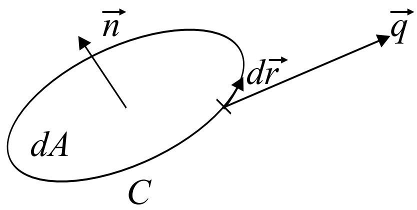

Figure 1.2.2: Circulation along a closed circle

What is the physical meaning of $\vec{\zeta}$? Consider a plane circular disc A bounded by the circle C of radius a, see Figure 1.2.2. By Stokes' theorem

$$\iint_A (\nabla \times \vec{q}) \cdot \vec{n}\, dA = \oint_C \vec{q} \cdot d\vec{r}$$

Now let $a \to 0$, then,

$$(\nabla \times \vec{q})_n \iint_A dA = \oint_C \vec{q} \cdot d\vec{r}$$

or,

$$\frac{1}{2}\zeta_n = \frac{1}{2}(\nabla \times \vec{q})_n = \frac{1}{a}\left[\frac{1}{2\pi a}\oint_C \vec{q} \cdot d\vec{r}\right]$$

The quantity

$$\left[\frac{1}{2\pi a}\oint_C \vec{q} \cdot d\vec{r}\right]$$

is the average tangential velocity along the circle. Hence $\zeta_n/2$ is the average angular speed of the fluid circling along C, i.e., the average rate of rotation. The line integral above is also known as the *circulation.*