

Lecture Notes on Fluid Dynamics
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1.9 Scaling and similarity parameters

The full set of Navier-Stokes equations is so complex that it is unrealistic and unrevealing to search for the most general solutions. Approximations valid for certain specific circumstances are much more useful. To make systematic approximations it is necessary to have a procedure that helps us discern precisely what is small and what is not. A standard procedure is to first find the scales relevant to the problem at hand. Normalization by these scales leads to dimensionless parameters which represent the relative importance of various parts of the full equations. Depending on the magnitudes of these parameters, suitable approximations can be devised which can lead to answers that capture the essence of the problem. As an illustration we use the incompressible fluid of constant density to explain this philosophy and approach

For a specific physical problem, such as wind blowing over a mountain, a ship moving in the sea, waves propagating on the sea, etc., there are usually some obvious scales of motion. The less obvious ones can also be derived by physical considerations. For the problems above the height of the mountain, or the length or draft of the ship, or the wavelength, is the obvious length scale L . The time from the start of the wind, or from the start of the ship motion, or the wave period, can be the time scale T . The speed of the oncoming wind, or of the ship, or of the wave propagation are the natural scales for the velocity U . If gravity is expected to be important the gravitational acceleration g can be used as the scale of body force per unit mass.

With these scales, let us introduce the dimensionless variables

$$\vec{q}' = \vec{q}/U, \quad t' = t/T, \quad \vec{x}' = x/L, \quad p' = p/P, \quad \vec{f}' = \vec{f}/g \quad (1.9.1)$$

where the pressure scale¹ P is yet unknown. Navier-Stokes equations now become

$$\nabla' \cdot \vec{q}' = 0 \quad (1.9.2)$$

$$\frac{L}{TU} \frac{\partial \vec{q}'}{\partial t'} + \vec{q}' \cdot \nabla' \vec{q}' = -\frac{P}{\rho U^2} \nabla' p' + \frac{gL}{U^2} \vec{f}' + \frac{\nu}{UL} \nabla'^2 \vec{q}'. \quad (1.9.3)$$

Since L, T are the characteristic scales, the dimensionless derivatives are of order unity,

$$\frac{\partial}{\partial x'} = O(1) \quad \frac{\partial}{\partial t'} = O(1)$$

¹Actually the scale of pressure variation, since p appears only as a gradient.

Several dimensionless parameters now appear in the normalized momentum equation. Each ratio is the measure of the magnitude of the term it multiplies. Usually at least one terms is expected to be dominant; it is convenient to arrange its coefficients to be unity. The dimensionless parameters then represent the importance of other terms relative to the dominant term. Thus (1.9.3) is of the form for inertia-dominated flows. The magnitude of the pressure gradient is then measured by the ratio $P/\rho U^2$, the instantaneous acceleration by L/TU , the body force by gL/U^2 and viscous stress by ν/U^2 , etc. Most of these parameters have been given special names of people who contributed greatly to their understanding. Thus

$$\begin{aligned} \frac{L}{TU} &= \text{Strouhal number} = St \sim \frac{\text{local acceleration}}{\text{convective acceleration}} \\ \frac{UL}{\nu} &= \text{Reynolds number} = Re \sim \frac{\text{inertia}}{\text{viscous force}} \\ \frac{U^2}{gL} &= (\text{Froude number})^2 = Fr^2 \sim \frac{\text{inertia}}{\text{body force}} \end{aligned}$$

It is now easy to decide how to make approximations. The following examples are of interest to many practical problems:

(i) High Reynolds number flows :

Let all other parameters be of order unity, i.e., local acceleration, pressure gradient and body force are as important as the convective inertia, but

$$Re = \frac{UL}{\nu} \gg 1. \quad (1.9.4)$$

which can be met by small viscosity, high speed, or a large body. Then the viscous stress term can be ignored. The crudest approximation is then governed by

$$\nabla \cdot \vec{q}' = 0 \quad (1.9.5)$$

$$\frac{L}{TU} \frac{\partial \vec{q}'}{\partial t'} + \vec{q}' \cdot \nabla' \vec{q}' = -\frac{P}{\rho U^2} \nabla' p' + \frac{gL}{U^2} \vec{f}'. \quad (1.9.6)$$

After restoring the dimensional variables, they are

$$\nabla \cdot \vec{q} = 0 \quad (1.9.7)$$

$$\frac{\partial \vec{q}}{\partial t} + \vec{q} \cdot \nabla \vec{q} = -\frac{1}{\rho} \nabla p + \rho \vec{f}. \quad (1.9.8)$$

which are called Euler's equations, for the inviscid flow of incompressible fluid of constant density.

If in addition, the Strouhal number is also very small, i.e., the time scale is large compared to L/U . Local acceleration can be neglected and the flow is essentially steady. In a steady flow of inviscid fluid, we must choose $P/\rho U^2 = O(1)$ meaning that ρU^2 must be the scale of

dynamical pressure. Physically fluid accelerates only if it is driven by some pressure gradient. Therefore the pressure scale is derived.

Note that all terms in the continuity equation have the same coefficients : unity, hence they are equally important. The law of mass conservation is a sacred cow in fluid dynamics and cannot be mutilated.

Since viscosity is ignored, the kinematic boundary conditions on a rigid body is that there is no normal flux.

$$\vec{q} \cdot \vec{n} \text{ on body} = \vec{Q} \cdot \vec{n} \text{ of body.} \quad (1.9.9)$$

There is no constraint on the tangential component of \vec{q} .

(ii) Low Reynolds-number flows:

If $Re = \frac{UL}{\nu} \ll 1$ which means either low velocity, small body or large viscosity. then viscous stresses must be dominant in the momentum equation, and inertia is insignificant. If, furthermore, there is no gravity, viscous stresses cannot stand alone and must be balanced by the pressure gradient. Hence, we conclude

$$\frac{P}{\rho U^2} = O\left(\frac{\nu}{UL}\right) \ll 1$$

or

$$P = O(\nu U/L). \quad (1.9.10)$$

Thus the scales of pressure variation are very different in high and low Reynolds number flows. Rearranging the momentum equation,

$$\frac{UL}{\nu} \frac{L}{TU} \frac{\partial \vec{q}'}{\partial t'} + \frac{UL}{\nu} \vec{q}' \cdot \nabla' \vec{q}' = -\nabla' p' + \frac{UL}{\nu} \frac{gL}{U^2} \vec{f}' + \nabla'^2 \vec{q}'. \quad (1.9.11)$$

For small Reynolds numbers, the crudest approximation of (1.9.11) is

$$0 = -\nabla' p' + \nabla'^2 \vec{q}'. \quad (1.9.12)$$

Thus, in dimensional form,

$$\nabla \cdot \vec{q} = 0 \quad (1.9.13)$$

$$0 = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{q}. \quad (1.9.14)$$

These are called Stokes equations.

On a surface of a rigid body, there can be no slippage, thus

$$\vec{q} \text{ on body} = \vec{Q} \text{ of body} . \quad (1.9.15)$$

Aside from being useful in estimating the relative importance of various terms, the dimensionless parameters have further significance of governing the *dynamic similarity*. Specifically even if two problems have quite different absolute scales, they are dynamically similar if their dimensionless parameters are the same. This similarity is the basis for simulating large-scale phenomena in a small-scale laboratory. Computation and presentation of theoretical discussions can also be simplified in dimensionless forms.

The discussions above are merely illustrative and by no means exhaustive. The approach of making approximations based on scaling and normalization is however useful in general.