Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
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2-3-gr-current.tex
Reference:

C. C. Mei, (1966), *J. Math. & Phys.* pp.

2.3 A gravity current

For the highly nonlinear equation, a relatively simple solution is that of a stationary (or permanent) wave which is profile advancing at a constant speed without changing its shape. Mathematically the profile is describable as

$$h(x,t) = h(x - Ct) = h(\sigma), \quad \sigma = x - Ct \tag{2.3.1}$$

By the chain rule of differentiation,

$$\frac{\partial h(x-Ct)}{\partial t} = \frac{dh}{d\sigma}\frac{\partial \sigma}{\partial t} = -C\frac{dh}{d\sigma}, \qquad \frac{\partial h(x-Ct)}{\partial x} = \frac{dh}{d\sigma}\frac{\partial \sigma}{\partial x} = \frac{dh}{d\sigma}$$

Hence (2.3.30) reduces to an ordinary differential equation,

$$-C\frac{dh}{d\sigma} + \frac{\rho g \cos\theta}{3\mu}\frac{d}{d\sigma}\left[h^3\left(\tan\theta - \frac{dh}{d\sigma}\right)\right] = 0 \tag{2.3.2}$$

Integrating once we get

$$-Ch + \frac{\rho g \cos\theta}{3\mu}\left[h^3\left(\tan\theta - \frac{dh}{d\sigma}\right)\right] = \text{constant}$$

Let the gravity current advance along a dry bed, then $h = 0$ is a part of the solution. The constant of integration must be set to zero. Introducing the dimensionless variables

$$h = H_c h', \quad \sigma = L_c \sigma', \quad \text{with} \;\; L_c = H_c/\tan\theta, \tag{2.3.3}$$

where H_c is the maximum depth far upstream, we get

$$-\frac{3C\mu}{\rho g H_c^2 \sin\theta} h' + h'^3\left(1 - \frac{dh'}{d\sigma'}\right) = 0, \tag{2.3.4}$$

Let the gravity current be uniform far upstream, then

$$h' \to 1, \quad \frac{dh'}{d\sigma'} \to 0, \quad \text{as} \;\; \sigma' \to -\infty. \tag{2.3.5}$$

It follows that

$$\frac{3C\mu}{\rho g H_c^2 \sin\theta} = 1$$

or,

$$\boxed{C = \frac{\rho g H_c^2 \sin\theta}{3\mu}} \tag{2.3.6}$$

and

$$h'\left[-1 + h'^2\left(1 - \frac{dh'}{d\sigma'}\right)\right] = 0, \tag{2.3.7}$$

One of the solution is $h' = 0$, representing the dry bed. For the nontrivial solution, we rewrite

$$d\sigma' = -\frac{h^2 dh}{1-h^2} = dh\left[1 - \frac{1}{2}\left(\frac{1}{1-h} + \frac{1}{1+h}\right)\right] \tag{2.3.8}$$

which can be integrated to give

$$\boxed{h' + \frac{1}{2}\log\left(\frac{1-h'}{1+h'}\right) = \sigma' - \sigma_o'} \tag{2.3.9}$$

This is an implicit relation between h' and σ', and represents a smooth surface decreasing monotonically from $h = 1$ at $\sigma' \sim -\infty$ to $h' = 0$ at the front $\sigma' = \sigma_o'$, as plotted in Figure 2.3.1. Note from (2.3.8) that $d\sigma'/dh' = 0$ when $h' = 0$, implying infinite slope at the tip of the gravity current. This infinity violates the original approximation that $dh'/d\sigma' = O(1)$. Fortunately it is highly localized and does not affect the validity of the theory elsewhere (see Liu & Mei, 1989, JFM).

Figure 2.3.1: Gravity current down an inclined plane

Eq. (2.3.6) tells us that the speed of the front is higher for a thicker layer, steeper slope or smaller viscosity. This relation can be confirmed by a quicker argument. In the fixed frame of reference, the total flux must be equal to CH. therefore C must be equal to the depth-averaged velocity $\overline{u}$ which is given by (2.3.19) with $\partial h/\partial x = 0$.

A similar analysis has been applied to a fluid-mud which is non-Newtionian characterized by the yield stress. Laboratory simulations have been reported by Liu & Mei (*J. Fluid Mech.* 207, 505-529.) who used a kaolinite/water mixture. Figure 2.3.2 shows the setup of the inclined flume and Figure 2.3.3 shows the recorded profiles of the gravity current along with the theory . The agreement is very good, despite the steep front where the approximation is locally invalid.

Figure 2.3.2: Experiment setup for a mud current down an inclined plane. From Liu & Mei 1989.

Figure 2.3.3: Profiles of a mud current down an inclined plane. From Liu & Mei 1989.