MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Civil and Environmental Engineering

1.731 Water Resource Systems

Lecture 9, Linear Programming Sensitivity Analysis, Oct. 5, 2006

Overview

An LPP depends on 3 types of **problem inputs**:

- **Right-hand side** values (e.g resource limits) b_i
- **Objective function** coefficients (e.g. unit costs) c_j
- **Technological** coefficients (e.g. resource requirements) A_{ij}

All may be uncertain or subject to change.
We wish to investigate **sensitivity** of optimal solution to these coefficients.

Suppose that optimal solution x^* lies at a feasible region **corner** defined by $\rho^* = m_A^* = n$ linearly independent constraint gradients, with a corresponding set of active constraints $\mathcal{C}(x^*)$.

When the problem inputs are changed slightly the active set $\mathcal{C}(x^*)$ may remain the same while the optimum solution may or may not change. As the input change becomes greater $\mathcal{C}(x^*)$ may also change. Each input type behaves somewhat differently.

Right-hand Side Values

As the right-hand side values change:

- The position but not the gradient of the corresponding constraint function changes.
- The shape of the feasible region changes.
- The optimum objective function value and optimum solution x^* change
- Eventually the active set $\mathcal{C}(x^*)$ also changes.

Focus on example when b_1 (water availability) changes:

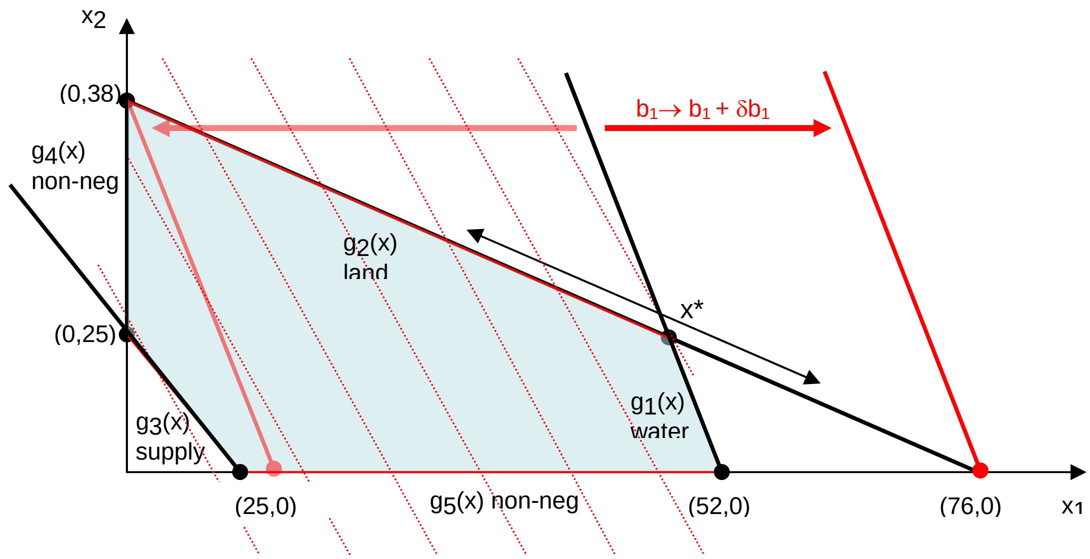
$$g_1(x) = 2x_1 + x_2 \leq b_1 = 104$$

change $b_1 \longrightarrow b_1 + \delta b_1$

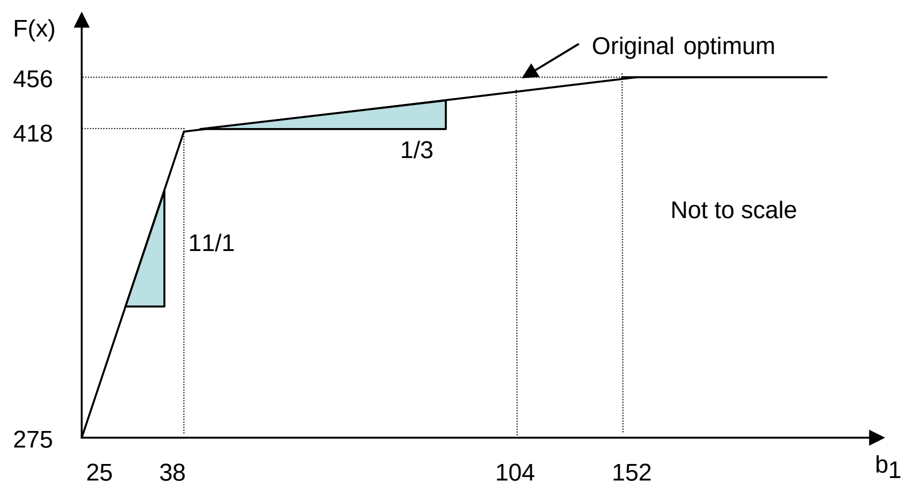
If $38 \leq b_1 + \delta b_1 \leq 152$ the water $g_1(x)$ constraints and land $g_2(x)$ constraints remain active so $\mathcal{C}(x^*) = \{1, 2\}$ does not change. Outside this range the land constraint is no longer active, so the constraint set changes to either $\mathcal{C}(x^*) = \{1, 4\}$ or $\mathcal{C}(x^*) = \{1, 5\}$.

As b_1 **increases** water is eventually no longer limiting and only Crop 1 is grown because it uses less land.

As b_1 **decreases** water is eventually too limited to permit all the land to be used and only Crop 2 is grown because it uses less water.



The objective function changes continuously as b_1 changes. Slope is constant and equal to λ_i so long as $\mathcal{C}(x^*)$ remains the same;

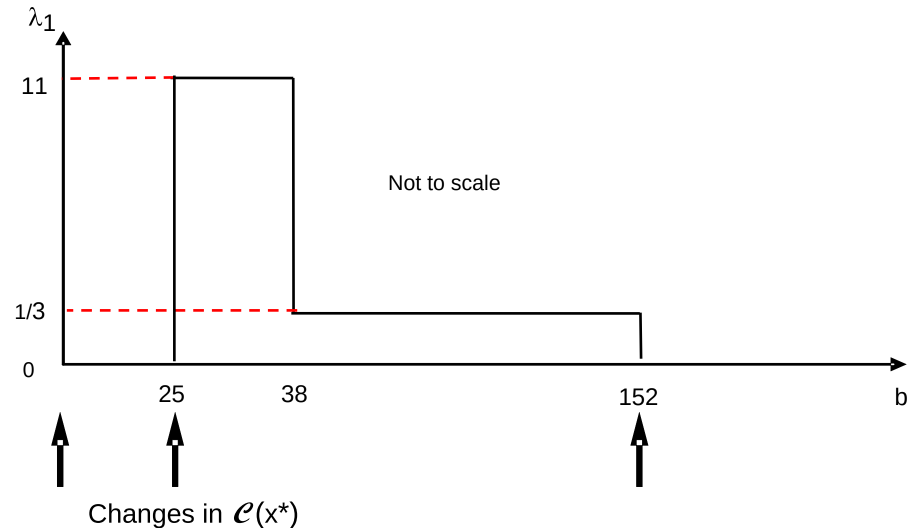


In general, slope of each segment = $\partial F(x)/\partial b_i = \lambda_i$ = **sensitivity of objective to b_i**. for a given $\mathcal{C}(x^*)$. For this example, this sensitivity has units of $/tonnes of water, which measures the value of additional water. Accordingly, λ_i is called the **shadow price** of water.

In the crop allocation example the $\partial F(x)/\partial b_i = \lambda_i$ vs. b_1 plot shows how the water shadow price **decreases** as available water **increases**. This plot is called a **derived demand**, since it indicates how the demand for water (price that farmer is willing to pay) changes as more water becomes available. For $b_1 > 152$ additional water cannot be used and shadow price drops to $\lambda_i = 0$.

Shadow price for a resource always **decreases** as more resource becomes available.

Note some constraint gradients become linearly dependent at point where $\mathcal{C}(x^*)$ changes.



Objective Function Coefficients

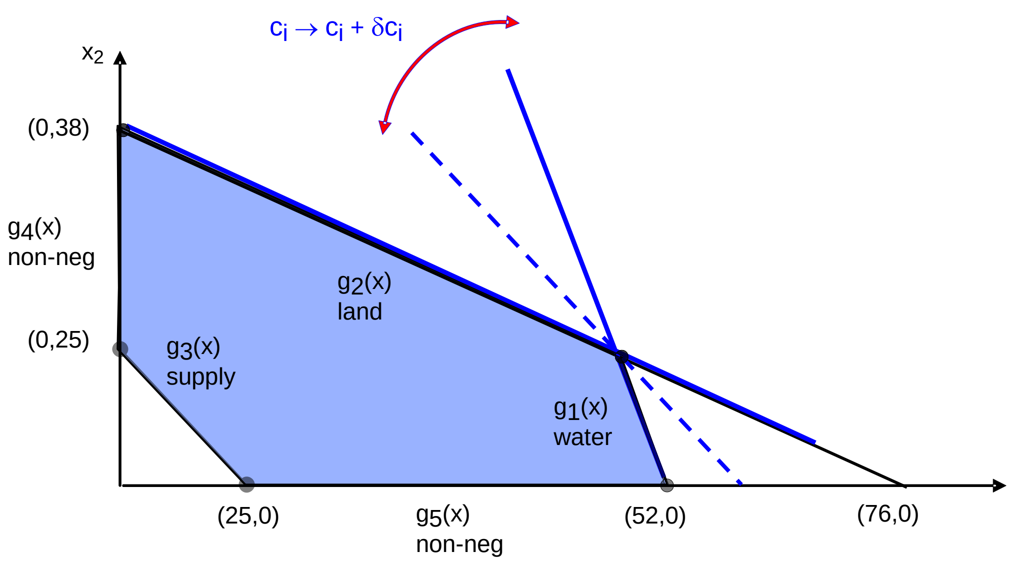
When a LPP objective function coefficient c_i changes

- The objective function contourn planes rotate.
- The feasible region does not change.
- The optimum objective function value changes but the optimum solution x^* does not change so long as the active constraint set $\mathcal{C}(x^*)$ remains the bsame
- Eventually $\mathcal{C}(x^*)$ also changes.

Focus on example when c_1 (price of Crop 1) changes:

$$F(x) = c_1 x_1 + c_2 x_2 = 6 x_1 + 11 x_2$$

change $c_1 \longrightarrow c_1 + \delta c_1$



If $11/2 \leq c_1 + \delta c_1 \leq 22$ the land $g_2(x)$ and water $g_1(x)$ constraints remain active so $\mathcal{C}(x^*) = \{1, 2\}$ does not change. Outside this range the active constraint set changes to either $\mathcal{C}(x^*) = \{2, 4\}$ or $\mathcal{C}(x^*) = \{1, 5\}$ and the optimum solution changes to either (0, 38) or (152, 0).

> As c_1 **increases** Crop 1 becomes more valuable and eventually all water is devoted to Crop 1 while some land goes unused.
>
> As b_1 **decreases** Crop 2 becomes more valuable and eventually all land is devoted to Crop 2 while some water goes unused.

The objective function changes continuously as c_1 changes. Slope is constant and equal to x_1^* so long as $\mathcal{C}(x^*)$.

In general, slope of each segment = $\partial F(x)/\partial c_i = x_i^*$ = **sensitivity of objective to c_i**. for a given $\mathcal{C}(x^*)$.

Objective function and some constraint gradients become linearly dependent at point where $\mathcal{C}(x^*)$ changes.

Technological Coefficients

When a LPP technological coefficient A_{ij} changes

- The corresponding constraint function gradients (and tangent planes) rotate.
- The shape of the feasible region changes.
- The optimum objective function value and optimum solution x^* change
- Eventually the active constraint set $\mathcal{C}(x^*)$ also changes.

Some constraint gradients become linearly dependent at point where $\mathcal{C}(x^*)$ changes.

When the technological coefficients are resource requirements (as in this example) coefficient changes favor one or another decision variable and the solution changes accordingly. Details can be worked out following an approach similar to that outlined above.

Parametric analysis

Some LPP optimization software automatically determines sensitivities and all values where the active constraint set $\mathcal{C}(x^*)$ over the feasible range of problem input values. In GAMS this must be done by looping through discrete values of the changing input, re-solving the optimization problem each time.