12.005 Lecture Notes 15

Elasticity

So far:

Stress $\rightarrow$ angle of repose vs accretionary wedge

Strain $\rightarrow$ reaction to stress $\rightarrow$ but how?

Constitutive relations

$$\tau_{ij} = \tau_{ij}\left(\varepsilon_{kl}\right); \quad \varepsilon_{ij} = \varepsilon_{ij}\left(\tau_{kl}\right)$$

For example,

- Elasticity
  - Isotropic
  - Anisotropic
- Viscous flow
  - Isotropic
  - Anisotropic
- Power law creep
- Viscoelasticity

Trade offs:

simplicity	$\leftrightarrow$	**realism**
constant		variable
isotropic		anisotropic
elastic, viscous		viscoelastic
history independent		history dependent

Tensors

Most physical quantities that are important in continuum mechanics like temperature, force, and stress can be represented by a tensor. Temperature can be specified by stating a single numerical value called a scalar and is called a zeroth-order tensor. A force, however, must be specified by stating both a magnitude and direction. It is an example of a first-order tensor. Specifying a stress is even more complicated and requires stating a magnitude and two directions—the direction of a force vector and the direction of the normal vector to the plane on which the force acts. Stresses are represented by second-order tensors.

Tensors are quantities independent of coordinate system.

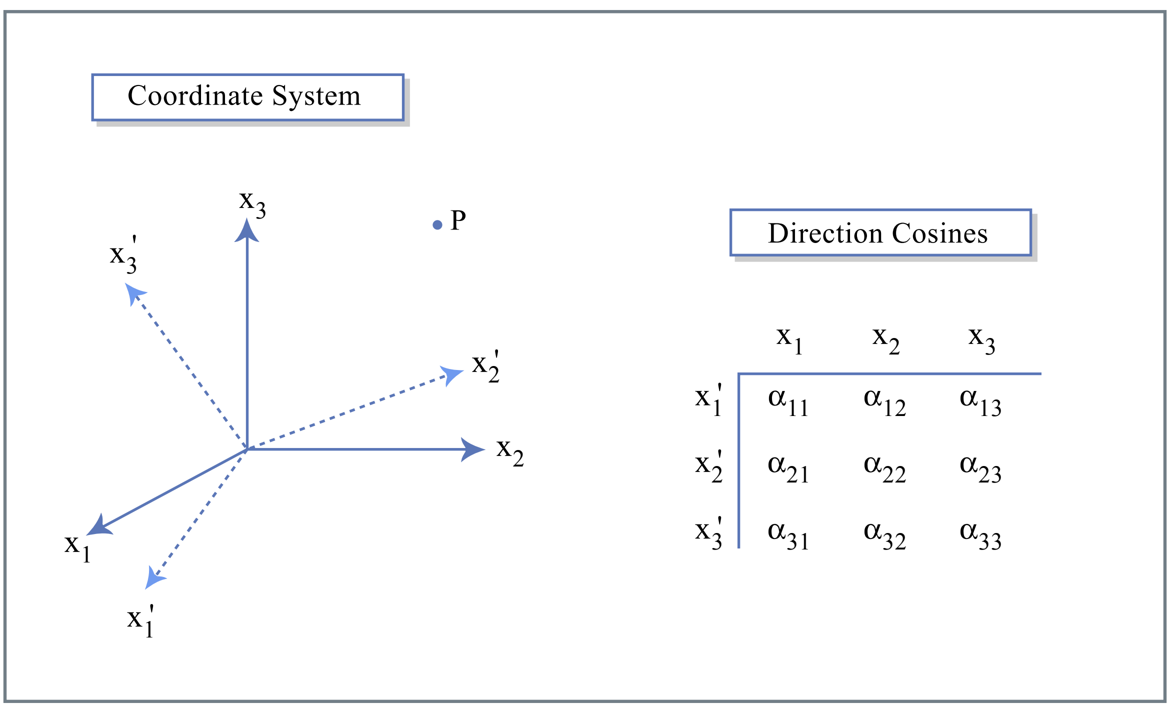


Figure 15.1

Figure by MIT OCW.

$$\alpha_{ij} = \cos\phi_{ij}$$

where ϕ_{ij} is the angle of primed to original.

$$x_i' = \alpha_{ij} x_j$$
$$x_i = \alpha_{ji} x_j'$$
$$\alpha_{ij} = \frac{\partial x_i'}{\partial x_j} = \frac{\partial x_j}{\partial x_i'}$$

Tensors:

a. 0th order (scalar) – quantity dependent only on position
b. 1st order (3^1 components) $A_i' = \alpha_{ij} A_j$
c. 2nd order ($3^2 = 9$ components) $A_{ij}' = \alpha_{is}\alpha_{jk} A_{sk}$
d. 3rd order ($3^3 = 27$ components) $A_{ijk}' = \alpha_{is}\alpha_{jt}\alpha_{kp} A_{stp}$
e. 4th order ($3^4 = 81$ components) $A_{ijkl}' = \alpha_{is}\alpha_{jt}\alpha_{kp}\alpha_{lq} A_{stpq}$

Linear Elastic Solid

$\tau_{ij} = c_{ijkl} e_{kl}$
c_{ijkl} is elastic modulus tensor
$c_{ijkl} = \underline{\text{constants}}$ ($\neq$ history, $\neq$ displacement, $\neq$ time)

$3^4 = 81$ components

$\tau_{ij} = \tau_{ji}$, $e_{ij} = e_{ji}$ → cuts to 36

Strain Energy

$$U = \frac{1}{2}\tau_{ij} e_{ij} \Rightarrow \tau_{ij} = \frac{\partial U}{\partial e_{ij}}$$

Write in terms of powers of e_{ij}

$U = \alpha + \cancel{\beta_{ij}} e_{ij} + \gamma_{ijkl} e_{ij} e_{kl}$

0 (to avoid spontaneous expansion, contraction)

$$\frac{\partial U}{\partial e_{ij}} = \left(\gamma_{ijkl} + \gamma_{klij}\right) e_{kl}$$

$$\downarrow$$

$$c_{ijkl} = c_{klij} \Rightarrow 21 \text{ individual components.}$$

21 components – data fitting – ugh!

Lots of available information, but lots of hard work.

Reality – 21 components (triclinic)

$\downarrow$

Simplicity – derive in homework

Isotropic

$$c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

λ and μ are Lame parameters.

$$\boxed{\tau_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}}$$

Can also express

$$e_{ij} = S_{ijkl}\tau_{kl}$$

$$\uparrow$$

Compliance tensor

Aside

$$e_{11} = S_{1132}\tau_{32}$$

$$\uparrow$$

triclinic, trigonal

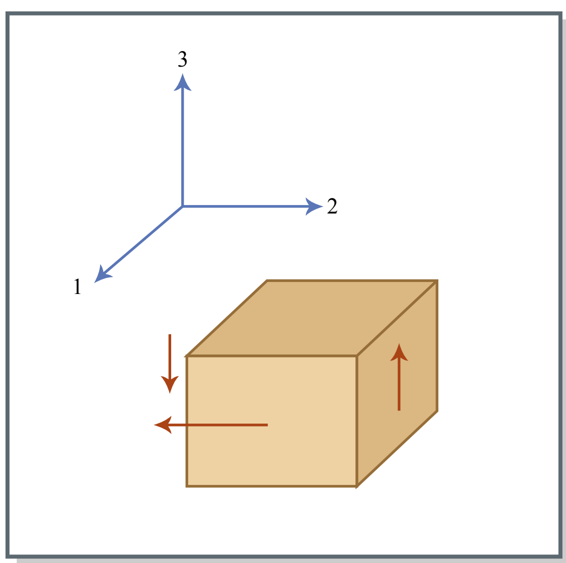


Figure 15.2

Figure by MIT OCW.

Sheer leads to lengthening.

Isotropic – can relate c_{ijkl} and S_{ijkl} directly

$$\tau_{ii} = \lambda e_{kk}\delta_{ii} + 2\mu e_{ii} = (3\lambda + 2\mu)e_{ii}$$

$$2\mu e_{ij} = \tau_{ij} - \frac{\lambda\delta_{ij}}{2\mu + 3\lambda}\tau_{kk}$$

Conventional moduli:

1. Hydrostatic comp.

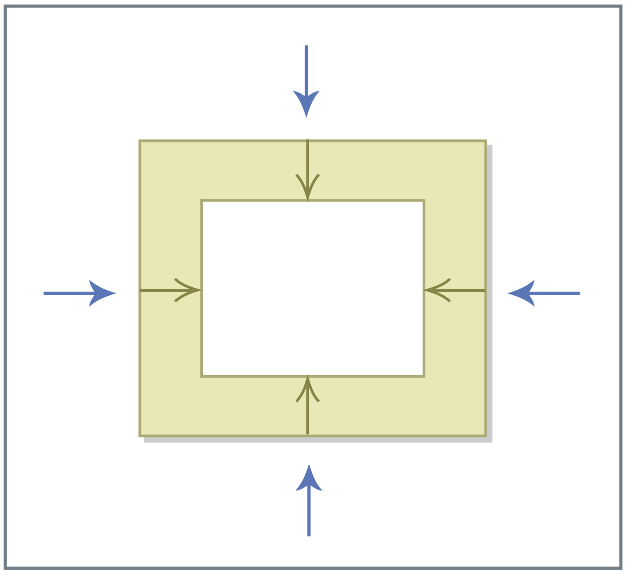


Figure 15.3

Figure by MIT OCW.

$$\tau_{ij} = -p\delta_{ij}$$

$$\tau_{ii} = -p\delta_{ii} = 3\lambda e_{kk} + 2\mu e_{ii}$$

$$= -3p = (3\lambda + 2\mu)e_{ii}$$

$$-\frac{p}{e_{ii}} = -\frac{VP}{\Delta V} \equiv K$$

where $K = \lambda + 2/3\mu$ is bulk modulus.

2. Uniaxial stress

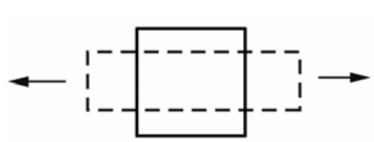


Figure by MIT OCW.

$$\tau_{11} = T$$

other $\tau_{ij} = 0$

$$2\mu e_1 = T - \frac{\lambda}{2\mu + 3\lambda} T$$

$$\frac{T}{e_1} \equiv E \ \ \text{(sometimes } Y\text{)}$$

where $E = \dfrac{\mu(2\mu + 3\lambda)}{\mu + \lambda}$ is Young's modulus

Hook's law:

$$T = Ee$$

$$\frac{e_{22}}{e_{11}} = \frac{e_{33}}{e_{11}} \equiv -\nu$$ This is called Poisson's ratio.

$$2\mu e_{22} = -\frac{\lambda}{2\mu + 3\lambda} \tau_{11} \quad \Rightarrow \quad \nu = \frac{\lambda}{2(\mu + \lambda)}$$

$$\theta = e_{11} + e_{22} + e_{33} = e_{11}(1 - 2\nu)$$

fluid: $\mu \to 0 \quad \Rightarrow \quad \nu \to \dfrac{1}{2}$

most material: $\nu = 0.2 - 0.3$

$$\nu = \frac{1}{4} \quad \Rightarrow \quad \lambda = \mu$$ It is Poisson solid.

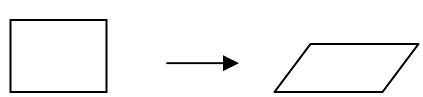
steel: ν ; $0.3 - 0.33$

seismically measured

$$v_p = \sqrt{\frac{\lambda + 2\mu}{\rho}} \qquad v_s = \sqrt{\frac{\mu}{\rho}}$$

compare $v_p, v_s \ \to \ \nu \ \to$ discriminate rock types

3. Simple shear



$$\tau_{12} = \tau_{21} = \tau$$

$$\tau_{12} = 2\mu e_{12} = 2Ge_{12}$$

where G is shear modulus.

Note: Among λ, μ, K, ν, E, G only two are independent.

Useful forms:

$$\tau_{11} = (\lambda + 2\mu)e_{11} + \lambda e_{22} + \lambda e_{33}$$
$$\tau_{22} = \lambda e_{11} + (\lambda + 2\mu)e_{22} + \lambda e_{33}$$
$$\tau_{33} = \lambda e_{11} + \lambda e_{22} + (\lambda + 2\mu)e_{33}$$

or

$$e_{11} = \frac{\tau_{11}}{E} - \frac{\nu\tau_{22}}{E} - \frac{\nu\tau_{33}}{E}$$
$$e_{22} = -\frac{\nu\tau_{11}}{E} + \frac{\tau_{22}}{E} - \frac{\nu\tau_{33}}{E}$$
$$e_{33} = -\frac{\nu\tau_{11}}{E} - \frac{\nu\tau_{22}}{E} + \frac{\tau_{33}}{E}$$