12.520 Lecture Notes #10

Finite Strain

The "paradox" of finite strain and preferred orientation – pure vs. simple shear

Suppose: 1) Anisotropy develops as mineral grains grow such that they are preferentially oriented with the "easy slip" direction facilitating deformation.

2) New grain growth is determined by the state of stress. (What else would a mineral grain "know" about?)

3) The state of stress for pure shear and simple shear are identical – pure and simple shear stress differ only by a rotation of 45° in coordinate system.

4) In pure shear there is only one slip direction, whereas in pure shear, 2 systems could be active.

Paradox: If the stress state is the same, how does the rock "know" to have 1 preferred orientation in simple shear, but not in pure shear?

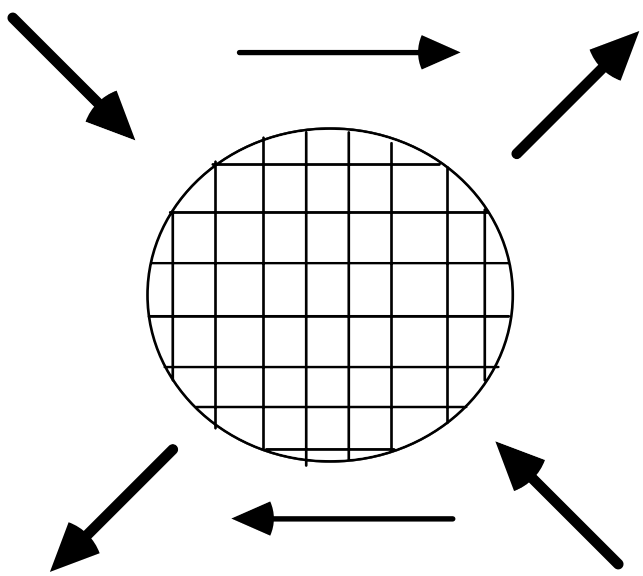

Fig. 10_1

Up to now, we have considered infinitesimal strain, where $e_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2 << 1$.

For the kinds of deformations common in structural geology, as well as for the deforming telephone book example shown in class, this condition of "infinitesimal" deformation no longer holds.

Life becomes complicated. In particular, it's important to know whether one is coming or going:

• **Reference frame**:

Consider a medium undergoing motion & deformation, with the displacement a (vector) function of position.

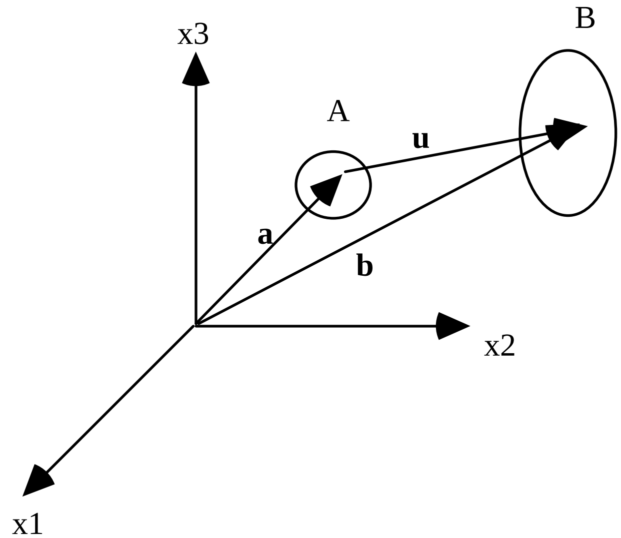

Fig. 10_2

$$b_j = f(a_i)$$

or

$$b_j = b_j(a_i)$$

Since the mapping a_i <—> b_j is one-to-one, we could also say

$$a_j = a_j(b_i).$$

It turns out (consider the telephone book example in class) that the description of strain depends on whether the reference frame is:

Lagrangian ∫ fixed to particles (the "a" frame)

$$u_i = b_i(a_j) - a_i$$

Example: elasticity -- the particles return to where they started

Eulerian ∫ fixed to space (the "b" frame)

$u_i = b_i - a_i(b_j)$

Example: fluid mechanics, geology -- no idea where the particles started or where they will end up; no stable frame fixed to the materials.

Finite strain depends on whether the reference frame is fixed to the particle or to "space:"

Lagrangian strain: $E_{ij} = (\partial u_i/\partial a_j + \partial u_j/\partial a_i + \partial u_l/\partial a_i \, \partial u_l/\partial a_j)/2$

Eulerian strain: $e_{ij} = (\partial u_i/\partial b_j + \partial u_j/\partial b_i - \partial u_l/\partial b_i \, \partial u_l/\partial b_j)/2$

When strain is small, the 3rd term is small^2, and both strains converge to the infinitesmal value.

Physical interpretation:

diagonal elements - e.g., elongation, given by $ds = (1 + E_1)\, ds_0$.

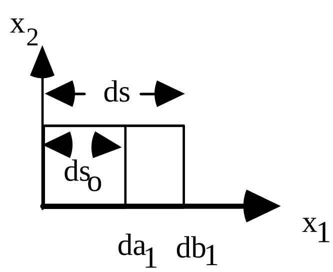

Fig. 10_3

For the Lagrangian system, $E_1 = (1 + 2E_{11})^{1/2} - 1$

off diagonal —>change in angles of coordinate axes:

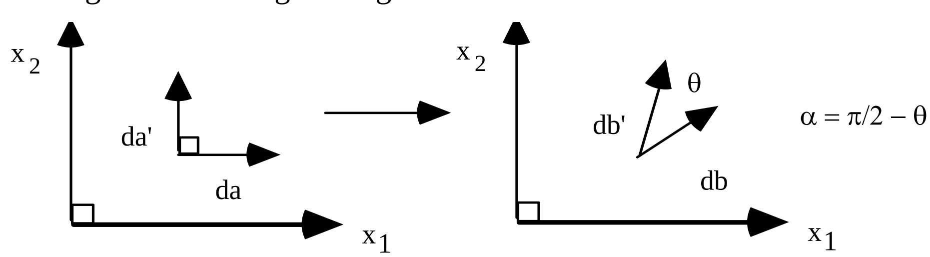

Fig. 10_4

$\sin a = 2E_{12}/[(1 + E_{11})(1 + E_{22})]1/2$

These quantities go to the infinitesimal values when $E_{ij} << 1$.

Because $e_{ij} \neq E_{ij}$, the principal values & principal directions of the Lagrangian and Eulerian strain tensors differ. We will discuss these later.

Consider two particles initially separated by da_i (note: not just a function of strain tensor) which undergo a displacement u_j, and are then separated by dx_i.

Lagrangian:

$$dx_i = da_i + \frac{\partial u_i}{\partial a_j} daj = \left[\delta_{ij} + \frac{\partial u_i}{\partial a_j}\right] da_j$$

This can be written, including the finite strain tensor,

$$dx_i = \left[\delta_{ij} + \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i} + \frac{\partial u_s}{\partial a_i}\frac{\partial u_s}{\partial a_j}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} - \frac{\partial u_j}{\partial a_i} - \frac{\partial u_s}{\partial a_i}\frac{\partial u_s}{\partial a_j}\right)\right] da_j$$
$$= \left[\delta_{ij} + \mathrm{E}_{ij} + \Omega_{ij}\right] da_j$$

Ω_{ij} — Lagrangian rotation tensor

if $\mathbf{E_{ij}} = \mathbf{0} \rightarrow$ rotation alone

Similarly, for the **Eulerian** case:

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial b_j} - \frac{\partial u_j}{\partial b_i} + \frac{\partial u_s}{\partial b_i}\frac{\partial u_s}{\partial b_j}\right)$$
$$da_i = \left[\delta_{ij} + e_{ij} + \omega_{ij}\right] dx_j$$

for small deformation: Ω_{ij} rotation needed to bring a fiber da_i into position.

For large deformation: no easy way to separate into pure rotation and pure shear—the two are intimately connected.

THIS IS OF UTMOST IMPORTANCE IN PROBLEMS LIKE THE DEVELOPMENT OF ROCK FABRIC. ALSO—REGIONAL TECTONICS.

Consider homogeneous strain

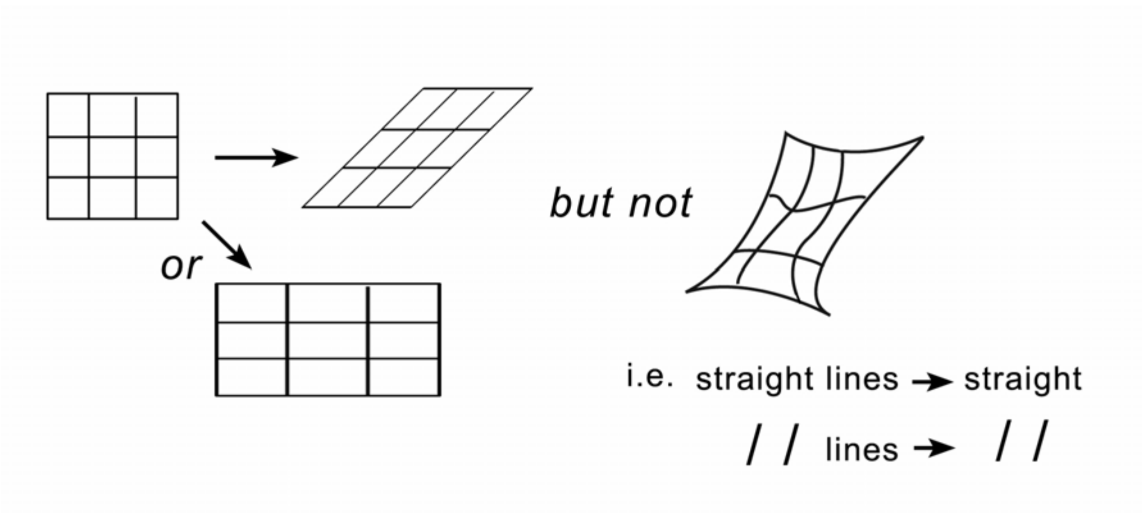

Fig. 10_5

This holds if

$$E_{ij} = \frac{1}{2}(\delta_{lm}\frac{\partial r_l}{\partial R_i}\frac{\partial r_m}{\partial R_j} - \delta_{ij}) \neq E_{ij}(R_i)$$

$$\Rightarrow \frac{\partial 1}{\partial 2} = \text{ constant } \Rightarrow \underset{\sim}{R} \,\&\, \underset{\sim}{r} \text{ linearly related}$$

Define $R_i \to x, y \;\; r_i \to x', y'$

In general

$$x' = ax + by$$

$$y' = cx + dy$$

$$E_{ij} = \frac{1}{2}\begin{bmatrix} a^2 + c^2 - 1 & ab + cd & 0 \\ ab + cd & b^2 + d^2 - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Examples

$$x' = kx$$
$$y' = y$$

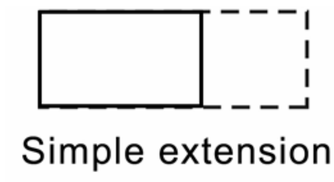

Fig. 10_6

$$x' = k_1 x$$
$$y' = k_2 y$$

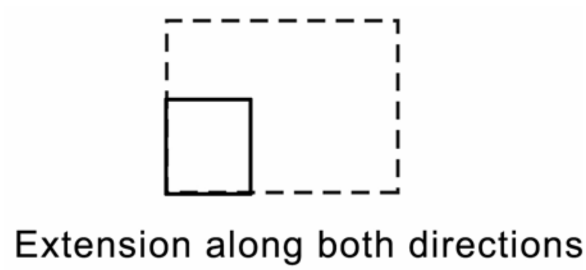

Fig. 10_7

$$x' = kx$$
$$y' = \frac{1}{k} y$$

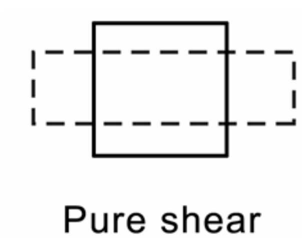

Fig. 10_8

$$x' = x + 2sy$$
$$y' = y$$

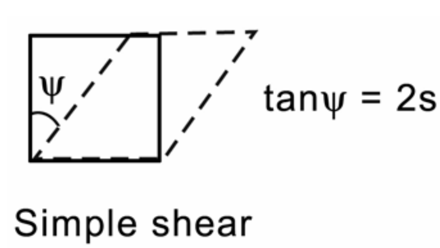

Fig. 10_9

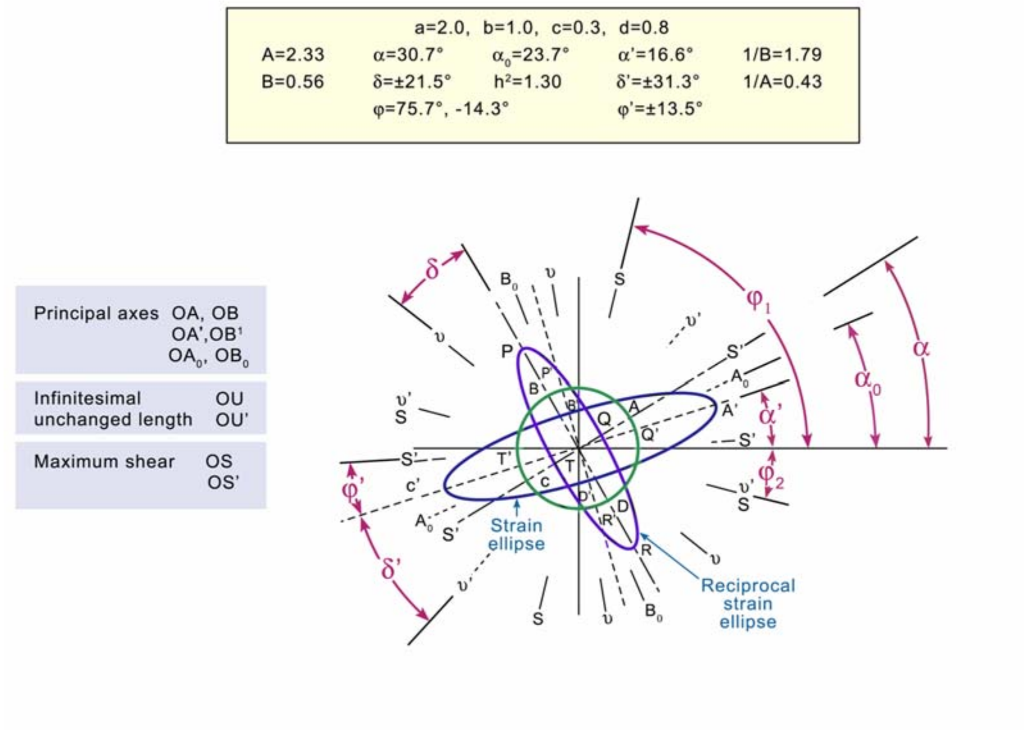

Fig. 10_10

Unstrained		Strained
$x = \frac{1}{h^2}(dx' - by')$		$x' = ax + by$
$y = \frac{1}{h^2}(-cx' + ay')$	$h^2 = ad - bc$	$y' = cx + dy$
	Principal Axes A, B	
	Roots of $R^4 - (a^2 + b^2 + c^2 + d^2)R^2 + h^4 = 0$	
$\tan 2\alpha = \frac{2(ab + cd)}{a^2 + c^2 - b^2 - d^2}$		$\tan 2\alpha' = \frac{2(ac + bd)}{a^2 + b^2 - c^2 - d^2}$
	No Length Change	
$\tan\delta = \pm[(1 - B^2)/(A^2 - 1)]^{1/2}$		$\tan\delta' = \pm\frac{B(A^2 - 1)^{1/2}}{A(1 - B^2)^{1/2}}$
relative to reciprocal axis		relative to principal axis
	Maximum Shear	
$\tan 2\varphi = \frac{b^2 + d^2 - a^2 - c^2}{2(ab + cd)}$		$\tan\varphi' = (B/A)$
relative to coord. axis		relative to principal axis
	Principal axis of infinitesimal strain	
	$\alpha_0 = \frac{\alpha + \alpha'}{2}$	