Lecture 7: Strain

Lectures 5 explored faults and brittle deformation across a failure plane. The next set of lectures deals with ductile deformation: stretching, compressing, and twisting materials into different shapes without breaking them. This lecture and lecture 9 treat the mathematical description of ductile deformation called strain. Lectures 11 and 12 describe the relationship between strain and stress.

1. Finite Strain and Infinitesimal Strain

Finite strain is characterized by big changes in the shape of a body. Infinitesimal strain is characterized by smaller changes. The next couple of lectures treat infinitesimal strain only because the small changes in shape cause squared and higher-order terms in the strain equation to be very small. These terms are negligible and make the strain equation simpler.

2. Infinitesimal Strain

Definition

Stress is a second-order tensor that provides a relationship between a normal vector and a traction vector. Strain is also a second-order tensor that provides a relationship between two vectors: a position vector dx_i and a displacement vector du_i. Consider the following picture:

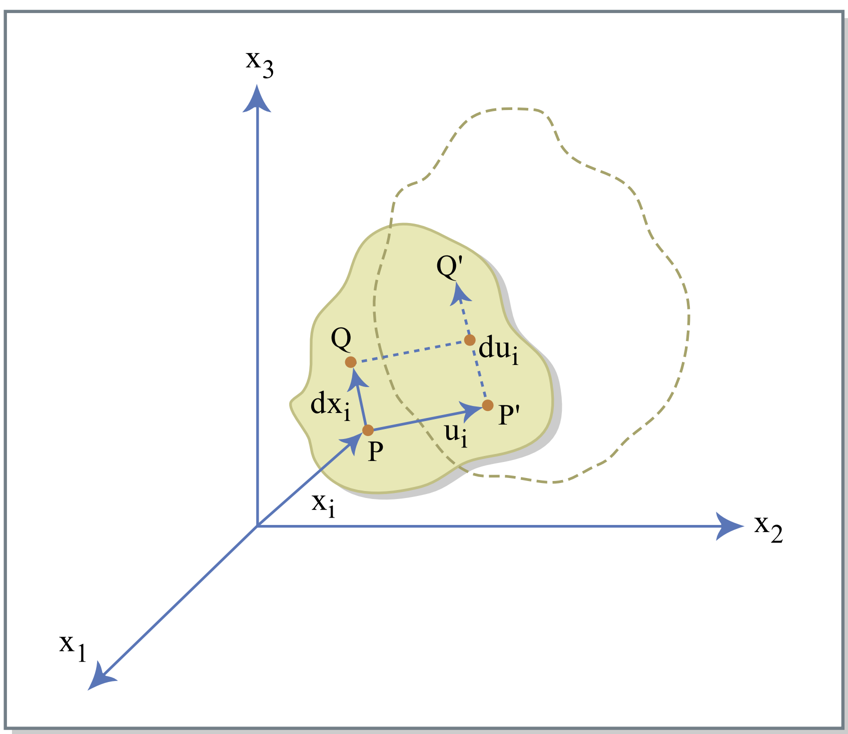

Figure 7.1

Figure by MIT OCW.

Assuming that the deformation is continuous, *du* is related to *dx* by the equation:

$$du_i = \frac{\partial u_i}{\partial x_1} dx_1 + \frac{\partial u_i}{\partial x_2} dx_2 + \frac{\partial u_i}{\partial x_3} dx_3 \quad i = 1,2,3$$

The matrix equation is written:

$$\begin{bmatrix} du_1 \\ du_2 \\ du_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

The nine-term matrix is the strain tensor.

<u>Interpreting the Strain Tensor</u>

The strain tensor looks complicated. What it means in terms of stretching, twisting, and rotating a body is not obvious. A few simple examples, however, help illustrate what its components mean in terms of different kinds of deformation.

a. Translations

In the case of rigid-body translations, *du* equals zero. The strain tensor has no sensitivity to them.

b. Rotations

Consider the following figure:

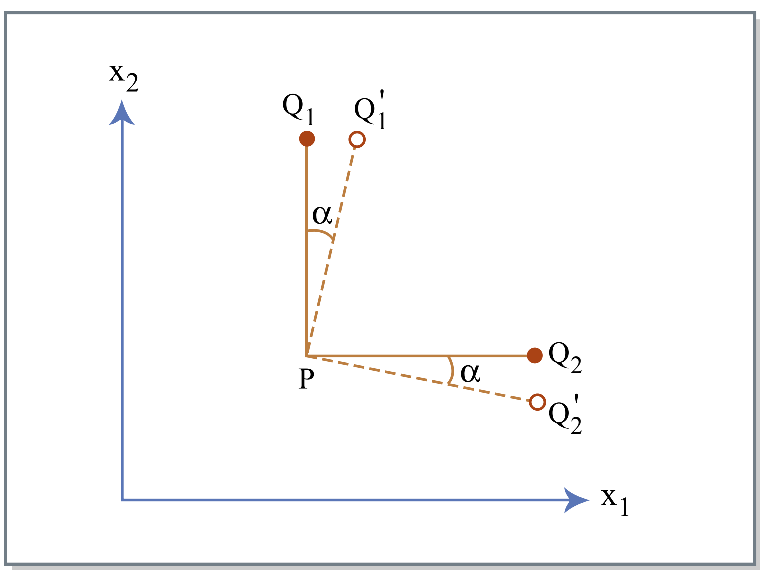

Figure 7.2

Figure by MIT OCW.

Two expressions for α can be written in terms of $\frac{\partial u_i}{\partial x_j}$:

$$\frac{\partial u_2}{\partial x_1} = -\tan\alpha$$
$$\frac{\partial u_2}{\partial x_1} = \tan\alpha$$

Because rotations in infinitesimal strain are small, the equations can be approximated using a small-angle identity:

$$\frac{\partial u_2}{\partial x_1} = -\alpha$$
$$\frac{\partial u_2}{\partial x_1} = \alpha$$

These equations can be combined to give a single expression for α:

$$\alpha = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right)$$

This expression can be generalized to represent any rotation:

$$\alpha = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = w_{ij}$$

The strain tensor can be rewritten to explicitly include the above expression:

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$= \quad \varepsilon_{ij} \quad + \quad w_{ij}$$

In this form, the second part of the strain tensor w_{ij} represents only rigid-body rotations. It is anti-symmetric. The first part ε_{ij} represents elongation, compression, and shear. It is symmetric.

c. Elongation and Compression

Consider the following figure:

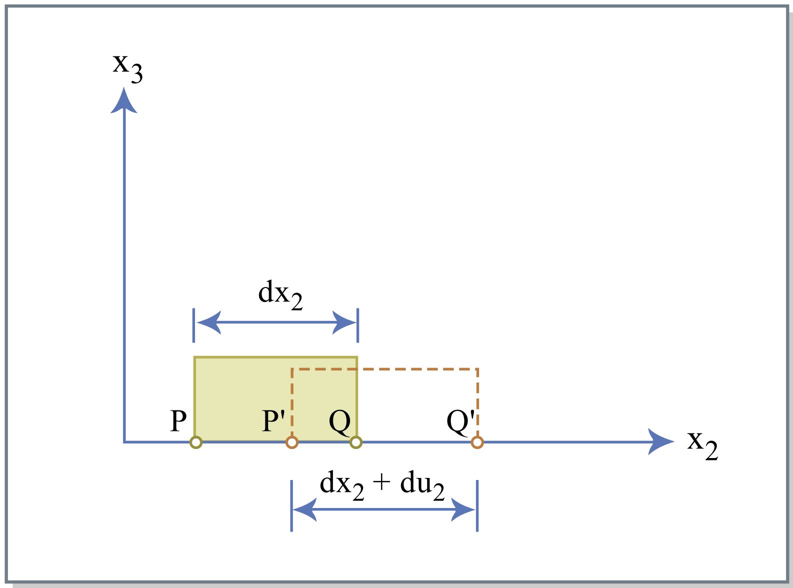

Figure 7.3
Figure by MIT OCW.

Since both dx and du are in the x_2 direction, the first part of the strain tensor ε_{ij} is given by

$$\varepsilon_{22} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{\partial x_2}\right) = \frac{\partial u_2}{\partial x_2}$$

Consequently, ε_{ij} i=j can be thought of as a measure of elongation in the direction i. It is equal to a change in length per unit length of a material. For example, if a metal rod of length l is stretched to a new length $l+\Delta l$, the elongation ε_{ij} is equal to $\frac{\Delta l}{l}$.

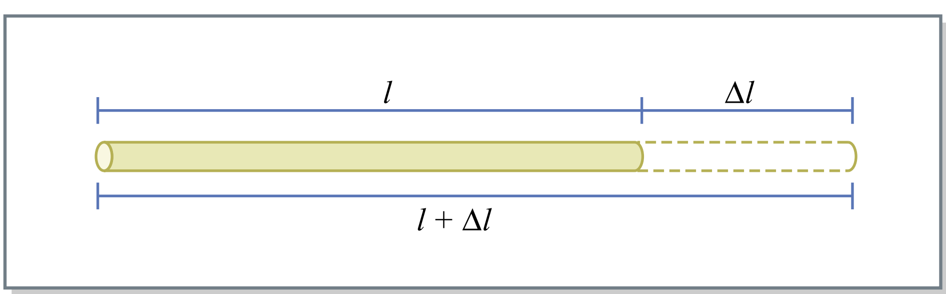

Figure 7.4
Figure by MIT OCW.

d. Shear

Consider the following picture:

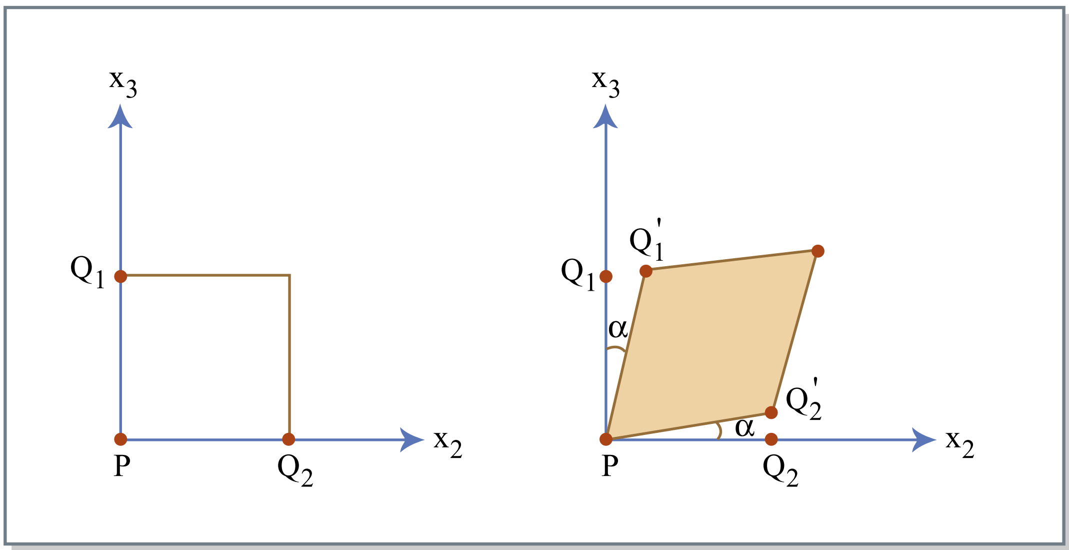

Figure 7.5

Figure by MIT OCW.

Just as in the case of rigid-body rotation, two expressions for φ can written in terms of $\frac{\partial u_i}{\partial x_j}$ using small-angle approximations:

$$\frac{\partial u_3}{\partial x_2} = \tan\varphi = \varphi$$
$$\frac{\partial u_2}{\partial x_3} = \tan\varphi = \varphi$$

These equations can be combined to give a single expression for φ:

$$\varphi = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) = \varepsilon_{23}$$

Consequently, while ε_{ij} i=j is a measure of elongation, ε_{ij} i≠j is a measure of shear.