Class 6: Early-Arrival Waveform Tomography
Mon, Sept 28, 2009

- Acoustic-wave equation in time and frequency domain
- Finite-difference wavefield simulation
- Time-domain versus frequency-domain inversion strategy
- Signal processing and wavelet extraction
- Computer exercise – invert synthetic waveform data

This class introduces a high-end imaging technology that is currently under research and development in academia and industry. Early-arrival waveform inversion is very promising in resolving hidden layers and complex velocity structures, and the computer capacity today is sufficient enough to deal with waveform tomography. This class shall offer a new commercial program that helps students to perform full wavefield modeling and waveform tomography.

Acoustic-wave equation

Ideally we would like to perform full waveform tomography by modeling with elastic wave equation, and inverting elastic wavefield for Vp, Vs, density, and Q at the same time. That sounds what it really should be done. Unfortunately, that tomography problem is too complicated. In stead, most of research attempts to solve a full-waveform tomography associated with **scalar acoustic wave equation** in time domain or in frequency domain.

We shall review two papers in this area, one for time-domain waveform tomography, and the other for frequency-domain waveform tomography.

References:

Sheng, J., A. Leeds, M. Buddensiek, and G. T. Schuster, 2006, Early arrival waveform tomography on near-surface refraction data: *Geophysics*, 71, No.4, U47-U57.
(Time-domain approach)

Ravaut, C., S. Opero, L. Improta, J. Virieux, A. Herrero, and P. Dell Aversana, 2004, Multiscale imaging of complex structures from multifold wide-aperture seismic data by frequency-domain full-wave tomography: application to a thrust belt, *Geophys. J. Int.* 159, 1032-1056.
(Frequency-domain approach)

Input: early-arrival seismic waveform data in time domain or frequency slices
Output: updated P-wave velocity model

Time-domain story:

Visco acoustic wave equation in time domain:

(1) $$\frac{1}{\kappa(\mathbf{r})}\frac{\partial^2 p(\mathbf{r},t|\mathbf{r}_s)}{\partial t^2} - \nabla \cdot \left[\frac{1}{\rho(\mathbf{r})}\nabla p(\mathbf{r},t|\mathbf{r}_s)\right] = s(\mathbf{r},t|\mathbf{r}_s),$$

$p(r,t|r_s)$ is the pressure field at position r at time t from a source at r_s, $k(r)$: bulk module, $\rho(r)$: density. $s(r,t|r_s)$ is the source function.

Scalar wave equation, assuming constant density, $c(r)^2=k(r)/\rho(r)= k(r)/\rho_0$:

(2) $$\frac{1}{c^2(r)}\frac{\partial^2 p(r,t\,|\,r_s)}{\partial^2 t} - \nabla^2 p(r,t\,|\,r_s) = s(r,t\,|\,r_s)$$

Question: can you derive equation (2) from (1) with the constant density assumption?

Forward modeling: synthetic data: convolution of Green's function with source wavelet:

(3)

$$p(r,t\,|\,r_s) = \int_{r'} G(r,t\,|\,r',0) * s(r',t\,|\,r_s)dr'$$

(4) Finite-difference solution to the following equation to obtain Green's function:

$$\frac{1}{c^2(r)}\frac{\partial^2 G(r,t\,|\,r',0)}{\partial^2 t} - \nabla^2 G(r,t\,|\,r',0) = \delta(r-r')\delta(t-0)$$

(5) Waveform Residuals:

$$\delta p(r_g,t\,|\,r_s) = \lfloor p_{obs}(r_g,t\,|\,r_s) - p_{calc}(r_g,t\,|\,r_s)\rfloor m(r_g,t\,|\,r_s)$$

(6) Objective Function: $\psi = ||\delta p\,||^2 + \tau\,||L(c)||^2$

(7) Gradient of Objective Function: $\partial\psi/\partial c = g(r) - \tau L^T L(c)$

$$g(r) = \frac{2}{c(r)} \sum_s \int_t \dot{p}(r,t \mid r_s)\, \dot{p}'(r,t \mid r_s)\, dt,$$

where:

$$\dot{p}(r,t \mid r_s) = \frac{\partial}{\partial t} p(r,t \mid r_s)$$

$$\dot{p}'(r,t \mid r_s) = \frac{\partial}{\partial t} p'(r,t \mid r_s)$$

p(r,t|r$_s$): forward-propagated wavefield
p'(r,t|r$_s$): back-projected waveform residuals

(8) Back projection of waveform residuals

$$p'(r,t \mid r_s) = \int_{r'} G(r,-t \mid r',0) * \delta s(r',t \mid r_s)\, dr',$$

where

$$\delta s(r',t \mid r_s) = \sum_g \delta(r' - r_g)\, \delta p(r_g,t \mid r_s)$$

(9) Nonlinear Conjugate Gradients:

$$d_k = -P_k g_k + \beta_k d_{k-1},$$

where:

P_k : geometric-spreading preconditioner;

$$\beta_k = \frac{g_k^T \bullet (P_k g_k - P_{k-1} g_{k-1})}{g_{k-1}^T \bullet P_{k-1} g_{k-1}} : \text{Polak – Ribiere formula}$$

(10) Update velocity model:

$$c_{k+1}(r) = c_k(r) + \lambda_k d_k(r),$$

where:

λ_k – – – Step Length

Frequency-domain story:

The 2-D visco-acoustic wave equation is written in the frequency domain as

$$\frac{\omega^2}{\kappa(x,z)} P(x,z,\omega) + \frac{\partial}{\partial x}\left(\frac{1}{\rho(x,z)}\frac{\partial P(x,z,\omega)}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\rho(x,z)}\frac{\partial P(x,z,\omega)}{\partial z}\right) = S(x,z,\omega) \quad (1)$$

where $\rho(x,z)$ is density, $\kappa(x,z)$ is the bulk modulus, ω is frequency, $P(x,z,\omega)$ is the pressure field and $S(x,z,\omega)$ is the source. Attenuation is easily implemented in this equation using complex

velocities $\tilde{c}$ (Toksoz & Johnston 1981).

$$\frac{1}{\tilde{c}} = \frac{1}{c}\left(1 + \frac{i}{2Q}\mathrm{sign}(\omega)\right) \quad (2)$$

where Q is the quality factor and ω is the angular frequency. The complex valued bulk modulus is given by $\tilde{\kappa}(x,z) = \rho(x,z)\tilde{c}(x,z)^2$.

Because density and quality factor are considered constant in this study, the visco-acoustic wave equation reduces to the scalar wave equation

$$\frac{\omega^2}{\tilde{c}^2(x,z)} P(x,z,\omega) + \triangle P(x,z,\omega) = S(x,z,\omega) \quad (3)$$

Forward modeling:

$$\mathbf{A}\mathbf{p} = \mathbf{s} \quad (4)$$

Inversion:

$$\delta\mathbf{m} = -\alpha\nabla_m E = -\alpha\mathrm{Re}\{\mathbf{J}^{\mathrm{T}}\delta\mathbf{d}^*\} \quad (6)$$

$$\frac{\partial\mathbf{p}}{\partial m_i} = -\mathbf{A}^{-1}\frac{\partial\mathbf{A}}{\partial m_i}\mathbf{p} \quad (7)$$

$$\delta m_i = \alpha\mathrm{Re}\left\{\mathbf{p}^{\mathrm{T}}\left[\frac{\partial\mathbf{A}^{\mathrm{T}}}{\partial m_i}\right]\mathbf{A}^{-1}\delta\mathbf{d}^*\right\}. \quad (8)$$

Stabilize the inversion:

$$\begin{aligned}\delta\mathbf{m} &= -\alpha(\mathrm{diag}\mathbf{H}_a + \epsilon\mathbf{I})^{-1}\mathbf{C}_m\nabla_m E \\ &= -\alpha(\mathrm{diag}\mathbf{H}_a + \epsilon\mathbf{I})^{-1}\mathbf{C}_m\mathrm{Re}\{\mathbf{J}^{\mathrm{T}}\delta\mathbf{d}^*\} \quad (9)\end{aligned}$$

Finite-difference modeling:

Model grid size, timestep, propagation velocity must meet CFL condition in time-domain explicit implementation:

CFL condition is a necessary condition for convergence while solving certain partial differential equations numerically. It arises when explicit time-marching schemes are used for the numerical solution. As a consequence, the timestep must be less than a certain time in many explicit time-marching computer simulations, otherwise the simulation will produce wildly incorrect results. The condition is named after Richard Courant, Kurt Friedrichs, and Hans Lewy who described it in their 1928 paper.

Perfect Matched Layer (PML) B.C.: see reference paper.

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