Final Review

1 Conservation Relations in Lagrangian form

- Mass:

$$\frac{dM}{dt} = 0 \tag{1}$$

- Linear mo:

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}_B + \mathbf{F}_S \tag{2}$$

- Energy

$$\frac{dE}{dt} = \dot{Q} - \dot{W} \tag{3}$$

2 Material derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \tag{4}$$

3 Velocity gradient tensor

$$\mathbf{G} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \tag{5}$$

$$\mathbf{G} = \frac{1}{2}\mathbf{G} + \frac{1}{2}\mathbf{G}^T + \frac{1}{2}\mathbf{G} - \frac{1}{2}\mathbf{G}^T \tag{6}$$

$$= \mathbf{e} + \frac{1}{2}\mathbf{r} \tag{7}$$

$$\begin{aligned} \mathbf{e} &\equiv \text{strain rate tensor} \\ \mathbf{r} &\equiv \text{rotation rate tensor} \end{aligned} \tag{8}$$

$$\mathbf{e} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} \end{bmatrix} \tag{9}$$

$$\mathbf{r} = \begin{bmatrix} 0 & -\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \\ \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_z \\ \omega_z & 0 \end{bmatrix} \tag{10}$$

$$\mathbf{M} \equiv \text{Linear uncertainty propagator}$$

$$\epsilon(t) = \mathbf{M}\epsilon(0) \tag{11}$$

$$[\mathbf{V}, \mathbf{\Lambda}] = \text{eig}(\mathbf{M}^T\mathbf{M}) \tag{12}$$

$$[\mathbf{U}, \mathbf{\Lambda}] = \text{eig}(\mathbf{M}\mathbf{M}^T) \tag{13}$$

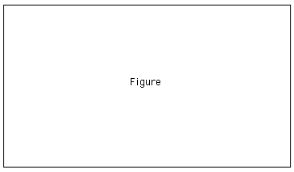

Figure 1: (`fig:ReviewLinUncerProp`) Linear uncertainty propagator.

4 Reynolds Transport Theorem

$$\frac{dC}{dt} = \frac{d}{dt}\iiint\limits_{\mathscr{V}} c \, d\mathscr{V} = \iiint\limits_{V} \frac{\partial c}{\partial t} dV + \iint\limits_{A} c \, \mathbf{u} \cdot d\mathbf{A} \tag{14}$$

$$\begin{aligned} C &\equiv \text{extensive (sum of parts)} \\ c &\equiv \text{intensive (stuff per volume)} \end{aligned}$$

5 Divergence (or Gauss) Theorem

$$\iiint\limits_{V} \nabla \cdot \mathbf{Q} \, dV = \iint\limits_{A} \mathbf{Q} \cdot d\mathbf{A} \tag{15}$$

Divergence on left, flux on right.

6 Another form of the RTT

$$\frac{dC}{dt} = \iiint\limits_{V} \left[\frac{\partial c}{\partial t} + \nabla \cdot (c \, \mathbf{u}) \right] dV \tag{16}$$

7 Yet Another form of the RTT

$$\frac{dC}{dt} = \iiint\limits_{V} \left[\frac{Dc}{Dt} + c \nabla \cdot \mathbf{u} \right] dV \tag{17}$$

$$\tag{18}$$

Use of RTT to go from Lagrange conservation laws (e.g. $\frac{dM}{dt} = 0$) to Eulearian (e.g. $\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0$)

8 Continuity

No approximation:

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0 \tag{19}$$

Boussinesq:

$$\nabla \cdot \mathbf{u} = 0 \tag{20}$$

Euler (but *can* have compressible Euler):

$$\nabla \cdot \mathbf{u} = 0 \tag{21}$$

9 Momentum

No approximation:

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{g} - \nabla p + \mu \nabla^2 \mathbf{u} + \frac{\mu}{3} \nabla (\nabla \cdot \mathbf{u}) \tag{22}$$

Boussinesq:

$$\rho_0 \frac{D\mathbf{u}}{Dt} = \rho' \mathbf{g} - \nabla p' + \mu \nabla^2 \mathbf{u} \tag{23}$$

Euler:

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{g} - \nabla p \tag{24}$$

10 Heat

No approximation: **(Jim – There are a couple approximations in there ;)**

$$\rho c_v \frac{DT}{Dt} = -p(\nabla \cdot \mathbf{u}) + \phi - \nabla \cdot \mathbf{q} \tag{25}$$

Boussinesq:

$$\frac{Dt}{Dt} = \frac{k}{pc_p} \nabla^2 T \tag{26}$$

11 State

Atmosphere:

$$p = \rho R T \tag{27}$$

Ocean:

$$\rho = \rho_0 (1 - \lambda_T (T - T_0) + \lambda_0 (S - S_0)) \tag{28}$$

$$\frac{DS}{Dt} = f \nabla^2 S \tag{29}$$

12 Entropy

No approximation and Boussinesq:

$$\mu > 0, k > 0 \tag{30}$$

13 Bernoulli

- Steady Flow (inviscid barotropic)

$$\frac{1}{2} \mathbf{u} \cdot \mathbf{u} + gz + \int \frac{dp}{\rho} = \text{constant along streamlines and vortex lines} \tag{31}$$

- Steady irrotational flow (inviscid barotropic):

$$\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + gz + \int \frac{dp}{\rho} \quad = \quad \text{constant everywhere} \tag{32}$$

- Irrotational flow (inviscid barotropic):

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\mathbf{u}\cdot\mathbf{u} + gz + \int \frac{dp}{\rho} \tag{33}$$

$$\mathbf{u} = \nabla\phi \tag{34}$$

Remember, $\nabla\cdot\mathbf{u} \Rightarrow \nabla\cdot\nabla\phi = 0 \Rightarrow \nabla^2\phi = 0$.

14 Solid body rotation

$$u_\theta = \Omega_0 r \tag{35}$$

$$\omega_z = 2\Omega_0 \tag{36}$$

$$\Lambda = \omega_z \pi r^2 \tag{37}$$

15 Point Vortex

$$u_\theta = \frac{\Gamma}{2\pi r} \tag{38}$$

$$\omega_z = 0 \quad \text{(Except at location of point vortex)} \tag{39}$$

16 Circulation

$$\Gamma = \int_c \mathbf{u}\cdot d\mathbf{s} \tag{40}$$

17 Stoke's Theorem

$$\int_c \mathbf{u}\cdot d\mathbf{s} = \iint_A (\nabla\times\mathbf{u})\cdot d\mathbf{A} \tag{41}$$

18 Kelvin's circulation theorem (non-rotating)

$$\frac{D\Gamma}{Dt} = 0 \tag{42}$$

Inviscid, barotropic, only conservative body forces.

19 Helmholtz vortex theorems

1. Vortex lines move with the fluid
2. Circulation of a vortex tube is constant along its length
3. A vortex tube can only end at a solid boundary, or form a closed loop
4. The circulation of a vortex tube is constant in time

20 Vorticity equation (non-rotating)

Incompressible, barotropic:

$$\frac{D\omega}{Dt} = \omega \cdot \nabla \mathbf{u} + \nu \nabla^2 \omega \tag{43}$$

Incompressible, baroclinic:

$$\frac{D\omega}{Dt} = \omega \cdot \nabla \mathbf{u} + \frac{1}{\rho^2} \nabla \rho \times \nabla p + \nu \nabla^2 \omega \tag{44}$$

21 Rotating frame

$$\begin{aligned}
\mathbf{u}_{fixed} &= \mathbf{u}_{rot} + \boldsymbol{\Omega} \times \mathbf{r} && (45)\\
\mathbf{a}_{fixed} &= \mathbf{a}_{rot} + 2\boldsymbol{\Omega} \times \mathbf{u}_{rot} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) && (46)\\
&= \mathbf{a}_{rot} + 2\boldsymbol{\Omega} \times \mathbf{u}_{rot} - \boldsymbol{\Omega}^2 \mathbf{R} && (47)
\end{aligned}$$

22 Momentum equation (rotating)

$$\begin{aligned}
\frac{D\mathbf{u}}{Dt} &= -\frac{1}{p} \nabla p + (\mathbf{g} + \boldsymbol{\Omega}^2 \mathbf{R}) - 2\boldsymbol{\Omega} \times \mathbf{u} + \nu \nabla^2 \mathbf{u} + \frac{\nu}{3} \nabla (\nabla \cdot \mathbf{u}) && (48)\\
&= -\frac{1}{p} \nabla p - \mathbf{g} - 2\boldsymbol{\Omega} \times \mathbf{u} + \nu \nabla^2 \mathbf{u} + \frac{\nu}{3} \nabla (\nabla \cdot \mathbf{u}) && (49)
\end{aligned}$$

Centrifugal force sucked into gravity term.

23 Coriolis force

- Turns stuff to the right (N.H.)
- Does no work

24 Vorticity equation (rotating earth-centric coordinates)

$$\frac{D\omega}{Dt} = (\omega + 2\boldsymbol{\Omega}) \cdot \nabla \mathbf{u} + \frac{1}{\rho^2} \nabla \rho \times \nabla p + \nu \nabla^2 \omega \tag{50}$$

25 Kelvin's circulation theorem (rotating)

$$\frac{D\Gamma_a}{Dt} = 0, \qquad \Gamma_a = \int_A (\omega + 2\mathbf{\Omega}) \cdot dA \tag{51}$$

26 Simplified momentum equations

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv + \nu\frac{\partial^2 u}{\partial z^2} \tag{52}$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu + \nu\frac{\partial^2 v}{\partial z^2} \tag{53}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{54}$$

Know scaling arguments that got us here.

27 Inviscid, simplified momentum equations

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv \tag{55}$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu \tag{56}$$

$$0 = -\frac{\partial p}{\partial z} - pg \tag{57}$$

28 f plane approximation

$$f = f_0 = 2\Omega \sin\phi_0 \tag{58}$$

29 β plane approximation

$$f = f_0 + \beta y = 2\Omega \sin\phi_0 + \frac{2\Omega\cos\phi_0}{r} y \tag{59}$$

30 Balances and flows from simplified, inviscid mo. eq.

1. Inertial oscillations:

$$-\frac{u_\theta^2}{r} = f u_\theta \tag{60}$$

2. Geostrophy

$$\frac{1}{p}\frac{\partial p}{\partial x} = fv \tag{61}$$

$$\frac{1}{p}\frac{\partial p}{\partial y} = -fu \tag{62}$$

3. Gradient wind (can be less than or greater than geostrophic wind):

$$-\frac{u_\theta^2}{r} = -\frac{1}{p}\frac{\partial p}{\partial r} + f u_\theta \tag{63}$$

$$u_\theta = f(R_0, \frac{1}{\rho f^2 r}, \frac{\partial p}{\partial r}) \tag{64}$$

   - Describes cyclonic/anticyclonic highs/lows
   - High pressure $\frac{\partial p}{\partial r}$ less than low pressure $\frac{\partial p}{\partial r}$
   - High pressure systems have gentle winds in center

4. Cyclostrophic wind

$$\frac{u_\theta^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r} \tag{65}$$

5. Isallobaric wind

$$u\frac{\partial u}{\partial x} = fv \tag{66}$$

$$\frac{\partial u}{\partial t} = fv \tag{67}$$

31 Geostrophy plus friction

$$\frac{1}{\rho}\frac{\partial p}{\partial x} - fv - \nu_H \nabla^2 u = 0 \tag{68}$$

Three way balance, cross-isobaric flow.

32 Balances and flows from simplified vorticity equation

- Taylor-Proudman (barotropic):

$$2\mathbf{\Omega} \cdot \nabla \mathbf{u} = 0 \tag{69}$$

$$\Rightarrow \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \tag{70}$$

$$\Rightarrow \text{vertical rigidity} \tag{71}$$

- Thermal wind (baroclinic):

$$2\mathbf{\Omega} \cdot \nabla \mathbf{u} + \frac{1}{\rho^2}\nabla \rho \times \nabla p = 0 \tag{72}$$

$$\Rightarrow \frac{\partial u}{\partial z} = -\frac{g}{f\rho}\mathbf{k} \times \nabla \rho \tag{73}$$

$$\Rightarrow \frac{\partial u}{\partial z} = \frac{g\rho_0\alpha}{f\rho}\mathbf{k} \times \nabla T \tag{74}$$

  This last step was made based on the relationship between p and T.

33 Ekman Layer

- Non-rotating boundary layers grow, Ekman layer does not.
- Mass transport:

$$\mathbf{M}_{Ek} = \frac{\tau \times \mathbf{k}}{f} \tag{75}$$

- Ekman spiral
- Ekman pumping and suction

$$W_{Ek} = \frac{1}{\rho f} \nabla \times \tau \cdot k \tag{76}$$

34 Sverdrup Transport

- Explained using Kelvin's circulation theorem: **(Jim – You appear to have two ρ in your notes)**

$$v = \frac{1}{h\rho} (\nabla \times \tau) \cdot \mathbf{k} \tag{77}$$

- Wind-driven circulation sensitive to curl of wind stress
- Return flow in the western boundary current

35 Shallow water equations

$$\frac{Du}{Dt} = -g\frac{\partial h}{\partial x} + fv \tag{78}$$

$$\frac{Dv}{Dt} = -g\frac{\partial h}{\partial y} - fu \tag{79}$$

$$\frac{\partial h}{\partial t} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{80}$$

36 Wave kinematics

$$k = \frac{2\pi}{\lambda x}, \qquad l = \frac{2\pi}{\lambda y}, \qquad \omega = \frac{2\pi}{T} \tag{81}$$

- Phase speed = $\frac{\omega}{k}$
- Group speed = $\frac{\partial \omega}{\partial k}$ (carries information).
- Dispersion relation relates ω, k, l, and provides graphical information about phase and group speed.

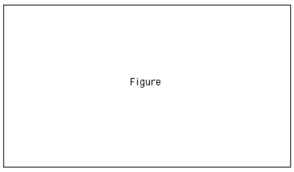

Figure 2: (`fig:ReviewWaveDispRel`) Wave dispersion relationship. Positive slope means positive group speed. Negative slope means negative group speed. Slope of line connecting a point on the curve to the origin gives phase speed.

37 Shallow water potential vorticity equation

$$\frac{D}{Dt}\left(\frac{\omega_z + f}{h}\right) = 0 \tag{82}$$

Obtained from cross-differentiating, subtracting and simplifying the shallow water equations. Note: fixed depth barotropic vorticity equation is $\frac{D}{Dt}(\omega_z + f) = 0$.

38 Potential vorticity

$$\frac{\omega_z + f}{h} = \text{constant} \tag{83}$$

Potential vorticity is conserved following the flow. "Flow over a mountain" example.

39 Shallow water gravity waves without rotation

- Obtained from non-rotating shallow water equations
- $\omega = \sqrt{g\bar{h}}K = cK$
- Non-dispersive

40 Inertia-gravity waves

- Obtained from rotating shallow water equations
- $\omega = 0, \pm\sqrt{f_0^2 + c^2K^2}$
- Dispersive, except in limit of large K
- Behave like non-rotating gravity waves at large K, $\omega = cK$
- Behave like inertial oscillations at small k, $\omega = f_0$

41 Kelvin waves

- Obtained from rotating shallow water equations with $v = 0$ (transverse velocity) and a lateral boundary
- $\omega = cK$
- Non-dispersive

42 Constant depth, barotropic Rossby waves

- Obtained from barotropic vorticity equation ($\frac{D}{Dt}(\omega_z + f) = 0$)
- Dispersion relationship (see figure):

$$\omega = -\frac{\beta k}{k^2 + l^2} \tag{84}$$

- Dispersive
- Phase and group speeds can be in opposite directions

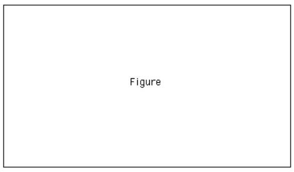

Figure 3: (`fig:ReviewBarotropicRossWave`) Constant depth, barotropic Rossby wave dispersion relationship.

43 Shallow water Rossby waves

- Obtained from conservation of potential vorticity
- $\omega = -\frac{\beta k}{k^2+l^2+\frac{1}{R_d^2}}$, $R_d = \frac{C}{f_0}$
- Dispersive
- Phase and group speeds can be in opposite directions
- R_d sets a length scale that forces a maximum phase speed

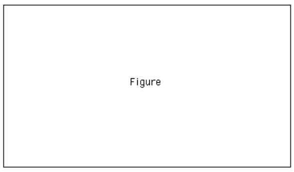

Figure 4: (`fig:ReviewShallowWaterRossWave`) Shallow water Rossby wave dispersion relationship.