Lecture 20

20.1 Administration

- Nothing this week

20.2 More on Rossby waves

Last time I waved my arms about the Rossby wave (see figure 20.1). A fluid line is displaced, generates relative vorticity due to Kelvin's Circulation Theorem $\Rightarrow \omega + 2\Omega =$ constant, $\omega_z + f$ in this case, and sets up a velocity field that props the disturbance westward. Let's describe the evolution of the Rossby wave. First we need an equation. Take Kelvin:

$$\frac{D}{Dt}\iint_A (\omega_z + 2\Omega \sin\phi) dA = 0 \tag{20.1}$$

$$\iint_A \frac{D}{Dt}(\omega_z + 2\Omega \sin\phi) dA = 0 \tag{20.2}$$

$$\iint_A \frac{D\omega_z}{Dt} dA + \omega_z \frac{DdA}{Dt} + \frac{D}{Dt}(2\Omega \sin\phi) dA + 2\Omega \sin\phi \frac{D}{Dt} dA = 0 \tag{20.3}$$

$$\iint_A \left(\frac{D\omega_z}{Dt} + 2\Omega \cos\phi \frac{D\phi}{Dt} \right) dA + (\omega_z + 2\Omega \sin\phi) \frac{D}{Dt} dA = 0 \tag{20.4}$$

Assume $\nabla \cdot \mathbf{u} = 0$, barotropic, and constant fluid thickness $\Rightarrow$ area must remain constant, so $\frac{D}{Dt} dA = 0$. A proof of this is as follows:

$$\delta v = h \delta A \Rightarrow \frac{D}{Dt}\delta v = \frac{Dh}{Dt}\delta A + h \frac{D}{Dt}\delta A = 0 \tag{20.5}$$

$$\Rightarrow h \frac{D}{Dt}\delta A = -\frac{Dh}{Dt}\delta A \Rightarrow \frac{D}{Dt}\delta A = 0 \tag{20.6}$$

$$\tag{20.7}$$

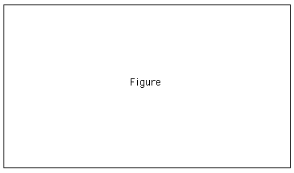

Figure 20.1: (`fig:Lec20RossbyWave1`) Rossby wave propagating westward.

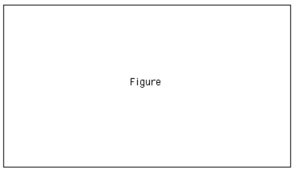

Figure 20.2: (`fig:Lec20LatY`) $\Delta\phi = \frac{\Delta y}{r}$; $\frac{\Delta\phi}{\Delta t} = \frac{\Delta y}{\Delta t}/r$; $\frac{D\phi}{Dt} = \frac{v}{r}$

Here we have made use of the continuity equation, and used $\frac{Dh}{Dt} = 0$, though we do not strictly need this last part. Thus we have

$$\iint_A \left(\frac{D\omega_z}{Dt} + 2\Omega\cos\phi \frac{D\phi}{Dt} \right) dA = 0 \tag{20.8}$$

We can find an expression for $\frac{D\phi}{Dt}$ as we did in the last lecture, see figure 20.2:

$$\iint_A \left(\frac{D\omega_z}{Dt} + \frac{2\Omega\cos\phi}{r} \right) dA = 0 \tag{20.9}$$

Note $\frac{2\Omega\cos\phi}{r} = \beta$

$$\iint_A \left(\frac{D\omega_z}{Dt} + \beta v \right) dA = 0 \tag{20.10}$$

Must be true for all dA.

$$\frac{D\omega_z}{Dt} + \beta v = 0 \tag{20.11}$$

Note at the beginning of this mess I could have written:

$$\begin{aligned} \frac{D}{Dt}(\omega_z + 2\Omega\cos\phi) &= 0 && (20.12)\\ \Rightarrow \frac{Dt}{Dt}(\omega_z + f) &= 0 && (20.13)\\ \Rightarrow \frac{D\omega_z}{Dt} + \frac{Df}{Dt} &= o && (20.14)\\ \Rightarrow \frac{D\omega_z}{Dt} + \beta v &= 0 && (20.15) \end{aligned}$$

Jim's note to self: For test. Why is this different from the vorticity equation under notation produced in classic? We can expand this

$$\frac{\partial\omega_z}{\partial t} + u\frac{\partial\omega_z}{\partial x} + v\frac{\partial\omega_z}{\partial y} + \beta v = 0 \tag{20.16}$$

This is a nonlinear equation, which makes analysis hard. **Jim – Bottom cut off).** A fairly standard trick is to assume variables can be broken into a mean state and a perturbation about the mean:

$$\begin{aligned} u &= \bar{u} + u' && (20.17)\\ v &= \bar{v} + v' && (20.18)\\ \omega_z &= \bar{\omega_z} + \omega_z' && (20.19)\\ & && (20.20) \end{aligned}$$

where $\mathbf{u} \gg u'$ and we have only a zonal flow such that $\bar{v} = 0$. Substitute:

$$\frac{\partial}{\partial t}(\bar{\omega_z} + \omega_z') + (\bar{u} + u')\frac{\partial}{\partial x}(\bar{\omega_z} + \omega_z') + (\bar{v} + v')\frac{\partial}{\partial y}(\bar{\omega_z} + \omega_z') + \beta(\bar{v} + v') = 0 \tag{20.21}$$

$$\frac{\partial \bar{\omega_z}}{\partial t} + \frac{\partial \omega_z'}{\partial t} + \bar{u}\frac{\partial \bar{\omega_z}}{\partial x} + \bar{u}\frac{\partial \omega_z'}{\partial x} + u'\frac{\partial \bar{\omega_z}}{\partial x} + u'\frac{\partial \omega_z'}{\partial x} + v'\frac{\partial \bar{\omega_z}}{\partial y} + v'\frac{\partial \omega_z'}{\partial t} + \beta v' = 0 \tag{20.22}$$

$$\frac{\partial \omega_z'}{\partial t} + \bar{u}\frac{\partial \omega_z'}{\partial x} + \beta v' = 0 \tag{20.23}$$

Again, this is noting that $\bar{v} = 0$ and $\bar{\omega}$ is a constant. The bar terms don't have to be constant. By definition the $^{-}$terms are a solution so they all drop out. Getting better. We're linear, but we have an equation that is a function of both ω_z' and v': 1 equation, 2 unknowns. We can get it in terms of a single variable by using the streamfunction (or perturbation streamfunction) since we have what amounts to a 2D fluid (in this case $\nabla \cdot \mathbf{u} = 0$, barotropic, fixed depth):

$$u' = -\frac{\partial \psi'}{\partial y} \qquad v' = \frac{\partial \psi'}{\partial x} \tag{20.24}$$

Recall:

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{20.25}$$

$$\omega_z' = \frac{\partial}{\partial x}\left(\frac{\partial \psi'}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \psi'}{\partial y}\right) \tag{20.26}$$

$$\omega_z' = \frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial y^2} \tag{20.27}$$

$$\omega_z' = \nabla^2 \psi' \tag{20.28}$$

Substitute:

$$\frac{\partial}{\partial t}\nabla^2\psi' + \bar{u}\frac{\partial}{\partial x}\nabla^2\psi' + \beta\frac{\partial \psi'}{\partial x} = 0 \tag{20.29}$$

Here is a linear equation in a single unknown... now we can do something. As always, start by assuming a form of the solutions **Jim – You may want to explain some about why we can assume this form of solutions)**:

$$\psi' + \Psi e^{i(kx+ly-\omega t)} \tag{20.30}$$

Remember:

- $\Psi \equiv$ amplitude
- $k \equiv$ zonal wave number
- $l \equiv$ meridional wave number
- $\omega \equiv$ frequency (note different from ω_z and ω)

Plug it in:

$$\frac{\partial \psi'}{\partial x} = ik\Psi e^{i(kx+ly-\omega t)} \tag{20.31}$$

$$\frac{\partial^2 \psi'}{\partial x^2} = -k^2\Psi e^{i(kx+ly-\omega t)} \tag{20.32}$$

$$\frac{\partial \psi'}{\partial y} = il\Psi e^{i(kx+ly-\omega t)} \tag{20.33}$$

$$
\begin{aligned}
\frac{\partial^2 \psi'}{\partial y^2} &= -l^2 \Psi e^{i(kx+ly-\omega t)} && (20.34)\\
\nabla^2 \psi' &= (-k^2 - l^2) e^{i(kx+ly-\omega t)} \Psi && (20.35)\\
\frac{\partial}{\partial x} \nabla^2 \psi' &= (-k^2 - l^2) ik e^{i(kx+ly-\omega t)} \Psi && (20.36)\\
\frac{\partial}{\partial t} \nabla^2 \psi' &= (-k^2 - l^2)(-i\omega) e^{i(kx+ly-\omega t)} \Psi && (20.37)
\end{aligned}
$$

Substitute:

$$
\begin{aligned}
(-k^2 - l^2)(-i\omega) e^{i(kx+ly-\omega t)} \Psi & && (20.38)\\
+ \ \bar{u}(-k^2 - l^2)(ik) e^{i(kx+ly-\omega t)} \Psi & && (20.39)\\
+ \qquad \beta(ik) e^{i(kx+ly-\omega t)} \Psi &= 0 && (20.40)\\
-\omega(-k^2 - l^2) + \bar{u}k(-k^2 - l^2) + \beta k &= 0 && (20.41)\\
(-\omega + \bar{u}k)(-k^2 - l^2) &= -\beta k && (20.42)\\
-\omega + \bar{u}k &= \frac{\beta k}{k^2 + l^2} && (20.43)\\
\omega &= \bar{u}k - \frac{\beta k}{k^2 + l^2} && (20.44)
\end{aligned}
$$

This is the *dispersion relation* for the system. Phase speed $\equiv c_x = \frac{\omega}{k} \Rightarrow c_x = \bar{u} - \frac{\beta}{k^2+l^2}$. In the absence of zonal flow, the phase propagation (the $+-$ vorticity patter propagation) is always to the west. Note that the phase speed is a function of the wavenumbers, so the Rossby wave is a *dispersive wave*. The longer wavelengths (smaller wave numbers) propagate westward at a speed greater than the shorter wavelengths. Indeed we can solve for the wavenumber that results in a stationary wave:

$$
\begin{aligned}
0 = \bar{u} - \frac{\beta}{k^2 + l^2} && (20.45)\\
k^2 + l^2 = \frac{\beta}{\bar{u}} && (20.46)
\end{aligned}
$$

For purely zonal wavelengths:

$$
k = \sqrt{\frac{\beta}{\bar{u}}} \qquad (20.47)
$$

for a stationary wave. What about the group velocity?

$$
\begin{aligned}
c_{gx} = \frac{\partial \omega}{\partial k} &= \frac{\partial}{\partial k}\left(\bar{u}k - \frac{\beta k}{k^2 + l^2}\right) && (20.48)\\
&= \bar{u} - \left[\frac{(k^2 + l^2)\beta - \beta k(2k)}{(k^2 + l^2)^2}\right] && (20.49)\\
&= \bar{u} - \frac{\beta k^2 + \beta l^2 - 2\beta k^2}{(k^2 + l^2)^2} && (20.50)\\
&= \bar{u} + \frac{\beta k^2 - \beta l^2}{(k^2 + l^2)^2} && (20.51)\\
&= \bar{u} + \frac{\beta(k^2 - l^2)}{(k^2 + l^2)^2} && (20.52)
\end{aligned}
$$

Thus the group velocity can be either east or west depending on the relative magnitudes of k and l. The last thing we want to look at is the dispersion relation for these waves:

$$
\omega = \bar{u}k - \frac{\beta k}{k^2 + l^2} \qquad (20.53)
$$

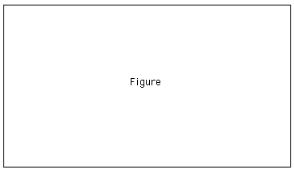

Figure 20.3: (`fig:Lec20ConcentricCircles`) Rossby wave dispersion relationship.

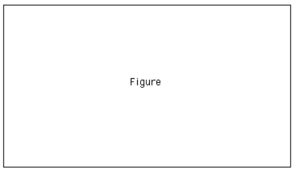

Figure 20.4: (`fig:Lec20SingleCircle`) Rossby wave dispersion relationship.

Let's assume $\bar{u} = 0$ or put yourself into a frame moving with $\bar{u}$.

$$\omega = -\frac{\beta k}{k^2 + l^2} \tag{20.54}$$

Plot solutions to $\frac{\omega}{\beta} = -\frac{k}{k^2+l^2}$ as a function of k and l.

Show concentric circles plot. (see figures 20.3 and 20.4).

Now for a fixed l plot $\frac{\omega}{\beta}$ vs k.

Show k vs $\frac{\omega}{\beta}$ plot. (see figures 20.5 and 20.6).

The slopes in these last plots is c_{gx}! Slope of line from origin to point is c_{px}!

20.3 Non-fixed depth

Remember that this description of Rossby waves has been for a barotropic fluid of constant depth. Now we want to talk about the impact of allowing a fluid of varying depth. It's going to take us a little while to get there but we'll see that the inclusion of a variable depth will change our expression of conservation of absolute vorticity.

$$\frac{D}{Dt}(\omega_z + f) = 0 \tag{20.55}$$

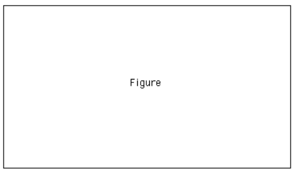

Figure 20.5: (`fig:Lec20klp4`) Rossby wave dispersion relationship, $l = .4$.

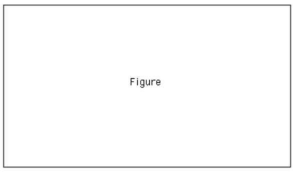

Figure 20.6: (`fig:Lec20kl0`) Rossby wave dispersion relationship, $l = 0$.

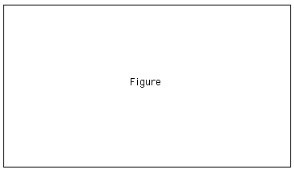

Figure 20.7: (`fig:Lec20ShallowWater`) Setup for shallow water equations, we will assume a flat bottom, but it is not necessary.

to an expression of conservation of angular momentum or so-called potential vorticity.

$$\frac{D}{Dt}(\frac{\omega_z + f}{h}) = 0 \tag{20.56}$$

We'll derive this expression in a couple of different ways. The first requires a new form of the momentum and continuity equations.

20.4 Shallow water equations

The setup for the shallow water equations is seen in figure 20.7. The fluid is barotropic, but depth of fluid can vary. Does this make sense? Yes!

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{20.57}$$

Note that if $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ don't balance then there *must* be a $\frac{\partial w}{\partial z}$. Thus a divergent horizontal velocity field $\Rightarrow \frac{\partial w}{\partial z} < 0 \Rightarrow$ a decrease in the height of the fluid (and vice-versa). Let's quantify $\frac{\partial w}{\partial z}$. Do this by vertically integrating continuity from the bottom to the free surface (easiest to go this way when have a single layer).

$$\int_0^h \frac{\partial u}{\partial x} dz + \int_0^h \frac{\partial v}{\partial y} dz + \int_0^h \frac{\partial w}{\partial z} dz = 0 \tag{20.58}$$

Since barotropic, $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ are independent of z:

$$\frac{\partial u}{\partial x} \int_0^h dz + \frac{\partial v}{\partial y} \int_0^h dz + \int_0^h dw = 0 \tag{20.59}$$

$$h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + w(h) - w(0) = 0 \tag{20.60}$$

The vertical velocity at $z = 0$ must be $w = 0$

$$h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + w(h) = 0 \tag{20.61}$$

The vertical velocity at $z = h$ is nothing more than the time rate of change of h:

$$w(h) = \frac{Dh}{Dt} = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} \tag{20.62}$$

Substitute

$$\frac{Dh}{Dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{20.63}$$

This is the *shallow water continuity equation.* Now for the momentum equations. First consider the w momentum equation; hydrostatic balance:

$$w: \qquad \frac{\partial p}{\partial z} = -\rho g \tag{20.64}$$

Integrate

$$\int_z^h dp = -\rho g \int_z^h dz \tag{20.65}$$

$$p(h) - p(z) = -\rho g(h - z) \tag{20.66}$$

$$p(z) = \rho g(h - z) + p(h) \tag{20.67}$$

$$\tag{20.68}$$

$p(h)$ is the pressure just above free surface, the atmospheric pressure p_a.

$$p(z) = \rho g(h - z) + pa \tag{20.69}$$

So, starting with the momentum equations:

$$x: \qquad \frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{20.70}$$

$$y: \qquad \frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{20.71}$$

can get $\frac{\partial p}{\partial x}$ and $\frac{\partial p}{\partial y}$ terms from differentiating hydrostatic equation (note z and $p_a \neq f(xy)$):

$$\frac{\partial p}{\partial x} = \rho g\frac{\partial h}{\partial x} \tag{20.72}$$

$$\frac{\partial p}{\partial y} = \rho g\frac{\partial h}{\partial y} \tag{20.73}$$

Substitute:

$$\frac{Du}{Dt} - fv = -g\frac{\partial h}{\partial x} \tag{20.74}$$

$$\frac{Dv}{Dt} + fu = -g\frac{\partial h}{\partial y} \tag{20.75}$$

$$\frac{Dv}{Dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{20.76}$$

With the continuity equations from before, these are *shallow water equations.* Sometimes will see

$$= -g\frac{\partial \eta}{\partial x} \tag{20.77}$$

$$= -g\frac{\partial eta}{\partial y} \tag{20.78}$$

$$\tag{20.79}$$

for momentum equations from $h = H + \eta$, as seen in figure 20.7. These equations are neat because

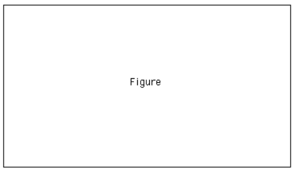

Figure 20.8: (`fig:Lec20StackedShallowWater`) Stacked shallow water layers.

- killed z motion equations by sucking into x and y motion equations
- removed the dependent variable w by fooling with continuity
- removed the independent variable z

Dependent: u, v and h
Independent: x, y and t
Note that the geostrophic form of the equations are:

$$u_g = -\frac{g}{f}\frac{\partial h}{\partial y} \qquad (20.80)$$

$$v_g = \frac{g}{f}\frac{\partial h}{\partial x} \qquad (20.81)$$

These are exactly the form you had to use for that PS problem where you had to find geostrophic and ageostrophic velocities. You can start to think about stratification by stacking shallow water layers on top of one another (see figure 20.8). Do derivation in terms of H and h' (could also have done one layer this way). 1$^{\text{st}}$ layer:

$$\frac{\partial p}{\partial z} = -p_1 g \qquad (20.82)$$

$$\int_{h'_1}^{z} dp = \int_{h'_1}^{z} -\rho_1 g dz \qquad (20.83)$$

$$p(z) - p(h'_1) = -\rho_1 g(z - h'_1) \qquad (20.84)$$

$$p(z) = \rho_1 g(h'_1 - z) + p_a, \qquad h'_2 - H_1 < z < h'_1 \qquad (20.85)$$

$$\frac{\partial p}{\partial x} = \rho_1 g\frac{\partial h'_1}{\partial x}, \qquad \frac{\partial p}{\partial y} = \rho_1 g\frac{\partial h'_1}{\partial y} \qquad (20.86)$$

At interface $z = h'_2 - H_1$

$$p(h'_2 - H_1) = \rho_1 g(h'_1 - h'_2 - H_1) \qquad (20.87)$$

2$^{\text{nd}}$ Layer:

$$\frac{\partial p}{\partial z} = -\rho_2 g \qquad (20.88)$$

$$\int_{h'_2 - H_1}^{z} dp = \int_{h'_2 - H_1}^{z} -\rho_2 g dz \qquad (20.89)$$

$$p(z) - p(h'_2 - H_1) = -\rho_2 g(z - h'_2 + H_1) \qquad (20.90)$$

$$p(z) = \rho_2 g(h'_2 - H_1 - z) + p(h'_2 - H_1), \qquad H_1 + H_2 < z < h'_2 - H_1 \qquad (20.91)$$

Substitute for $p(h'_2 - H_1)$ from first layer

$$p(z) = p_2 g(h'_2 - H_1 - z) + p_1 g(h'_1 - h'_2 + H_1) \qquad (20.92)$$

$$p(z) = \rho_2 g h'_2 - \rho_1 g h'_2 - \rho_2 g H_1 + \rho_1 g H_1 + \rho_1 g h'_1 - \rho_2 g z \qquad (20.93)$$

Define $\frac{p_2-\rho_1}{\rho_2} g = g'$ =reduced gravity.

$$\begin{aligned} p(z) &= \rho_2 g' h_2' - \rho_2 g' H_1 + \rho_1 g h_1' - \rho_2 g z && (20.94) \\ \frac{\partial p}{\partial z} &= \rho_2 g' \frac{\partial h_2'}{\partial x} + \rho_1 g \frac{\partial h_1'}{\partial x} && (20.95) \\ \frac{\partial p}{\partial y} &= \rho_2 g' \frac{\partial h_2'}{\partial y} + \rho_1 g \frac{\partial h_1'}{\partial y} && (20.96) \end{aligned}$$

Continuity, first layer:

$$\begin{aligned} \int_{h_2'-H_1}^{h_1'} \frac{\partial u}{\partial x} dz + \int_{h_2'-H_1}^{h_1'} \frac{\partial v}{\partial y} dz + \int_{h_2'-H_1}^{h_1'} \frac{\partial w}{\partial z} dz &= 0 && (20.97) \\ \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) (h_1' - h_2' + H_1) + w(h_1') - w(h_2' - H_1) &= 0 && (20.98) \\ \frac{D}{Dt}(h_1' + H_1 - h_2') + (h_1' - h_2' + H_1) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) &= 0 && (20.99) \\ \frac{\partial}{\partial t}(h_1' - h_2') + H_1 \left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right) &= 0 && (20.100) \end{aligned}$$

To make the last step we linearized. Continuity, second Layer:

$$\begin{aligned} \int_{-H_1-H_2}^{h_2'-H_1} \frac{\partial u}{\partial x} dz + \int_{-H_1-H_2}^{h_2'-H_1} \frac{\partial v}{\partial y} dz + \int_{-H_1-H_2}^{h_2'-H_1} \frac{\partial w}{\partial z} dz &= 0 && (20.101) \\ \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) (h_2' - H_1 + H_1 + H_2) + w(h_2' - H_1) - w(-H_1 + H_2) &= 0 && (20.102) \\ \frac{D}{Dt}(h_2' - H_1) + (h_2' + H_2) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) &= 0 && (20.103) \\ \frac{\partial}{\partial t}(h_1') + H_2 \left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right) &= 0 && (20.104) \end{aligned}$$

Again, this last step is the result of linearization. We end up with equations for level 1:

$$\begin{aligned} \frac{Du_1}{Dt} - f v_1 &= -g \frac{\partial h_1'}{\partial x} && (20.105) \\ \frac{Dv_1}{Dt} + f u_1 &= -g \frac{\partial h_1'}{\partial y} && (20.106) \\ \frac{D}{Dt}(h_1' - h_2') + (h_1' + H_1 - h_1') \left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \right) &= 0 && (20.107) \end{aligned}$$

And level 2:

$$\begin{aligned} \frac{Du_2}{Dt} - f v_2 &= -g \frac{\partial h_2'}{\partial x} - \frac{\rho_1}{\rho_2} g \frac{\partial h_1'}{\partial x} && (20.108) \\ \frac{Dv_2}{Dt} + f u_2 &= -g \frac{\partial h_2'}{\partial y} - \frac{\rho_1}{\rho_2} g \frac{\partial h_1'}{\partial y} && (20.109) \\ \frac{Dh_2'}{Dy} + (h_2' + H_2) \left(\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} \right) &= 0 && (20.110) \end{aligned}$$

Imagine enforcing a rigid lid $\Rightarrow h_1' = 0$. Level 2 equations become

$$\frac{Du_2}{Dt} - f v_2 = -g' \frac{\partial h_2'}{\partial x} \qquad (20.111)$$

$$\frac{Dv_2}{Dt} + fu_2 = -g'\frac{\partial h_1'}{\partial y} \tag{20.112}$$

$$\frac{Dh_2'}{Dt} + (h_2' + H_2)\left(\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y}\right) = 0 \tag{20.113}$$

Just like a single layer shallow water but $g = g'$ We'll come back to this. More on this later if we have **(Jim – The rest was cut off).** Back to the shallow water equations:

$$x: \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial h}{\partial x} \quad \mathbf{1} \tag{20.114}$$

$$y: \quad \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial h}{\partial y} \quad \mathbf{2} \tag{20.115}$$

To get the associated vorticity equation, let's subtract $\frac{\partial \mathbf{2}}{\partial x}$ from $\frac{\partial \mathbf{1}}{\partial y}$

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial \mathbf{2}}{\partial x} - \frac{\partial \mathbf{1}}{\partial y} \tag{20.116}$$

Put derivation on doc cam (Jim – I kept your equation references here, that maybe should be changed to LaTeXreferences).

$$x: \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial h}{\partial x} \quad \mathbf{1}$$

$$y: \quad \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial h}{\partial y} \quad \mathbf{2}$$

$$\frac{\partial}{\partial y}(\mathbf{1}): \quad \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) - f\frac{\partial v}{\partial y} - v\frac{\partial f}{\partial y} = -g\frac{\partial}{\partial y}\left(\frac{\partial h}{\partial x}\right) \quad \mathbf{3}$$

$$\frac{\partial}{\partial x}(\mathbf{2}): \quad \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right) + \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) + f\frac{\partial u}{\partial x} + u\frac{\partial f}{\partial y} = -g\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial y}\right) \quad \mathbf{4}$$

$$\mathbf{3}-\mathbf{4}: \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} = 0$$

Note $\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$, thus

$$\begin{aligned} &\frac{\partial \omega_z}{\partial t} + u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) \\ &\quad + \frac{\partial u}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{\partial v}{\partial y}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 0 + \beta v = 0 \end{aligned}$$

The β bit was derived as follows

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y} 2\Omega \sin\phi \\ &= 2\Omega\cos\phi \frac{\partial \phi}{\partial y} \\ &= 2\Omega\cos\phi\left(\frac{1}{r}\right) \\ \frac{\partial f}{\partial y} &= \beta \end{aligned}$$

Writing things in terms of ω:

$$\frac{\partial \omega_z}{\partial t} + u\frac{\partial \omega_z}{\partial y} + v\frac{\partial \omega_z}{\partial y} + \omega_z \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v = 0 \tag{20.117}$$

$$\frac{D\omega_z}{Dt} + (\omega_z + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v = 0 \tag{20.118}$$

From shallow water continuity equation:

$$\frac{Dh}{Dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{20.119}$$

$$\Rightarrow \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -\frac{1}{h}\frac{Dh}{Dt} \tag{20.120}$$

Substitute:

$$\frac{D\omega_z}{Dt} - \left(\frac{\omega_z + f}{h}\right)\frac{Dh}{Dt} + \beta v = 0 \tag{20.121}$$

Note that:

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} \tag{20.122}$$

$$\frac{Df}{Dt} = v\frac{\partial f}{\partial y} = \beta v \tag{20.123}$$

Substitute:

$$\frac{D\omega_z}{Dt} - \left(\frac{\omega_z + f}{h}\right)\frac{Dh}{Dt} + \frac{Df}{Dt} \quad = \quad 0 \tag{20.124}$$

$$\frac{D}{Dt}(w_z + f) - \left(\frac{\omega_z + f}{h}\right)\frac{Dh}{Dt} \quad = \quad 0 \tag{20.125}$$

Multiply through by $-\frac{h}{h^2}$

$$\frac{(\omega_z + f)\frac{Dh}{Dt} - h\frac{D}{Dt}(\omega_z + f)}{h^2} = 0 \tag{20.126}$$

$$\frac{D}{Dt}\left(\frac{\omega_z + f}{h}\right) = 0 \tag{20.127}$$

This is the *shallow water potential vorticity equation.*