# Waves Primer II

Waves are:

1. Important
2. Easy
3. Neat

## 23.1 Basic wave form

The wave form that you will almost always use is

$$a = Re[A_c e^{i(kx+ly-\omega t)}] \tag{23.1}$$

Where

- $a \equiv$ variable
- $A_c \equiv$ wave amplitude (complex in this case)
- $kx + ly - \omega t \equiv$ phase (note, could also have a $mz$)
- $k$, $l \equiv$ wave numbers
- $\omega \equiv$ frequency
- $t \equiv$ time

Since $e^{i\phi} = \cos\phi + i\sin\phi$ we get

$$a = Re[A_c(\cos(kx+ly-\omega t) + i\sin(kx+ly-\omega t))] \tag{23.2}$$

Defining $A_c = A_1 + iA_2$

$$a = Re[A_1(\cos(kx+ly-\omega t) + A_1 i\sin(kx+ly-\omega t) \tag{23.3}$$
$$+iA_2\cos(kx+ly-\omega t) - A_2\sin(kx+ly-\omega t)] \tag{23.4}$$
$$a = A_1\cos(kx+ly-\omega t) - A_2\sin(kx+ly-\omega t) \tag{23.5}$$

Define

$$A_1 = A\cos\phi \tag{23.6}$$
$$A_2 = -A\sin\phi \tag{23.7}$$

Such that

$$a = A\cos\phi\cos(kx+ly-\omega t) - A\sin\phi\sin(kx+ly-\omega t) \tag{23.8}$$

Two angle formulas say:

$$a = A\cos(kx+ly-\omega t+\phi) \tag{23.9}$$

Where $\phi$ is a reference phase. The wave number is the number of waves per $2\pi$ (see figure 23.1).

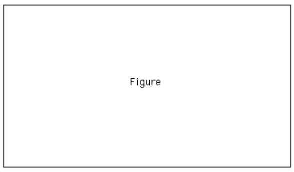


Figure 23.1: (`fig:WaveNumber`) Wavelengths in the $x$ and $y$ directions as well as the overall wave length. Wave number is the number of waves in $2\pi$.

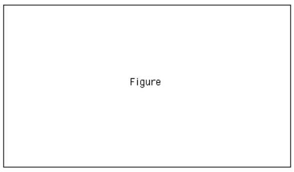


Figure 23.2: (`fig:WaveLengths`) Wavelengths in the $x$ and $y$ directions as well as the overall wave length. What is $\lambda$ in terms of $\lambda_x$ and $\lambda_y$?

## 23.2 Wavelength

Consider a wave of the form:

$$a = A\cos(kx + ly - \omega t + \phi) \tag{23.10}$$

where $\lambda$ is the distance over which signal repeats self in $x$ direction $\Rightarrow$ distance over which $kx$ increases by $2\pi \Rightarrow k\lambda_x = 2\pi$:

$$\lambda_x = \frac{2\pi}{k} \quad \text{and} \quad \lambda_y = \frac{2\pi}{l} \tag{23.11}$$

This is how component-wise wavelengths and wavenumbers are related.

What about the $\lambda$ that gives the shortest crest to crest distance (see figure 23.2)? We have that

$$\sin\theta = \frac{\lambda_y}{\sqrt{\lambda_x^2 + \lambda_y^2}} \tag{23.12}$$

and

$$\sin\theta = \frac{\lambda}{\lambda_x} \tag{23.13}$$

Equating these last two equations yields

$$\begin{aligned}
\frac{\lambda}{\lambda_x} &= \frac{\lambda_y}{\sqrt{\lambda_x^2 + \lambda_y^2}} && (23.14)\\
\lambda &= \frac{\lambda_x \lambda_y}{\sqrt{\lambda_x^2 + \lambda_y^2}} && (23.15)\\
\lambda^2 &= \frac{\lambda_x^2 \lambda_y^2}{\lambda_x^2 + \lambda_y^2} && (23.16)\\
\lambda^2\lambda_x^2 + \lambda^2\lambda_y^2 &= \lambda_x^2 + \lambda_y^2 && (23.17)
\end{aligned}$$

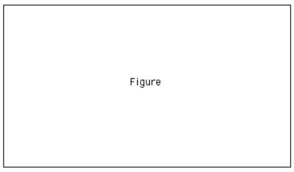


Figure 23.3: (`fig:WaveNumberVersusLength`) Wave number versus wave lengths.

$$\lambda^2 + \frac{\lambda^2 \lambda_y^2}{\lambda_x^2} = \lambda_y^2 \tag{23.18}$$

$$\frac{\lambda^2}{\lambda_y^2} + \frac{\lambda^2}{\lambda_x^2} = 1 \tag{23.19}$$

$$\frac{1}{\lambda^2} = \frac{1}{\lambda_y^2} + \frac{1}{\lambda_x 2} \tag{23.20}$$

Since $\lambda_x = \frac{2\pi}{k}$ and $\lambda_y = \frac{2\pi}{l}$ we get

$$\frac{1}{\lambda^2} = \frac{k^2}{4\pi^2} + \frac{l^2}{4\pi^2} \tag{23.21}$$

$$4\pi^2 = (k^2 + l^2)\lambda^2 \tag{23.22}$$

$$\lambda^2 = \frac{4\pi^2}{k^2 + l^2} \tag{23.23}$$

$$\lambda = \frac{2\pi}{\sqrt{k^2 + l^2}} \tag{23.24}$$

We define $k$ to be the total wave number and equal to $\sqrt{k^2 + l^2}$. Note that while $\mathbf{k} = k\mathbf{i} + l\mathbf{j}$, $\lambda \neq \lambda_{x\mathbf{i}} + \lambda_{y\mathbf{j}}$. Thus, $K$ is the magnitude of the wavenumber vector. Note smaller wavenumber implies longer waves.

**Show PIX** (See figure 23.3)

## 23.3 Frequency

Frequency is the number of periods per $2\pi$. Similar to the wavelength definition, the period is the time over which the signal repeats. For

$$a = A\cos(kx + ly - \omega t + \phi) \tag{23.25}$$

this means the time over which $\omega t$ increases by $2\pi$

$$\Rightarrow \omega T = 2\pi \tag{23.26}$$

$$T = \frac{2\pi}{\omega} \tag{23.27}$$

Note that a smaller frequency implies a longer period.

**Show PIX** (See figure 23.4) **Jim – Picture doesn't show anything.**

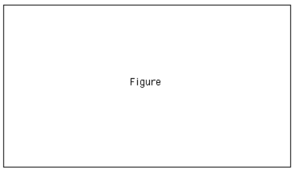


Figure 23.4: (`fig:WaveFrequencyVersusPeriod`) Wave frequency versus wave period.

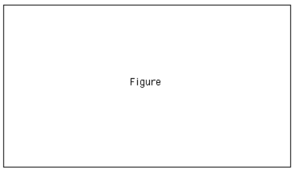


Figure 23.5: (`fig:WavePhaseSpeed`) A wave's phase speed is the speed of individual crests and troughs.

## 23.4 Phase speed

Phase speed is the speed of individual crests and troughs. Consider a moving crest over a time $\Delta t = t_2 - t_1$ (see figure 23.5). Express $\Delta x$ as a fraction of wavelength in $x$:

$$\Delta x = \alpha \lambda_x \tag{23.28}$$

In one period, $\Delta x = \lambda_x$; in two periods $\Delta x = 2\lambda_x$, etc. Thus $\lambda$ is the fraction of period traveled over $\delta t$.

$$\alpha = \frac{\Delta t}{T} \tag{23.29}$$

So:

$$\Delta x = \frac{\Delta t}{T}\lambda_x = \Delta t \frac{\omega}{2\pi} \cdot \frac{2\pi}{k} = \frac{\omega \Delta t}{k} \tag{23.30}$$

Note that this means

$$\frac{\Delta x}{\Delta t} = \frac{\omega}{k} \tag{23.31}$$

In the limit of $\Delta t \to 0$, this gives an expression for the speed of a wave crest in the $x$-direction. We call if the phase speed

$$c_x = \frac{\Delta x}{\Delta t} = \frac{\omega}{k} \tag{23.32}$$

Similarly

$$c_y = \frac{\omega}{l} \tag{23.33}$$

Just as in the wavenumber calculation (see figure 23.6):

$$c = \frac{c_x c_y}{\sqrt{cx^2 + cy^2}} \tag{23.34}$$

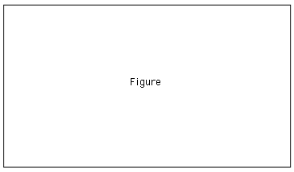


Figure 23.6: (`fig:WaveAbsolutePhaseSpeed`) Phase speeds in the $x$ and $y$ directions as well as the overall phase speed. What is $c$ in terms of $c_x$ and $c_y$?

Turn the crank and you get

$$c = \frac{\omega}{\sqrt{k^2 + l^2}} \tag{23.35}$$

$$c = \frac{\omega}{K} \tag{23.36}$$

As for $\lambda$, $\mathbf{c} \neq c_x\mathbf{i} + c_y\mathbf{j}$.

**Show MOVIE**

## 23.5 Group speed

The group speed is the speed at which energy (or information) travels. Consider the superposition of two waves

$$a = A_1 \cos(k_1 x + l_1 y - \omega_1 t) + A_2 \cos(k_2 x + l_2 y - \omega_2 t) \tag{23.37}$$

where

$$A_1 = A_2, \qquad k_1 \text{ and } k_2 \gg k_1 - k_2, \qquad \omega_1 \text{ and } \omega_2 \gg \omega_1 - \omega_2 \tag{23.38}$$

Trigonometry tells us

$$a = 2A \cos\left(\frac{\Delta k}{2} x - \frac{\Delta \omega}{2} t\right) \cos(\bar{k}x - \bar{\omega}t) \tag{23.39}$$

where $\Delta k = k_1 - k_2$, $\bar{k} = \frac{(k_1 + k_2)}{2}$ and similarly for $\omega$. Thus we can write

$$a = \tilde{A} \cos(\bar{k}x - \bar{\omega}t) \tag{23.40}$$

The $\tilde{A}$ is a wavy amplitude modeulation of $\cos(\bar{k}x - \bar{\omega}t)$. Note that since $\Delta k \ll \bar{k}$ and $\Delta\omega \ll \bar{\omega}$, amplitude modulation has much longer wavelength and much longer periods than $\cos(\bar{k}x - \bar{\omega}t)$. As before the phase speed is $\frac{\bar{\omega}}{\bar{k}}$, but the speed of the amplitude modulation goes like

$$\frac{\Delta\omega/2}{\Delta k/2} = \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk} \tag{23.41}$$

in the limit $\Delta k \to 0$. $c_g$ is the group velocity which is equal too $\frac{d\omega}{dk}$. In multiple-dimensions we have

$$c_{g_x} = \frac{\partial \omega}{\partial k} \tag{23.42}$$

$$c_{g_y} = \frac{\partial \omega}{\partial l} \tag{23.43}$$

$$\mathbf{c}_g = \nabla_k \omega \tag{23.44}$$

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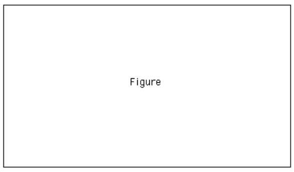


Figure 23.7: (`fig:WaveDispersion`) Schematic of dispersion.

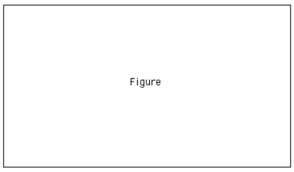


Figure 23.8: (`fig:WaveDispersion1`) One example of a dispersion relationship.

`groupspeedk.avi` $\frac{\Delta\omega}{\Delta k}$ same in each, $\frac{\omega}{k}$ changes.

`groupspeeddeltak.avi` $\frac{\Delta\omega}{\Delta k}$ changes, $\frac{\omega}{k}$ same. Note $\frac{\omega}{k} = \frac{\Delta\omega}{\Delta k}$ in bottom.

`groupspeedomega.avi` $\frac{\Delta\omega}{\Delta k}$ same, $\frac{\omega}{k}$ changes. Optically, larger $\frac{\omega}{k}$ seems faster group speed, but not? **(Jim – part of the text got cut off on the right of the page.)**

`groupspeeddeltaomega.avi` $\frac{\Delta\omega}{\Delta k}$ changes, $\frac{\omega}{k}$ same.

`groupspeeddirection.avi` Note that $-\frac{\omega}{k}$ looks faster when it is not.

`groupspeedphasespeed.avi` Shows $-\frac{\Delta\omega}{\Delta k}$, $+\frac{\omega}{k}$.

## 23.6 Dispersion relatioship

In general, $k$, $l$, $\omega$, and $A$ are not independent of one another, and

- phase speed $= c = c(k,l) = \frac{\omega(k,l)}{K}$
- group velocity $= c_y = c_y(k,l) = \frac{\partial\omega(k,l)}{\partial k}$

Waves with different frequencies will travel at different speeds, smearing out coherent signals. This is called *dispersion* (see figure 23.7). The way in which frequency changes as a function of $k$ and $l$ is called the *dispersion relation.* The classic pictures are seen in figures 23.8 and 23.9.

In general, these dispersion relations are obtained by substituing an $a = Ae^{i(kx+ly-\omega t+\phi)}$ into the equation of interest and reducing (all the $e^{i(kx+ly-\omega t+\phi)}$'s drop out) to obtain $\omega = f(k,l)$.

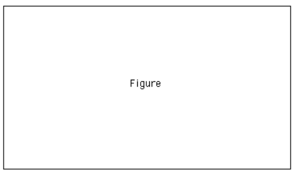


Figure 23.9: (`fig:WaveDispersion2`) Another example of a dispersion relationship.