1. Consider the case where a fluid is contained in the infinite plane with a mean depth D. A uniform flow $\vec{u} = U_0\,\hat{i}$ is directed parallel to the x-axis.

   a) Formulate the linearized gravity wave problem for *steady* waves and find the condition U_0 must satisfy for a wave-like solution to exist.

   b) Finally, specialize to the shallow-water case and put a wall at $y = y_B = \cos kx$. Find the boundary condition for v at y_B.

2. Consider the linearized, Boussinesq equations of motion on the f-plane in the hydrostatic approximation, with N=constant:

$$
\begin{cases}
u_t - fv = -\dfrac{1}{\rho_0} p_x \\
v_t + fu = -\dfrac{1}{\rho_0} p_y \\
0 = -\dfrac{1}{\rho_0} p_z - \dfrac{g\rho}{\rho_0} \\
u_x + v_y + w_z = 0 \\
\rho_t + w\dfrac{d\rho_0}{dz} = 0
\end{cases}
$$

(Notice that even if $w_t << g$ in the third momentum equation, $w_t \neq 0$!!)
Following a procedure like the one developed in class, reduce the system to a unique equation for the vertical velocity w.

Assume a plane wave as solution

$$w = e^{-i\omega t + ikx + ily + imz}$$

and derive the dispersion relationship. Write it in terms of the angle θ of the polar system in wave-number space as well as in terms of the wave numbers (k,l,m).
Discuss the two limiting cases, $N=0$ and $f=0$. Provide a short discussion comparing this solution with the one derived for the non-hydrostatic case.

3. Consider the case of a non-rotating stratified but compressible fluid for which the Boussinesq approximation does not hold. The momentum and mass conservation equations are still

$$\frac{du^*}{dt} = -\frac{1}{\rho^*}\frac{\partial p^*}{\partial x} \qquad (1)$$

$$\frac{dv^*}{dt} = -\frac{1}{\rho^*}\frac{\partial p^*}{\partial y} \qquad (2)$$

$$\frac{dw^*}{dt} = -\frac{1}{\rho^*}\frac{\partial p^*}{\partial z} - g \qquad (3)$$

$$\frac{\partial \rho^*}{\partial t} + \underline{\nabla} \cdot \left(\rho^* \vec{u}^*\right) = 0 \qquad (4)$$

but the energy equation is now:

$$\frac{d\rho^*}{dt} = \frac{1}{c_s^2}\frac{dp^*}{dt} \qquad (5)$$

where c_s is the sound speed.

Assume as previously

$$\begin{cases} \overrightarrow{u^*} = \overrightarrow{u_0} + \vec{u} \equiv \vec{u} \\ p^* = p_0(z) + p \\ \rho^* = \rho_0(z) + \rho \end{cases}$$

where the basic state is at rest ($\overrightarrow{u_0} \equiv 0$) and is in hydrostatic equilibrium.

a) Transform eq. (4) in such a way as to use eq. (5) for $\frac{d\rho^*}{dt}$. This will be the new equation (4).

b) Linearize all the equations of motion, remembering that the Brünt-Vaisala frequency is now:

$$N^2(z) = -\frac{g}{\rho_0}\frac{d\rho_0}{dz} - \frac{g^2}{c_s^2}.$$

c) Equations (1), (2), (3) will be analogous to those derived in class for the incompressible case. Equations (4) and (5) will be different, containing terms involving the sound speed, c_s. Derive again for the entire set two equations in (*w*,*p*).

d) In the two equations obtained for (*w*,*p*), assume the coefficients to be constant and the time dependence of $e^{-i\omega t}$, which will allow you to derive in a simple way a final equation for *p*, by eliminating *w*.

e) In the *p*-equation assume a sinusoidal wave as a solution. Derive the dispersion relationship. What is its limit when $c_s \to \infty$?

f) Optional: as the surfaces of constant ω are surfaces of revolution around the *m*-axis, expressing $m = m(\omega)$, find the projections of the ω=constant surfaces in the (m, k_H) plane in the two limits $\frac{\omega}{N} << 1$ and $\frac{\omega}{N} >> 1$.

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