FINAL EXAM
Please do individually

1. Long gravity waves obey the dispersion relation

$$\omega^2 = gH\left(k^2 + l^2\right)$$

Where H is the water depth and g is the gravity acceleration. Suppose that a packet of waves with $\mathbf{k}=(k_0, l_0)$ propagates from a region of uniform depth H_0 into water which slowly deepens as *x* increases, but the depth is independent of *y*.

   a) Find the depth to which the waves can propagate. (i.e., where is the turning point where $c_{gx}=0$)?
   b) Find the shape of the ray (function of (x, y)) in the vicinity of the turning depth H_T. (hint: Taylor expand in the neighborhood of H_T.)

2. Now consider the same packet from problem 1 to propagate into water that slowly shallows as *x* increases. The waves have an initial amplitude of η_0 in the region prior to the sloping topography (where $H=H_0$).

   a) Find the relation for the variation of wave amplitude as the waves move into shallower water. When the water depth becomes very small how does the amplitude depend on *H* ? (you will need the energy equation).

   b) How does the amplitude behave in the neighborhood of the turning point in problem 1?

   Remember that energy, averaged over a wave period, or wavelength, is $E=\rho g\eta^2/2$.

3. Consider a nonrotating stratified fluid with constant *N*. At time t=0 a small region of the fluid is disturbed near the origin x=0, z=0. This generates internal waves that radiate away from the disturbance site. The disturbance generates a broad band of wave frequencies, but the wavelengths are limited to a maximum of λ_m.

   a) If the fluid is taken to be inviscid find the region of the fluid that could contain waves at some later time $t > 0$. For simplicity take the flow to be two-dimensional (x, z). Consider the trajectory of the ray emanating from the source at (0,0).

   b) After a long time the effects of viscosity can be expected to damp the waves. For constant viscosity ν the equation governing linear internal waves is (in (x, z)).

$$\nabla^2 w_{tt} + N^2 w_{xx} = \nu \nabla^4 w_t$$

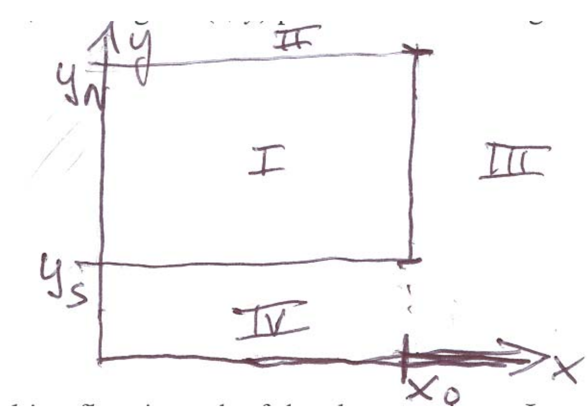
From this equation determine the dispersion relation. Examine the limit where $[\nu(k^2+m^2)]^2 >> 4N^2\cos^2\theta$. To which type of regime does this limit correspond? Here $\cos\theta = k/(k^2+m^2)^{1/2}$ and k and m are the horizontal and vertical wavenumbers, respectively. Write the resulting solution. When does the minimum decay occur? Now examine the opposite limit $[\nu(k^2+m^2] << 4N^2\cos^2\theta$ and write the corresponding solution. To which type of regime does this limit correspond?

4. Consider a layer of incompressible fluid of mean depth D bounded above by a rigid lid, and rotating with a constant frequency $f/2$. For y<0, $f\nabla h_B / D = \beta_1 \hat{j}$ and for y>0, $f\nabla h_B / D = \beta_2 j$, where β_1 and β_2 are constants. In the context of linear quasi-geostrophic theory discuss what happens to a wave with wavenumber $\vec{k}_I = \hat{i} k_I + \hat{j} l_I$ incident upon the line y=o from the region y<0. Find the reflected and transmitted wave amplitudes and wavenumbers. Note that β_1 and β_2 are not equal. When is the reflection total? (Hint: what is preserved in the reflection/transmission?)

5. Consider a layer of uniform density on the beta-plane ($f=f_0+\beta y$) and mean thickness H. The motion is governed by the following equation:

$$fu = -g\eta_y$$
$$fv = g\eta_x$$
$$\eta_t + H(u_x + v_y) = -\lambda\eta + W(x,y,t).$$

Where η is the departure of the layer depth from the rest value H, λ is a constant representing the effects of mixing, W is a prescribed vertical mass flux. At t=0 the vertical mass flux $W = w_0\delta(x-x_0)$ for $y_S < y < y_N$ and W=0 elsewhere is turned on and held fixed, dividing the (x, y) plane in different regions as sketched below:



Find the resulting flow in each of the above regions. Interpret your results in terms of Rossby wave dynamics: which type of Rossby wave do you have? Consider a time $t_1>0$: how far has the wave propagated at y_S, y_N and in between?

6. Consider a two-layer dry compressible atmosphere. The lower layer, $0 < z < D$, is isothermal with temperature T_1, and the upper layer, $D < z < \infty$ has temperature T_2, also a constant. For each layer $N_1^2 =$constant; $N_2^2 =$constant.

   a) Find the basic state density and pressure profiles given the density $\rho=\rho_0$ at z=0. What is the density jump at z=D? What is the condition to have a stable atmosphere?

   b) Under what conditions could a linear baroclinic (i.e., vertical propagation) Rossby wave (with horizontal and temporal structure $\sim e^{i(kx+ly-\sigma t)}$ exist? You will need the vertical structure equation derived in class for both layers but do not need to solve the complete problem to give an answer.

   c) Now solve the problem for waves trapped in both layers ($q_1^2 < 0; q_2^2 < 0$). Specify the vertical structure from the vertical structure equation in both layers. You need to use the boundary conditions at z=0; z=D; $z\rightarrow+\infty$ (express the pressures and vertical velocities in terms of the stream function as done in class). Without fully solving the equations obtained from the boundary conditions, solve them for $\lambda=0$: what mode are you obtaining? What are the vertical profiles of $\varphi_1(z)$, $\varphi_2(z)$ in this case?

7. Consider a two-dimensional basin with homogeneous density ($\rho = \rho_0 =$ constant) of depth D and length L in x-direction. The basin infinitely long in y, i.e. $\frac{\partial}{\partial y} = 0$

   a) Find the linear free surface gravity oscillation and the dispersion relationship. Describe the corresponding surface displacement patterns.

   b) Even though the general solution is a linear superposition of all the normal modes of the basin, consider one individual mode. Now a travelling pressure disturbance is forcing the basin

$$P_a = P_0 \cos(kx - \omega t) = P_0 \,\mathrm{Re}\left[e^{i(kx-\omega t)}\right]$$

   The following pressure fixes only the frequency response of the basin through the surface boundary condition on the velocity potential. Find the amplitude A_N of the n-normal mode. For what value of ω will P_a produce a larger response? How will the response depend on K?