Slowly varying media: Ray theory

Suppose the medium is not homogeneous (gravity waves impinging on a beach, i.e. a varying depth). Then a pure plane wave whose properties are constant in space and time is not a proper description of the wave field.

However, if the changes in the background occur on scales that are long and slow compared to the wavelength and period of the wave, a plane wave solution may be locally appropriate. (Fig. 2.1) This means: $\lambda << L_m$ where L_m is the length scale over which the medium changes. Consider the local plane wave

$$\phi(\vec{x},t) = a(\vec{x},t)e^{i\theta(\vec{x},t)}$$

a varies on the scale L_m while θ varies on the scale λ.

$$\frac{\partial\theta}{\partial x_i} = 0(\frac{1}{\lambda}); \ \frac{1}{a}\frac{\partial a}{\partial x_i} = 0(\frac{1}{L_m})$$

$$\Rightarrow \ \underline{\nabla}\phi = ae^{i\theta} \bullet \underline{\nabla}\theta + 0(\frac{\lambda}{L_m})$$

with $$\theta = \vec{k} \bullet \vec{x} - wt$$

Define the local wavenumber and the local frequency as:

$$\vec{k} = \underline{\nabla}\theta\,|_t \qquad \omega = -\frac{\partial\theta}{\partial t}\,|_x$$

From these definitions it follows that:

$\underline{\nabla} \times \vec{k} = 0$ the local wave number is irrotational.

Conservation of crests in a slowly varying medium.

Suppose we go from point A to point B over the curve C_1.

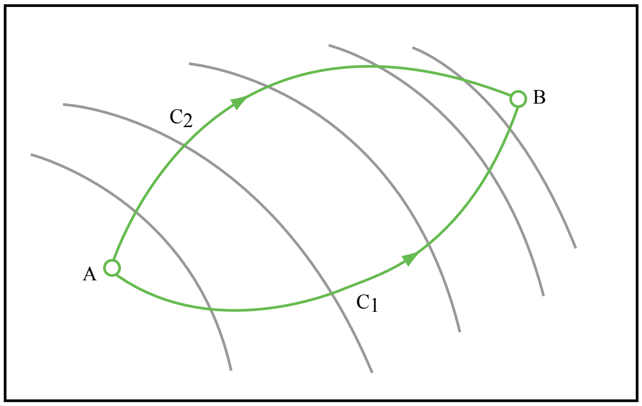

Figure by MIT OpenCourseWare.

Figure 1

slowly varying wave fronts

The number of wave cress we pass along C_1 is

$$n_{c_1} = \frac{1}{2\pi}\int_A^B \vec{k}\bullet d\vec{s} = \frac{1}{2\pi}\int_{c_1} \vec{k}\bullet d\vec{s}$$

The number of wave crests we pass along C_2 is

$$n_{c_2} = \frac{1}{2\pi}\int_A^B \vec{k}\bullet d\vec{s} = \frac{1}{2\pi}\int_{c_2} \vec{k}\bullet d\vec{s}$$

Before for plane waves $\omega = \Omega(\vec{k})$ only, now $\omega = \Omega(\vec{k},\vec{x},t)$.

As $(\omega,\vec{k})$ are slowly varying functions of space/time, the dispersion relation is explicitly dependent on space/time. Now we can introduce the group velocity in another way

$$\frac{\partial\omega}{\partial t}\Big|_{\vec{x}} = \frac{\partial\Omega}{\partial t}\Big|_{\vec{k},\vec{x}} + \sum_i \frac{\partial\Omega}{\partial k_i}\Big|_{\vec{x},t}\frac{\partial k_i}{\partial t}\Big|_{\vec{x}} =$$

$$= \frac{\partial\Omega}{\partial t}\Big|_{\vec{k},\vec{x}} + c_{gi}\frac{\partial k_i}{\partial t}\Big|_{\vec{x}}$$

Where we use the summation convention over repeated indices,

and $\quad c_{gi} = \dfrac{\partial\Omega}{\partial k_i} \quad$ by definition i = 1, 2, 3 = x,y,z

$\vec{c}_g = \nabla_{\vec{k}}\Omega$ group velocity

The difference is:

$$n_{c_1} - n_{c_2} = \frac{1}{\pi}[(\int_{c_1} - \int_{c_2})\vec{k} \bullet d\vec{s}] = \frac{1}{\pi}[(\int_{c_1 - c_2} + \int)\vec{k} \bullet d\vec{s}] = \frac{1}{2\pi}\oint_{c_{total}} \vec{k} \bullet d\vec{s} = \iint_A \underline{\nabla} x\vec{k} \bullet \hat{n} dA \equiv 0$$

$\hat{n}$ = unit vector normal to C

We have used Stokes theorem relating the line integral of the tangential component of $\vec{k}$ to the area integral of its curl over the area bounded by the closed contour C. The increase of phase is the same on C_1 and C_2. This means the number of crests along C_1 is the same as the number of crests along C_2, that is the number of cress inside the area A is conserved. Crests are neither created nor destroyed inside A. The crests have no ends, so the number of crests within a wave group will be the same for all time. This is obviously true only for slowly varying plane waves.

From the definition of $\vec{k}$ and w it follows:

$$\frac{\partial\vec{k}}{\partial t} + \underline{\nabla}\omega = 0 \tag{1}$$

We have seen that the number of crests we cross from A to B is the same along any path connecting A and B. Then:

$$n = \frac{1}{2\pi}\int_A^B \vec{k} \bullet d\vec{s}$$

$$\frac{\partial n}{\partial t} = \frac{1}{2\pi}\int_A^B \frac{\partial\vec{k}}{\partial t} \bullet d\vec{s} = -\frac{1}{2\pi}\int_A^B \underline{\nabla}\omega \bullet d\vec{s} = \frac{1}{2\pi}(\omega(A) - \omega(B)]$$

This says that the rate of change of the number of wave crests between A and B is equal to the frequency of crest inflow at A minus the frequency crest outflow at B.

Crests are neither created nor destroyed in the smoothly varying function ϕ. The number in any local region increases or decreases solely due to the arrival of pre-existing crests at A, not to the creation or destruction of existing crests.

We now introduce the dynamics by asserting that the wavenumber and frequency must be related by a dispersion relation in the same way as for a plane wave.

Since by eq. (1)

$$\frac{\partial k_i}{\partial t} = -\frac{\partial \omega}{\partial x_i} \qquad \text{we have}$$

$$\frac{\partial \omega}{\partial t} = \frac{\partial \Omega}{\partial t} - c_{g_i}\frac{\partial \omega}{\partial x_i} \qquad \text{or}$$

$$\frac{\partial \omega}{\partial t} + \vec{c}_g \bullet \underline{\nabla}\omega = \frac{\partial \Omega}{\partial t}\Big|_{\vec{k},\vec{x}} \qquad (1) \quad \text{equation for } \omega$$

Similarly from (1)

$$\frac{\partial k_i}{\partial t}\Big|_{\vec{x}} + \frac{\partial \Omega}{\partial x_i}\Big|_{\vec{k},t} + \frac{\partial \Omega}{\partial k_j}\Big|_{\vec{x},\vec{t}}\ \frac{\partial k_j}{\partial x_i} = 0$$

$$\frac{\partial k_i}{\partial t} + \frac{\partial \Omega}{\partial kj}\frac{\partial k_j}{\partial x_i} = -\frac{\partial \Omega}{\partial x_i} \qquad \text{or}$$

$$\frac{\partial \vec{k}}{\partial t} + \vec{c}_g \bullet \underline{\nabla}\vec{k} = -\underline{\nabla}\Omega\Big|_{\vec{k},t} \qquad (2)$$

The "ray equation" gives the velocity at which the wave packet, or wave group, moves:

$$\vec{c}_g = \frac{d\overline{x}}{dt} \qquad \text{or} \quad c_{gx} = \frac{dx}{dt}; \quad c_{gy} = \frac{dy}{dt}$$

in two dimensions. Then the ray path in the (x,y) plane is

$$\frac{dy}{dx} = \frac{c_{gy}}{c_{gx}} \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + \vec{c}_g \bullet \underline{\nabla}$$

$$\vec{c}_g = \frac{d\vec{x}}{dt} \qquad \text{(I)}$$

$$\frac{\partial \omega}{\partial t} + c_g \bullet \underline{\nabla}\omega = \frac{\partial \Omega}{\partial t}\Big|_{\vec{t},\vec{x}} \qquad \text{(II)}$$

$$\frac{\partial \vec{k}}{\partial t} + \vec{c}_g \bullet \underline{\nabla}\vec{k} = -\underline{\nabla}\Omega\Big|_{\vec{k},t} \qquad \text{(III)}$$

$\Omega = \Omega(\vec{k},\vec{x},t)$ has an explicit parametric dependence on $(\vec{x},t)$, for instance when waves enter in water of changing depth. The ray equations give the evolution of the local wavenumber $\vec{k}$ and the local frequency ω as we move along the ray, i.e. we move with the wave packet at the local group velocity $\vec{c}_g$. Is a plane wave a particular solution of the ray theory formulation? Suppose the medium is homogeneous, no changes in $(\vec{x},t)$

$\omega = \Omega(\vec{k})$ only

Solution: plane wave $\phi = ae^{i(\vec{k}\bullet\vec{x}-\omega t)}$

where $(\vec{k},\omega)$ do not change but are constant in space

Initial condition $\phi(\vec{x}) = ae^{i\vec{k}\bullet\vec{x}}$ gives $\vec{k}(t=0)$

As $\frac{\partial \vec{k}}{\partial x_2} \equiv 0$ and $\frac{\partial \Omega}{\partial x_i} \equiv 0$

The ray equation (III) gives

$\frac{\partial \vec{k}}{\partial t} = 0$: $\vec{k}$ never changes along the ray and remains equal to $\vec{k}$ (t=0). .

$\omega = \Omega(\vec{k})$ gives ω at t = 0

As $\frac{\partial\omega}{\partial x_i} = 0$; $\frac{\partial\Omega}{\partial t} = 0$ eq. (II) gives

$$\frac{\partial\omega}{\partial t} = 0 \qquad \omega = \omega(t=0)$$

The frequency never changes along the ray. Thus the plane wave solution in a homogeneous medium is entirely consistent with the ray theory formulation.

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