8. Internal waves modified by rotation – unbounded fluid

The equations of motion, linearized, now are:

(1) $\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho_o}\frac{\partial p}{\partial x}$ $\rho_o = \rho_o(z)$, $p_o = p_o(z)$

(2) $\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho_o}\frac{\partial p}{\partial y}$ f = constant – f-plane

(3) $\frac{\partial w}{\partial t} = -\frac{1}{\rho_o}\frac{\partial p}{\partial z} - \frac{\rho g}{\rho_o}$ Basic state:

(4) $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ $\frac{\partial p_o}{\partial z} = -\rho_o g$

(5) $\frac{\partial \rho}{\partial t} + w\frac{d\rho_o}{\partial z} = 0$

$\frac{\partial}{\partial t}$ of (1) and f x (2)

$$\frac{\partial^2 u}{\partial t^2} - f\frac{\partial v}{\partial t} = -\frac{1}{\rho_o}\frac{\partial^2}{\partial x \partial t} \quad \Rightarrow \quad \frac{\partial^2 u}{\partial t^2} + f^2 u = \frac{-1}{\rho_o}\frac{\partial^2 p}{\partial x \partial t} - \frac{f}{\rho_o}\frac{2p}{\partial y}$$

fx (1) and $\frac{\partial}{\partial t}$ of (2)

$$f\frac{\partial v}{\partial t} = -f^2 u - \frac{f}{\rho_o}\frac{\partial p}{\partial y} = \frac{\partial^2 u}{\partial t^2} + \frac{1}{\rho_o}\frac{\partial^2 p}{\partial x \partial t} \Rightarrow \frac{\partial^2 v}{\partial t^2} + f^2 v = -\frac{1}{\rho_o}\frac{\partial^2 p}{\partial y \partial t} + \frac{f}{\rho_o}\frac{\partial p}{\partial x}$$

$$\frac{\partial^2 u}{\partial t^2} + f^2 u = -\frac{1}{\rho_o}\frac{\partial^2 p}{\partial x \partial t} - \frac{f}{\rho_o}\frac{\partial p}{\partial y}$$

these give (u,v) if we know p

$$\frac{\partial^2 v}{\partial t^2} + f^2 v = -\frac{1}{\rho_o}\frac{\partial^2 p}{\partial y \partial t} + \frac{f}{\rho_o}\frac{\partial p}{\partial x}$$

Adopt the procedure followed with f = 0

Take $\frac{\partial}{\partial t}$ of incompressibility equation (4)

$$\frac{\partial}{\partial t}\frac{\partial u}{\partial x}+\frac{\partial}{\partial t}\frac{\partial v}{\partial y}+\frac{\partial^2 w}{\partial z\partial t}=0$$

using the horizontal momentum eqs.

$$\frac{\partial}{\partial x}\left[fv-\frac{1}{\rho_o}\frac{\partial p}{\partial x}\right]+\frac{\partial}{\partial y}\left[-fu-\frac{1}{\rho_o}\frac{\partial p}{\partial y}\right]+\frac{\partial^2 w}{\partial z\partial t}=0$$

or $$\frac{\partial^2 w}{\partial z\partial t}+f(v_x-u_y)=\frac{1}{\rho_o}\nabla_H^2 p \qquad \text{(I)} \qquad \text{(like we did for } f=0)$$

$\zeta = v_x - u_y$

The equation for ζ is simply obtained forming the vorticity eq. from the horizontal momentum eqns.

$$\frac{\partial}{\partial t}(v_x-u_y)-f\frac{\partial w}{\partial z}=0 \qquad \text{(II)} \qquad \text{Three functions } (p, \zeta, w) \text{ and 2 eqs.}$$

Notice that if $f = 0$ $\quad \frac{\partial}{\partial t}(v_x-u_y)=0 \qquad \zeta = 0$ for all times

However, neither ζ nor $\frac{\partial w}{\partial z}$ can be eliminated from (I) and (II). So we must follow a different approach:

Evaluate the horizontal divergence from horizontal momentum eqs.

$$\frac{\partial}{\partial t}u_x-fv_x=-\frac{1}{\rho_o}\frac{\partial^2 p}{\partial x^2}$$

$$\frac{\partial}{\partial t}v_y+fu_y=-\frac{1}{\rho_o}\frac{\partial^2 p}{\partial y^2}$$

gives

$$\frac{\partial}{\partial t}(u_x+v_y)-f(v_x-u_y)=-\frac{1}{\rho_o}\nabla_H^2 P$$

$$\frac{\partial^2}{\partial^2 t}(u_x+v_y)-f\frac{\partial}{\partial t}(v_x-u_y)=-\frac{1}{\rho_o}\frac{\partial}{\partial t}\nabla_H^2 p$$

But from (II) $\frac{\partial}{\partial t}(v_x - u_y) = f\frac{\partial w}{\partial z}$

$u_x + v_y = -w_z$ from (4)

$$\left[\frac{\partial^2}{\partial t^2} + f^2\right]\frac{\partial w}{\partial z} = +\frac{1}{\rho_o}\frac{\partial}{\partial t}\nabla_H^2 p \quad \text{(III)}$$

From (3) and (5)

$$\rho_o \frac{\partial^2 w}{\partial t^2} = -\frac{\partial^2 p}{\partial z \partial t} - g\frac{\partial \rho}{\partial t} = \frac{\partial^2 p}{\partial z \partial t} + g\frac{d\rho_o}{dz}w$$

or

$$\frac{\partial^2 w}{\partial t^2} + \left(\frac{-g}{\rho_o}\frac{d\rho_o}{dz}\right)w = -\frac{1}{\rho_o}\frac{\partial^2 p}{\partial z \partial t}$$

$$\frac{\partial^2 w}{\partial t^2} + N^2 w = -\frac{1}{\rho_o}\frac{\partial^2 p}{\partial z \partial t} \quad \text{(IV)}$$

Rewrite (III) and (IV)

$$\rho_o\left(\frac{\partial^2}{\partial t^2} + f^2\right)\frac{\partial w}{\partial z} = \frac{\partial}{\partial t}\nabla_H^2 p$$

$$\rho_o\left(\frac{\partial^2 w}{\partial t^2} + N^2 w\right) = -\frac{\partial^2 p}{\partial z \partial t}$$

Take ∇_H^2 of the second one:

$$\nabla_H^2\left[\rho_o\left(\frac{\partial^2 w}{\partial t^2} + N^2 w\right)\right] = -\frac{\partial}{\partial z}\left[\frac{\partial}{\partial t}\nabla_H^2 p\right] = \frac{-\partial}{\partial z}\rho_o(\frac{\partial^2}{\partial t^2} + f^2)\frac{\partial w}{\partial z}]$$

or:

$$\rho_o \nabla_H^2\left[\frac{\partial^2 w}{\partial t^2} + N^2 w\right] + \frac{\partial}{\partial z}\left[\rho_o\frac{\partial^2}{\partial t^2}\frac{\partial w}{\partial z} + \rho_o f^2\frac{\partial w}{\partial z}\right] = 0$$

$$\frac{\partial^2}{\partial t^2}\nabla_H^2 w + N^2\nabla_H^2 w + \frac{\partial^2}{\partial t^2}\left[\frac{1}{\rho_o}\frac{\partial}{\partial z}\left(\rho_o\frac{\partial w}{\partial z}\right)\right] + \frac{f^2}{\rho_o}\frac{\partial}{\partial z}\left(\rho_o\frac{\partial w}{\partial z}\right) = 0$$

or:

$$\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{1}{\rho_o}\frac{\partial}{\partial z}\left(\rho_o\frac{\partial w}{\partial z}\right)\right] + \frac{f^2}{\rho_o}\frac{\partial}{\partial z}\left(\rho_o\frac{\partial w}{\partial z}\right) + N^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = 0$$

Similarly to what we did for the case of no rotation

$$\frac{1}{\rho_o}\frac{\partial}{\partial z}\left(\rho_o\frac{\partial w}{\partial z}\right) = \frac{1}{\rho_o}\frac{d\rho_o}{dz}\frac{\partial w}{\partial z} + \frac{\partial^2 w}{\partial z^2}$$

and $\left(\frac{1}{\rho_o}\frac{d\rho_o}{dz}\frac{\partial w}{\partial z}\right) / \frac{\partial^2 w}{\partial z^2} << 1$ which is also the Boussinesq approximation as in the non-rotating case

Then the master equation simplifies to:

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + f\frac{\partial^2 w}{\partial z^2} + N^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = 0$$

becoming the old one if $N = 0$

Look for $w = w_o e^{i(kx+ly+mz-\omega t)}$ we get the dispersion relationship

$$\omega^2 = \frac{f^2 m^2 + N^2(k^2 + l^2)}{k^2 + l^2 + m^2}$$

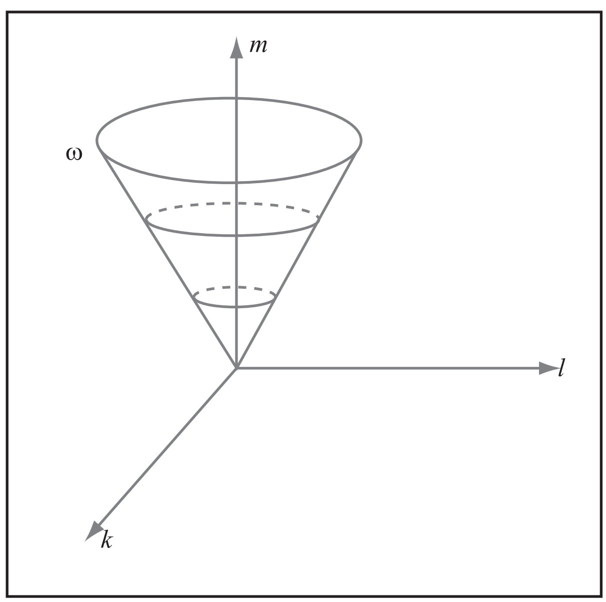

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Figure 1.

or

$$m^2 = \left[\frac{N^2 - \omega^2}{\omega^2 - f^2}\right](k^2 + l^2)$$

Remember:

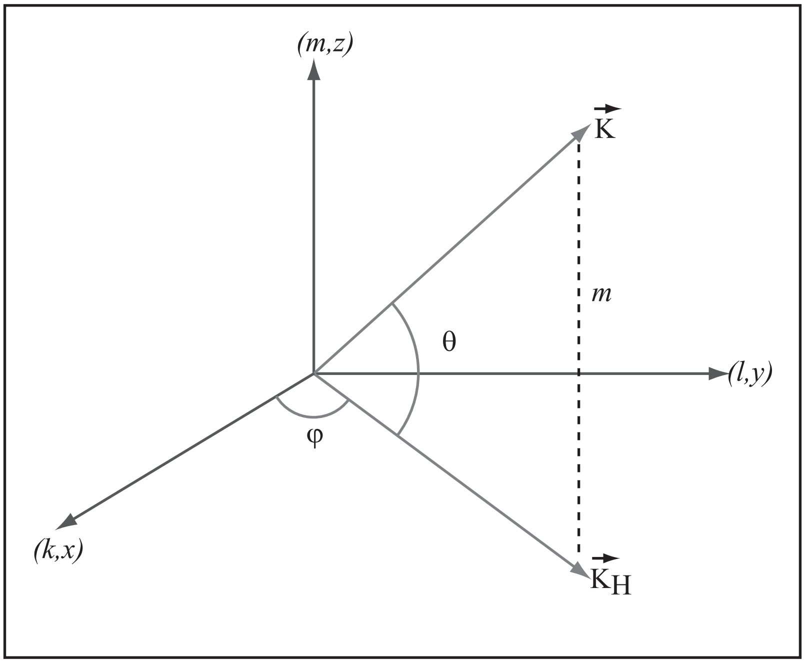

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Figure 2.

Rewrite the dispersion relationship as:

$$\omega^2 = f^2 \frac{m^2}{K^2} + N^2 \frac{k_H^2}{K^2}$$

$$m = K \sin\theta \quad k_H = K \cos\theta$$

$$\omega^2 = f^2 \sin^2\theta + N^2 \cos^2\theta$$ which becomes $\omega = \pm N\cos\theta$ if $f = 0$

In the atmosphere and ocean N>>f and $\frac{N}{f} = 0(100)$ so that the dispersion curves

previously shown still holds. Again, from

$\underline{\nabla} \bullet \vec{u} = 0$ and $\vec{u} = \vec{u}_o e^{i(\vec{k}\bullet\vec{x} - \omega t)}$ $\quad \vec{K} \bullet \vec{u}_o = 0$

$\vec{u}$ is in planes perpendicular to $\vec{K} \rightarrow$ transverse waves phase lines are lines of constant p:

$\underline{\nabla}p = \vec{K}p_o$ is parallel to $\vec{K}$ and perpendicular to $\vec{u}$.

Other useful forms of the dispersion relation

$$\omega^2 = f^2\sin^2\theta + N^2\cos^2\theta$$

are

a) $N^2 - \omega^2 = N^2 - f^2\sin^2\theta - N^2\cos^2\theta = N^2(1-\cos^2\theta) - f^2\sin^2\theta = N^2\sin^2\theta - f^2\sin^2\theta$

$$N^2 - \omega^2 = (N^2 - f^2)\sin^2\theta$$

b) $\omega^2 - f^2 = f^2\sin^2\theta + N^2\cos^2\theta - f^2 = -f^2(1-\sin^2\theta) + N^2\cos^2\theta = N^2\cos^2\theta - f^2\cos^2\theta$

$$\omega^2 - f^2 = (N^2 - f^2)\cos^2\theta$$

Consider the two limiting cases:

a) $f = 0$

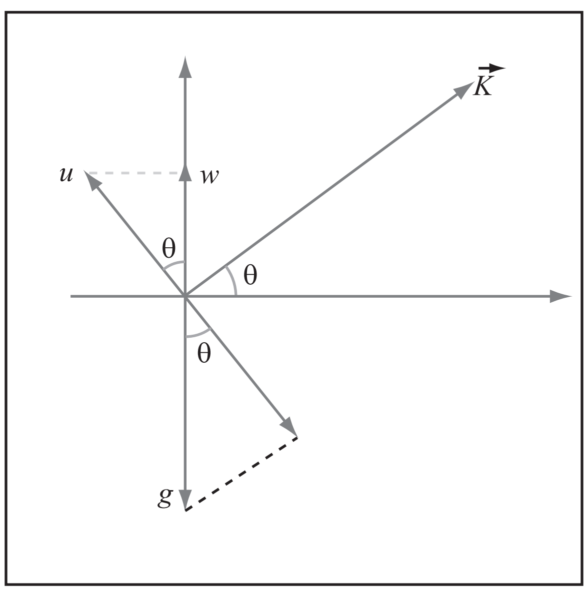

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Figure 3.

$u_t = -g\dfrac{\rho}{\rho_o}\cos\theta$ buoyancy along u

acceleration in u = buoyancy force along u

$$\rho_t + w\rho_{oz} = 0 \quad \rightarrow \quad \rho_t + u\cos\theta\rho_{oz} = 0$$

$$u_{tt} = -\frac{g}{\rho_o}\cos\theta\rho_t = -\frac{g}{\rho_o}(-u\cos\theta\rho_{oz})\cos\theta$$

$$u_{tt} + \left(-\frac{g}{\rho_o}\frac{d\rho_o}{dz}\right)u\cos^2\theta = 0$$

$$u_{tt} + N^2\cos^2\theta u = 0$$ u is a linear oscillator with $\omega = N\cos\theta$

b) If N = 0

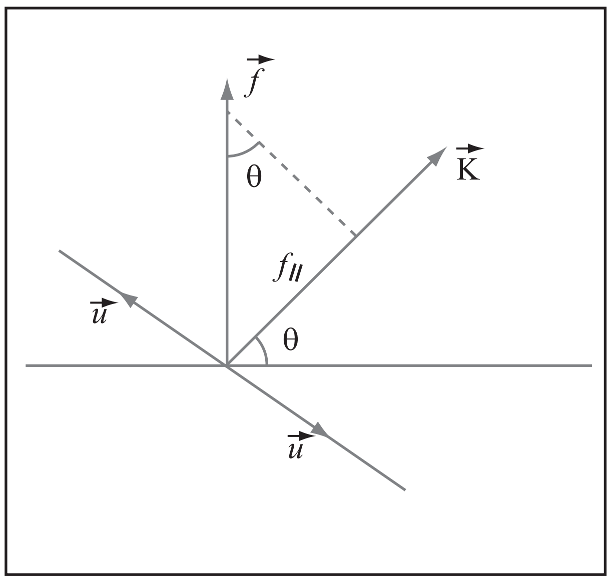

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Figure 4

then the momentum eq. in any direction normal to $\vec{k}$ is:

$$\vec{u}_t + (\vec{f}\times\vec{u}) = 0 \qquad \vec{f}_{//} \text{ component}//\vec{K}$$

or

$$\vec{u}_t + \vec{f}_{//}\times\vec{u} = 0 \qquad f_{//} = f\sin\theta$$

$$\vec{u}_H + (f\sin\theta)\hat{k}'\times\vec{u} = 0 \qquad \hat{k}' = \text{unit vector in the direction of } \vec{K}.$$

The motion occurs in circles around $\vec{K}$ at $\omega = f\sin\theta$

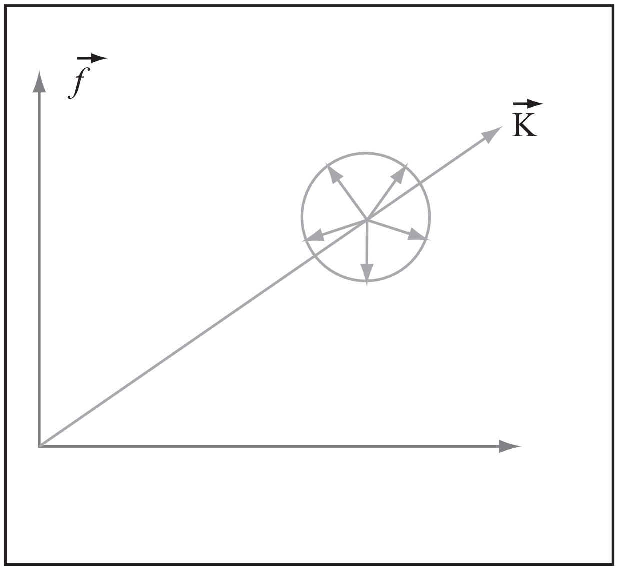

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Figure 5.

Again $\vec{c}_g$ is by definition the gradient of ω in wavenumber space. Like before, it is perpendicular to the conical surfaces of constant ω.

Now:

$$\vec{c}_g = \frac{N^2 - f^2}{\omega K} - \cos\theta \sin\theta [\sin\theta \cos\phi, \sin\theta \sin\phi, -\cos\theta \ ]$$

The magnitude of $\vec{c}_g$ is now

$$\frac{(N^2 - f^2)}{\omega K} - \cos\theta \sin\theta$$

Notice that if $f = 0$ the magnitude becomes

$$\frac{N^2 \cos\theta \sin\theta}{\omega K} = \frac{N^2 \cos\theta \sin\theta}{N \cos\theta K} = \frac{N}{K} \sin\theta \qquad \text{with } \omega = N \cos\theta$$

the previous relationship.

Like before, upward propagation of phase implies downward propagation of energy and vice versa. The particle motions with f≠0, N≠0 are a combination of the two extreme cases.

Assuming $w = w_o \; e^{i(kx+ly+mz-\omega t)}$

$$p = p_o e^{i(k\times+ly+mz-\omega t)} \; ; \; \rho = \rho_o e^{i(k\times+ly+mz-ut)}$$

$$u = \vec{u}_o e^{i(kx+ly+z-\omega t)}$$

the relationship between w and p follows from

$$\frac{\partial^2 w}{\partial t^2} + N^2 w = -\frac{1}{\rho_o}\frac{\partial^2 p}{\partial z \partial t}$$ which gives

$$w = \frac{-m\omega}{N^2-\omega^2}\frac{p_o}{\rho} = -\frac{K\omega}{(N^2-f^2)\sin\theta}\frac{p}{\rho_o}$$

where we have used

$$N^2\text{-}\omega^2 = (N^2\text{-}f^2)\sin^2\theta$$

$$m = K \sin\theta$$

From

$$\frac{\partial^2 u}{\partial t^2} + f^2 u = -\frac{1}{\rho_o}\frac{\partial^2 p}{\partial \times \partial t} - \frac{f}{\rho_o}\frac{\partial p}{\partial y}$$

$$\frac{\partial^2 v}{\partial t^2} + f^2 v = -\frac{1}{\rho_o}\frac{\partial^2 p}{\partial y \partial t} + \frac{f}{\rho_o}\frac{\partial p}{\partial x}$$

The relations follow between (u, v) and p:

$$u = \frac{k\omega + ilf}{\omega^2 - f^2}\frac{p}{\rho_o}; \quad v = \frac{l\omega + ikf}{\omega^2 - f^2}\frac{p}{\rho_o}$$

If the x-axis is chosen to be in the direction of the horizontal component of the wave vector $K \equiv K_H$; $l = 0$

$$u = \frac{k_H \omega}{\omega^2 - f^2} \frac{p}{\rho_o}; \quad v = \frac{-ik_H f}{\omega^2 - f^2} \frac{p}{\rho_o}$$

using $\omega^2 - f^2 = (N^2 - f^2)\cos^2\theta$ and $K_H = K\cos\theta$

we finally get

$$u = \frac{K\omega}{(N^2 - f^2)\cos\theta} \frac{p}{\rho_o} = -\tan\theta w$$

$$v = \frac{-iKf}{(N^2 - f^2)\cos\theta} \frac{p}{\rho_o} = \frac{-ifu}{\omega} = +\frac{if}{\omega}\tan\theta w$$

The perturbation density ρ is obtained from

$$\frac{\partial \rho}{\partial t} + w\frac{\partial \rho_o}{dz} = 0$$

$$\Rightarrow \quad \frac{\rho}{\rho_o} = \frac{iN^2}{g\omega} W$$

$$N^2 = -\frac{g}{\rho_o}\frac{d\rho_o}{dz}$$

Take now the real parts

$\omega = w_o - \cos(kx+my-\omega t)$ $\qquad$ $k \equiv k_H$; $l = 0$

$$\begin{cases} u = -\tan\theta \,\mathrm{Re}(w) = -\tan\theta w_o \cos(kx + mz - \omega t) \\ v = \mathrm{Re}(\frac{if}{\omega}\tan\theta w) = -\frac{f}{\omega}\tan\theta \sin(kx + mz - \omega t) \end{cases}$$

$$\begin{cases} \rho = \mathrm{Re}(\frac{iN^2}{g\omega}w) = -\frac{N^2}{g\omega}w\rho_o \sin(kx+mz-\omega t) \rightarrow \frac{-N^2}{g\omega}\rho_o w_o \sin(kx+mz-\omega t) \\ p = -\frac{\rho_o(N^2-f^2)\sin\theta}{k\omega}\mathrm{Re}(w) = -\frac{\rho_o(N^2-f^2)\sin\theta}{k\omega}w_o \cos(kx+mz-\omega t) \end{cases}$$

Total energy <E> averaged again over one wavelength will obey the same energy equation as the Coriolis force does no work and hence does not contribute to the energy equation.

$$< kE >= \frac{1}{2}\rho_o < u^2 + v^2 + w^2 >$$

$$PE = \rho w g \qquad \Rightarrow \quad PE = \rho g \frac{dz}{dt}$$

From the adiabatic equation

$$\frac{\partial \rho}{\partial t} + w\frac{d\rho_o}{dz} = 0 \qquad \Rightarrow \qquad w = -\frac{1}{d\rho_o / dz}\frac{\partial \rho}{\partial t}$$

From $N^2 = -\frac{g}{\rho_o}\frac{d\rho_o}{dz} \qquad \Rightarrow \qquad -\frac{1}{d\rho_o / dz} = +\frac{g}{\rho_o N^2}$

and $\quad w = \frac{g}{\rho_o N^2}\frac{\partial \rho}{\partial t}$ hence

$$PE = \frac{g^2}{\rho_o N^2}\rho\frac{\partial \rho}{\partial t} = \frac{g^2}{2\rho_o N^2}\frac{\partial \rho^2}{\partial t}$$

The total energy averaged over one wavelength is the same as in the non-rotating case

$$< E >= \frac{1}{2}\rho_o w_o^2 \frac{K^2}{K_H^2} = \frac{1}{2}\rho_o(\frac{w_o}{\cos\theta})^2$$

However in the rotating case energy is not equi-partitioned

<KE> is increased by rotation

$$\frac{<KE>}{<PE>} = \frac{\omega^2 + f^2 \sin^2\theta}{\omega^2 - f^2 \sin^2\theta}$$

<PE> is decreased

The energy flux is again $<F> = <E> \vec{c}_g$

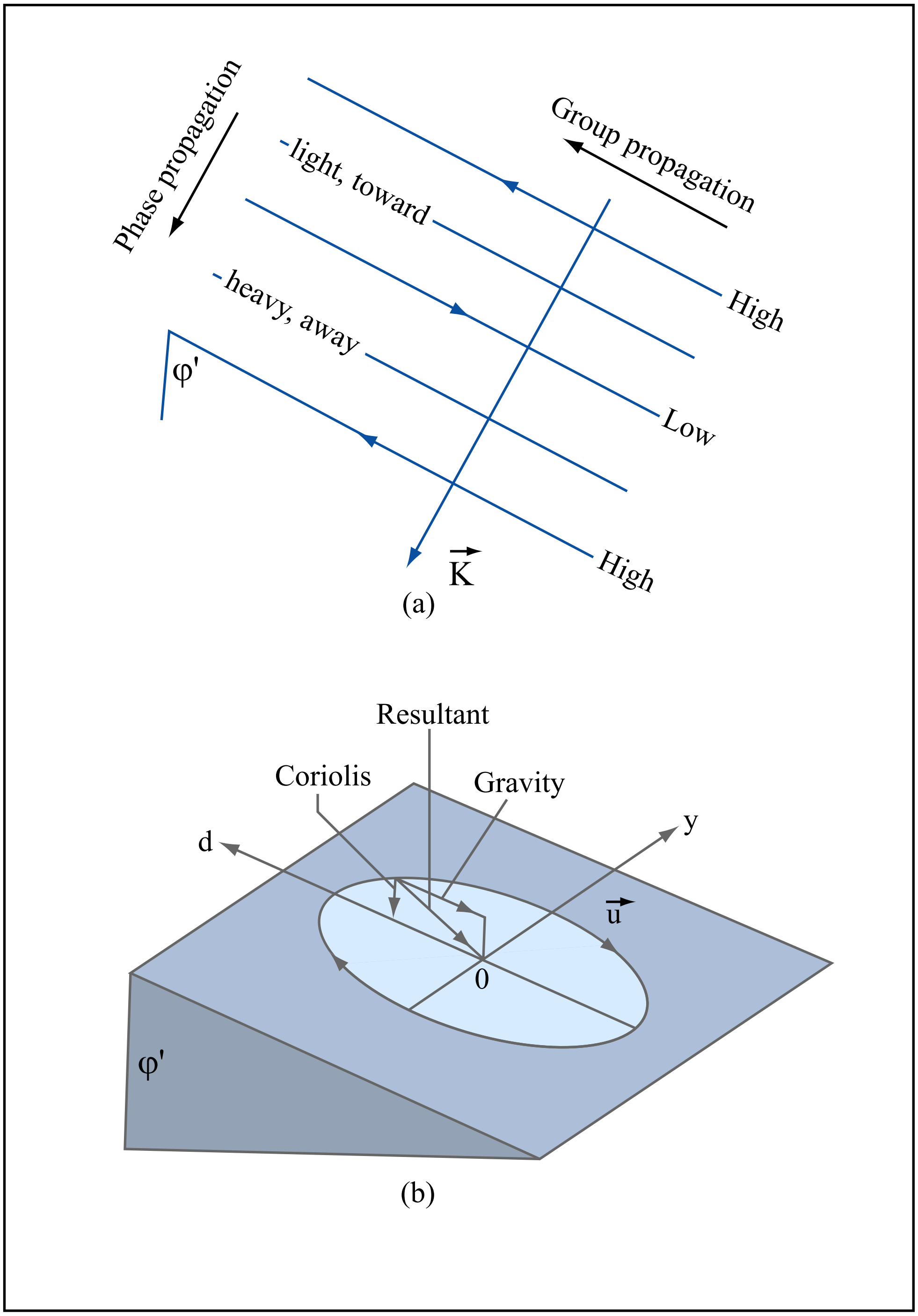

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Figure 6.

12.802 Wave Motion in the Ocean and the Atmosphere
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