

## 11. Laplace tidal equations on the sphere

Thin layer of fluid  $D/L \ll 1$

linearized

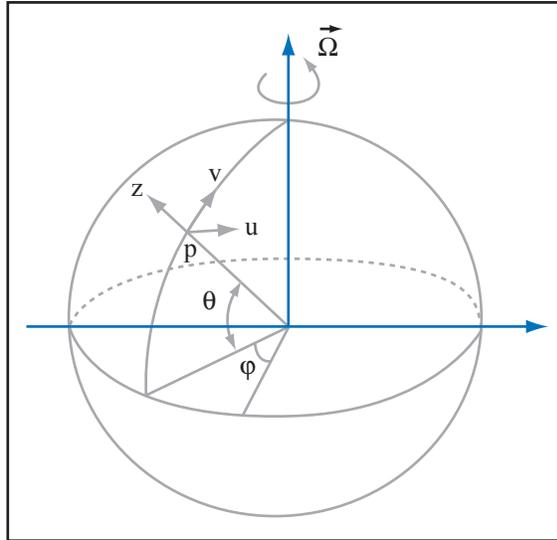


Figure by MIT OpenCourseWare.

$\phi$  = longitude (average) of P,  $\theta$  = latitude;

$u$  = east-west velocity

$R$  = earth's radius

$v$  = south-north velocity

$w$  = radial velocity upward

$$\frac{\partial u}{\partial t} - 2\Omega \sin \theta v + 2\Omega \cos \theta w = -\frac{1}{\rho_0 R \cos \theta} \frac{\partial p}{\partial \phi}$$

$$\frac{\partial v}{\partial t} + 2\Omega \sin \theta u = -\frac{1}{\rho_0 R} \frac{\partial p}{\partial \theta}$$

$$\frac{\partial w}{\partial t} - 2\Omega \cos \theta u = -\frac{1}{\rho_0} \frac{\partial p}{\partial z} - \frac{g p}{\rho_0} + \Omega^2 \cos^2 \theta r$$

$$\frac{\partial p}{\partial t} + w \frac{\partial p_0}{\partial z} = 0 \quad \rightarrow \text{adiabatic motion}$$

$$\frac{\partial u}{\partial \phi} + \frac{\partial}{\partial \theta} (v \cos \theta) + R \cos \theta \frac{\partial w}{\partial z} = 0$$

The equations have been linearized and, as usual, we have split

$$\rho_{\text{total}} = \rho_o(z) + \rho(\phi, \theta, z, t)$$

$$\text{with } \frac{d\rho_o}{dz} = -\rho_o g$$

$$p_{\text{total}} = p_o(z) + p(\phi, \theta, z, t)$$

The third momentum equation contains the centrifugal force which can be written as the gradient of the centrifugal potential

$$\nabla \left( \frac{1}{2} \Omega^2 \cos^2 \theta r^2 \right) \quad r = \text{distance from center of the earth}$$

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial z}$$

which can be combined with the gravitational potential:

$$\nabla \left( gr + \frac{1}{2} \Omega^2 \cos^2 \theta r^2 \right) = \nabla g' \rightarrow \text{ellipsoidal surfaces}$$

$g'$  varies by about 0.3% from the pole to the equator. In this way we can eliminate the centrifugal component which is very small and we keep spherical coordinates because the variation in  $g'$  is very small.

More important are the two components of the Earth's rotation,  $2\Omega \cos\theta$ , tangent to the earth's surface.

In the first momentum equation

$$\frac{W}{U} = \frac{D}{L} \ll 1 \quad \text{from continuity}$$

Thus, on the basis of scale analysis we can ignore the tangent component of the Earth's rotation in the first momentum equation.

Then energy arguments make it necessary to ignore the tangent component also in the third momentum equation. In fact if we keep it and we form the energy equation, we find the Coriolis force doing work, which is impossible.

Neglecting  $2\Omega\cos\theta w$  and  $2\Omega\cos\theta u$  is often called the traditional approximation. We want, however to justify it again through scale analysis. The scale for the pressure field is determined by the geostrophic balance (second momentum eq. as the 2 terms are both  $0(1)$ ).

$$P^* = \rho_0 2UL$$

then 
$$\frac{\partial P^*}{\partial z} = 0\left(\rho_0 2 \frac{\Omega UL}{D}\right)$$

Then in the third momentum equation:

$$\frac{\rho_0 2\Omega u \cos\theta}{P_z} = 0\left(\frac{D}{L}\right) \ll 1$$

And we can ignore the tangent component of the Earth's rotation also in the third momentum, eq.

Then

$$u_t - 2\Omega \sin\theta v = -\frac{1}{\rho_0 R \cos\theta} p_\phi$$

$$v_t + 2\Omega \sin\theta u = -\frac{1}{\rho_0 R} p_\theta$$

$$0 = -p_z - g\rho \quad \rightarrow \text{hydrostatic}$$

$$u_\phi + (v\cos\theta)_\theta + R\cos\theta w_z = 0$$

$$\rho_t + w \frac{d\rho_0}{dz} = 0$$

Where in the third momentum equation we have neglected the vertical acceleration as for large scale motions  $|w_t| \ll g$ .

As already done, we can write the adiabatic equation as

$$\rho_t - w \frac{\rho_0 N^2}{g} = 0 \quad \text{with} \quad N^2 = \frac{-g}{\rho_0} \frac{d\rho_0}{dz}$$

$$g\rho = -p_z$$

combining the two we eliminate  $\rho$   $g\rho_t = w\rho_0 N^2 = -p_{zt}$

$$p_{zt} + \rho_0 N^2 w = 0 \quad \text{knowing } p_z, \text{ we know } \rho$$

So now we have 4 equations in (u,v,w, p)

$$u_t - 2\Omega \sin \theta v = -\frac{1}{\rho_0 R \cos \theta} P_\phi$$

$$v_t + 2\Omega \sin \theta u = -\frac{1}{\rho_0 R} P_\phi$$

$$u_\phi + (v \cos \theta)_\theta + R \cos \theta w_z = 0$$

$$p_{zt} + \rho_0 N^2 w = 0$$

We consider a flat bottom only  $z = -D$ , because in this case we can make a separation of variables into a function of  $z$  and one of  $(\phi, \theta)$ :

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} U(\phi, \theta, t) \\ V(\phi, \theta, t) \end{bmatrix} F(z) \quad \text{same vertical dependence for (u,v) as they}$$

$$w = W(\phi, \theta, t) G(z) \quad \text{have similar structure}$$

$$\frac{P}{\rho_0} = gP(\phi, \theta, t)F(z) \quad \text{--> needed from geostrophic balance}$$

(U is not a velocity scale now, is a variable!)

Substituting into the z horizontal momentum eqn.,  $F(z)$  drops out:

$$V_t - fV = -\frac{g}{R \sin \theta} P_\phi$$

$$f = 2\Omega \sin \theta$$

$$V_t + fV = -\frac{g}{R} P_\theta$$

The continuity eq. is more complicated

$$\left[ \frac{U_\phi}{R \cos \theta} + \frac{(V \cos \theta)_\theta}{R \cos \theta} \right] F(z) + W G_z = 0$$

or

$$\frac{1}{W} \left[ \frac{U_\phi}{R \cos \theta} + \frac{(V \cos \theta)_\theta}{R \cos \theta} \right] = -\frac{G_z}{F} = -\frac{1}{h} \quad \text{separations constant}$$

independent of z                      dependent only on z

$\frac{1}{h}$  = dimensions of depth =  $\frac{1}{F} \frac{dG}{dz}$  called "equivalent depth"

Then the horizontal structure equation for continuity is:

$$\frac{U_\phi}{R \cos \theta} + \frac{(V \cos \theta)_\phi}{R \cos \theta} + \frac{W}{h} = 0$$

The adiabatic equation with the separation of variables becomes

$$g P_t F_z(z) + N^2 W G(z) = 0$$

or

$$P_t + W \frac{N^2}{g} \frac{G}{F_z} = 0$$

But from

$$\frac{G_z}{F} = \frac{1}{h} \rightarrow G_{zz} = \frac{F_z}{h} \Rightarrow F_z = h b_{zz}$$

and we have

$$P_t + W \frac{G}{G_{zz}} \frac{N^2}{gh} = 0$$

or

$$\frac{P_t}{W} = - \frac{G}{G_{zz}} \frac{N^2}{gh} = \text{arbitrary constant}$$

Notice that while before the separation constant  $h$  had the dimensions of a depth now

$$\frac{G}{G_{zz}} \left( - \frac{g}{\rho_0} \frac{d\rho_0}{dz} \right) \frac{1}{gh} = 0(\lambda^2 \cdot \frac{1}{\lambda^2}) = \text{dimensionless}$$

The separation constant is dimensionless and can be arbitrarily chosen. Choose

$$\frac{P_t}{W} = 1 \rightarrow P_t = W \quad \text{adiabatic equation}$$

the

$$- \frac{G}{G_{zz}} \frac{N^2}{gh} = 1 \rightarrow G_{zz} + \frac{N^2}{gh} G = 0$$

Rewrite all equations:

$$1. \quad U_t - fV = - \frac{g}{R \cos \theta} P_\phi$$

$$2. \quad V_t + fU = - \frac{g}{R} P_\theta$$

$$3. \quad \frac{U_\phi}{R \cos \theta} + \frac{(V \cos \theta)_\theta}{R \cos \theta} + \frac{W}{h} = 0$$

Laplace's tidal equations:

The equations for the horizontal structure

$$4. \quad P_t = W$$

$$5. \quad G_{zz} + \frac{N^2}{gh} G = 0 \quad F = hG_z$$

Notice that the vertical dependence is contained only in the last equation which with the appropriate boundary conditions is an eigenvalue problem in which  $h_n^{-1}$  is the eigenvalue.

Now we need to specify the boundary condition. At  $z = -D$   $w = 0 \Rightarrow G(z) \equiv 0$

At the free surface  $\eta$  the kinematic condition  $w = \frac{\partial \eta}{\partial t}$  gives  $WG(\eta) = \partial \eta / \partial t$

$$P_{\text{total}} = p_{\text{atm}} \equiv 0 \quad p_{\text{total}} = p_o(z) + p \quad \text{at } z = \eta$$

$$\frac{p_{\text{total}}}{\rho_o} = \frac{p_o(\eta)}{\rho_o} + gPF(\eta) \quad \text{expand around } z = 0$$

$$\approx \frac{p_o(0)}{\rho_o} + \frac{1}{\rho_o} \frac{dp_o}{dz} \Big|_{z=0} \eta + \dots gPF(0) = 0$$

$$gPF(0) = g\eta$$

$$\frac{\partial}{\partial t} : \quad g \frac{\partial \eta}{\partial t} = gP_t F(0) \quad \text{or}$$

$$gWG(0) = gP_t F(0) \quad \text{at } z = 0$$

But from 4, adiabatic equation,  $P_t = W$

Therefore,

$$G(0) = F(0) = hG_z(0) \quad \text{from } F = hG_z$$

$$\text{or} \quad G_z - \frac{g}{h} G = 0 \quad \text{at } z = 0$$

So, finally, the eigenvalue problem for  $G$  is

$$G_{zz} + \frac{N^2}{gh} G = 0 \quad \text{Vertical structure equation}$$

$$G = 0 \quad \text{at } z = -D$$

$$G_z - \frac{G}{h} = 0 \quad \text{at } z = 0$$

Notice that there is an infinite number of horizontal structure equations, each set being identical except that instead of the total water depth  $-D$ , each system has an equivalent depth  $h_n$  corresponding to the  $n$ -eigenvalue of the problem.

Generally, the problem must be solved numerically. Let us examine the simplest case  $N^2 = \text{constant}$ . Notice that, if we had kept  $\frac{\partial w}{\partial t}$  in the third momentum eq. and assumed a time-dependent  $e^{-i\omega t}$  for all the functions, we would have obtained the following eigenvalue problem:

$$G_{zz} + \frac{N^2 - \omega^2}{gh} G = 0$$

$$G = 0 \quad \text{at} \quad z = -D$$

$$G_z - \frac{1}{h} G = 0 \quad \text{at} \quad z = 0$$

For  $N^2 = \text{constant}$  a solution  $G(z)$  satisfying the bottom boundary condition at  $z = -D$  is

$$G = A \sin[m(z+D)] \quad m^2 = \frac{N^2}{gh} \quad \text{the wavenumber, quantized}$$

The eigenvalue relation for  $h$  is obtained from the surface boundary condition and gives

$$\tan(mD) = mh = \frac{N^2}{gm} \quad \text{Infinite number of denumerable}$$

solutions

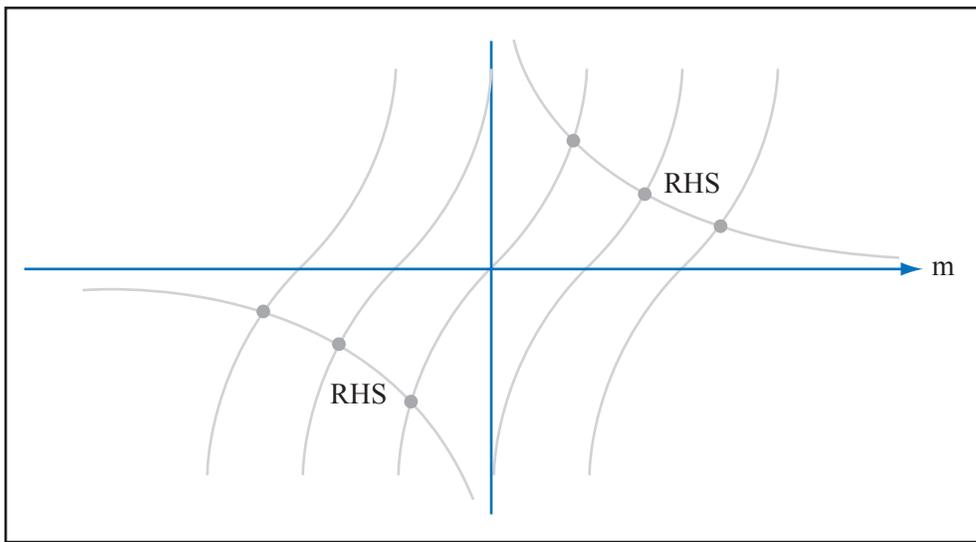


Figure by MIT OpenCourseWare.

Write the dispersion relationships as:

$$\tan(mD) = \left( \frac{N^2 D}{g} \right) \frac{1}{mD}$$

Notice that  $\left[ \frac{N^2 D}{g} = -\frac{D}{\rho_o} \frac{d\rho_o}{dz} \approx -\frac{\Delta\rho_o}{\rho_o} \ll 1 \right]$

Then we can consider 2 limits

$mD = 0(1)$  or large  $m$ . Then RHS of above  $\approx 0 (\ll 1)$  and the solutions are

$$\tan(mD) = 0$$

$$mD = n\pi \quad n = 1, 2, 3 \quad \text{and} \quad m = \frac{n\pi}{D}$$

As  $m^2 = \frac{N^2}{gh}$  the equivalent depth for mode  $n$  is

$$gh_n = \frac{N^2 D}{n^2 \pi^2}$$

For this mode the horizontal equations will have a long wave-gravity wave speed

$$c_n = \sqrt{gh_n} \ll \sqrt{gD}$$

These speeds are the long wave speeds for internal gravity modes, much slower than the

phase speed  $\sqrt{gD}$ .

The model structures for each n are simply

$$G_n = \sin\left[\frac{n\pi(z+D)}{D}\right]$$

$$F_n = \cos\left[\frac{n\pi(z+D)}{D}\right] \quad \text{Baroclinic modes}$$

Lowest mode

Suppose now that  $mD \rightarrow 0$ , very small m, very large  $\lambda_z$ .

$$\text{Then } \tan(mD) \approx mD = \frac{N^2}{gm} \rightarrow m^2 = \frac{N^2}{gD}$$

$$\text{But } m^2 = \frac{N^2}{gh} \rightarrow h_0 = D \quad \text{total depth}$$

This is the lowest mode, the barotropic one

$$\text{Then } m_0 = \sqrt{\frac{N^2}{gD}} \quad m_0 \rightarrow 0$$

$$F_0 = \cos[m_0(z+D)] \quad \text{is almost z-independent}$$

Because of the possibility of variable separation, we can simply study the wave solutions of the horizontal structure equations, remembering that for each solution there is a vertical structure function associated.

Combining 3 and 4, we can finally write the Laplace tidal equations as

$$U_t - fV = -\frac{g}{R \cos\theta} P_\phi$$

$$V_t + fU = -\frac{g}{R} P_\theta \quad \text{I}$$

$$P_t + h \left[ \frac{V_\phi}{R \cos\theta} + \frac{(V \cos\theta)_\theta}{R \cos\theta} \right] = 0$$

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