Equatorial beta-plane

Equations

We consider Mercator coordinates

$$x = a\lambda \quad , \qquad y = a \ln\left(\frac{1+\sin\theta}{\cos\theta}\right)$$

which have the properties that the scale factors in the x and y direction are the same

$$ds = a\cos\theta d\lambda = \cos\theta\, dx$$

$$ds = a\, d\theta = a\frac{1}{\frac{\partial y}{\partial \theta}}dy = \frac{1+\sin\theta}{\cos\theta[1+\frac{\sin\theta+\sin^2\theta}{\cos^2\theta}]}dy = \cos\theta dy$$

with this form, we have $h_1 = h_2 = \cos\theta = \mathrm{sech}(y/a)$ and

$$\frac{\partial}{\partial t}u - fv = -\cosh(y/a)\frac{\partial}{\partial x}P$$

$$\frac{\partial}{\partial t}v + fu = -\cosh(y/a)\frac{\partial}{\partial y}P$$

$$\frac{\partial}{\partial t}P + gH\cosh^2(y/a)\frac{\partial}{\partial x}[\mathrm{sech}(y/a)u] + gH\cosh^2(y/a)\frac{\partial}{\partial y}[\mathrm{sech}(y/a)v] = 0$$

where $f = 2\Omega\sin\theta = 2\Omega\tanh(y/a)$. If we multiply the first and second equations by $\mathrm{sech}(y/a)$ and define $\tilde{\mathbf{u}} = \mathrm{sech}(y/a)\mathbf{u}$, we have

$$\frac{\partial}{\partial t}\tilde{u} - f\tilde{v} = -\frac{\partial}{\partial x}P$$

$$\frac{\partial}{\partial t}\tilde{v} + f\tilde{u} = -\frac{\partial}{\partial y}P$$

$$\frac{\partial}{\partial t}P + gH\cosh^2(y/a)\frac{\partial}{\partial x}\tilde{u} + gH\cosh^2(y/a)\frac{\partial}{\partial y}\tilde{v} = 0$$

(Note - this trick will not work as well with the nonlinear equations, but the equatorial beta-plane approx. is still used for those as well. It's not really necessary, since we will be dropping all the y^2/a^2 terms in any case.) We now expand

$$f \simeq 2\Omega\frac{y}{a}(1 - \frac{1}{3}\frac{y^2}{a^2}) \quad , \qquad \mathrm{sech}^2(y/a) = 1 + \frac{y^2}{a^2}$$

and drop the order $\frac{y^2}{a^2}$ terms to get the "equatorial beta-plane" equations

$$\frac{\partial}{\partial t}\tilde{u} - \beta y\tilde{v} = -\frac{\partial}{\partial x}P$$

$$\frac{\partial}{\partial t}\tilde{v} + \beta y\tilde{u} = -\frac{\partial}{\partial y}P$$

$$\frac{\partial}{\partial t}P + gH\frac{\partial}{\partial x}\tilde{u} + gH\frac{\partial}{\partial y}\tilde{v} = 0$$

We shall drop the tildes and use these hereafter.

Single equation for v

From the momentum equations, we can eliminate u to find

$$\frac{\partial^2 v}{\partial t^2} + \beta^2 y^2 v = \beta y \frac{\partial P}{\partial x} - \frac{\partial^2 P}{\partial y \partial t} \tag{1}$$

We can also eliminate it from the x-momentum and the mass equation to get a second relationship

$$\frac{\partial^2 v}{\partial y \partial t} + \beta y \frac{\partial v}{\partial x} = \frac{\partial^2 P}{\partial x^2} - \frac{1}{gH}\frac{\partial^2 P}{\partial t^2} \tag{2}$$

Now we eliminate P

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{gH}\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2}{\partial t^2} + \beta^2 y^2\right) v = \left(\beta y \frac{\partial}{\partial x} - \frac{\partial^2}{\partial y \partial t}\right)\left(\frac{\partial^2}{\partial y \partial t} + \beta y \frac{\partial}{\partial x}\right) v$$

or

$$\frac{\partial^4}{\partial x^2 \partial t^2} v + \beta^2 y^2 \frac{\partial^2}{\partial x^2} v - \frac{1}{gH}\frac{\partial^4}{\partial t^4} v - \frac{\beta^2 y^2}{gH}\frac{\partial^2}{\partial t^2} v = \beta^2 y^2 \frac{\partial^2}{\partial x^2} v - \frac{\partial^4}{\partial y^2 \partial t^2} v - \beta \frac{\partial^2}{\partial x \partial t} v$$

giving

$$\left(\nabla^2 - \frac{\beta^2 y^2}{gH}\right)\frac{\partial}{\partial t} v_t + \beta \frac{\partial}{\partial x} v_t - \frac{1}{gH}\frac{\partial^3}{\partial t^3} v_t = 0$$

One solution is $\frac{\partial}{\partial t} v{=}0$; we'll come back to that. The other sets of solutions satisfy

$$\left(\nabla^2 - \frac{\beta^2 y^2}{gH}\right)\frac{\partial}{\partial t} v + \beta \frac{\partial}{\partial x} v - \frac{1}{gH}\frac{\partial^3}{\partial t^3} v = 0$$

For solutions with $\exp(\imath k x - \imath \omega t)$ dependence, we get a y-structure equation

$$\left[\frac{\partial^2}{\partial y^2} - k^2 + \frac{\omega^2}{gH} - \frac{\beta k}{\omega} - \frac{\beta^2 y^2}{gH}\right] v = 0 \tag{3}$$

This is the quantum-mechanical harmonic oscillator equation.

Solutions

Gravest mode

We begin with the gravest mode

$$v = \exp(-\frac{1}{2}\frac{y^2}{R^2})$$

which has

$$\frac{\partial^2 v}{\partial y^2} = -\frac{1}{R^2}v + \frac{y^2}{R^4}v$$

and will be a satisfactory solution if

$$R^4 = gH/\beta^2 \quad \textit{and} \quad \frac{\omega^2}{gH} - \frac{\beta k}{\omega} - k^2 = \frac{1}{R^2} \tag{4}$$

The quantity

$$R = (gH)^{1/4}\beta^{-1/2}$$

is called the "equatorial deformation radius" and is the fundamental disturbance scale in the north-south direction.

This cubic has three roots,

$$\omega = -\sqrt{gH}k$$

and

$$\omega = \frac{1}{2}\sqrt{gH}k \pm \frac{1}{2}\sqrt{gH}k\left[1 + \frac{4}{k^2R^2}\right]^{1/2} = \frac{1}{2}\beta R^2 k \pm \frac{1}{2}\beta R^2 k\left[1 + \frac{4}{k^2R^2}\right]^{1/2}$$

The first root turns out to be spurious; if we consider the x-momentum equation and the mass equation, we see that

$$\begin{pmatrix} -\imath\omega & \imath k \\ \imath k g H & -\imath\omega \end{pmatrix}\begin{pmatrix} u \\ P \end{pmatrix} = \begin{pmatrix} \beta y v \\ -gH\frac{\partial v}{\partial y} \end{pmatrix}$$

The two equations are linearly dependent when $\omega^2 = k^2 gH$. See also equation (2). For there to be a non-trivial v solution, we must have

$$-\omega\frac{\partial v}{\partial y} + k\beta y v = 0 \quad \textit{or} \quad \frac{\partial v}{\partial y} = \frac{\beta k}{\omega}yv$$

which is true. However, we have still lost a degree of freedom, so that the remaining equation (1) does not have a pressure which is well-behaved at $\pm\infty$.

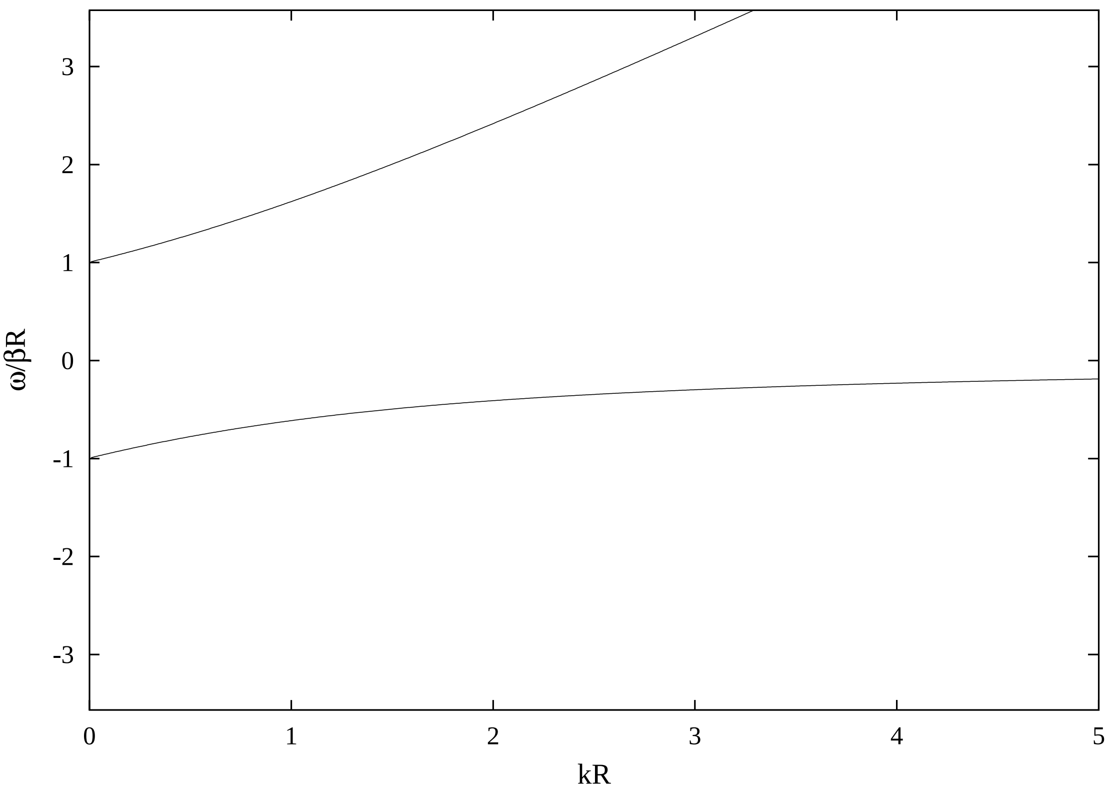
The $\omega^2 \neq gHk^2$ cases have all fields well-defined. This solution gives a dispersion relation as shown in figure (1). Note that the high frequency limit of (4) gives

$$\omega \simeq \left[gH(k^2 + R^{-2})\right]^{1/2}$$

and the low frequency limit is

$$\omega = -\frac{\beta k}{k^2 + R^{-2}}$$

much like the mid-latitude WKB approximations. This wave is called the Yanai wave or the mixed Rossby-gravity wave, since the high-frequency positive branch looks like a gravity wave, while the low-frequeny branch has the characteristics of a Rossby mode. If we take the convention of positive frequency but allow the wavenumber k to span the whole real axis, the connections among the modes becomes clearer (see figure 3).



Higher modes

There are an infinite set of solutions to the structure equation (3)

$$v = H_n(\frac{y}{R})\exp(-\frac{1}{2}\frac{y^2}{R^2})$$

where the H_n are the Hermite polynomials and

$$\frac{\omega^2}{gH} - \frac{\beta k}{\omega} - k^2 = \frac{1}{R^2}(2n+1)$$

The polynomials satisfy

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} = e^{x^2/2}\left(x - \frac{d}{dx}\right)^2 e^{-x^2/2}$$

which imply the two recursion relations

$$H_{n+1} = 2xH_n - 2nH_{n-1} \quad , \quad H'_n = 2nH_{n-1}$$

and therefore

$$\frac{\partial^2}{\partial y^2} v = -\frac{2n+1}{R^2} v + \frac{y^2}{R^2} v$$

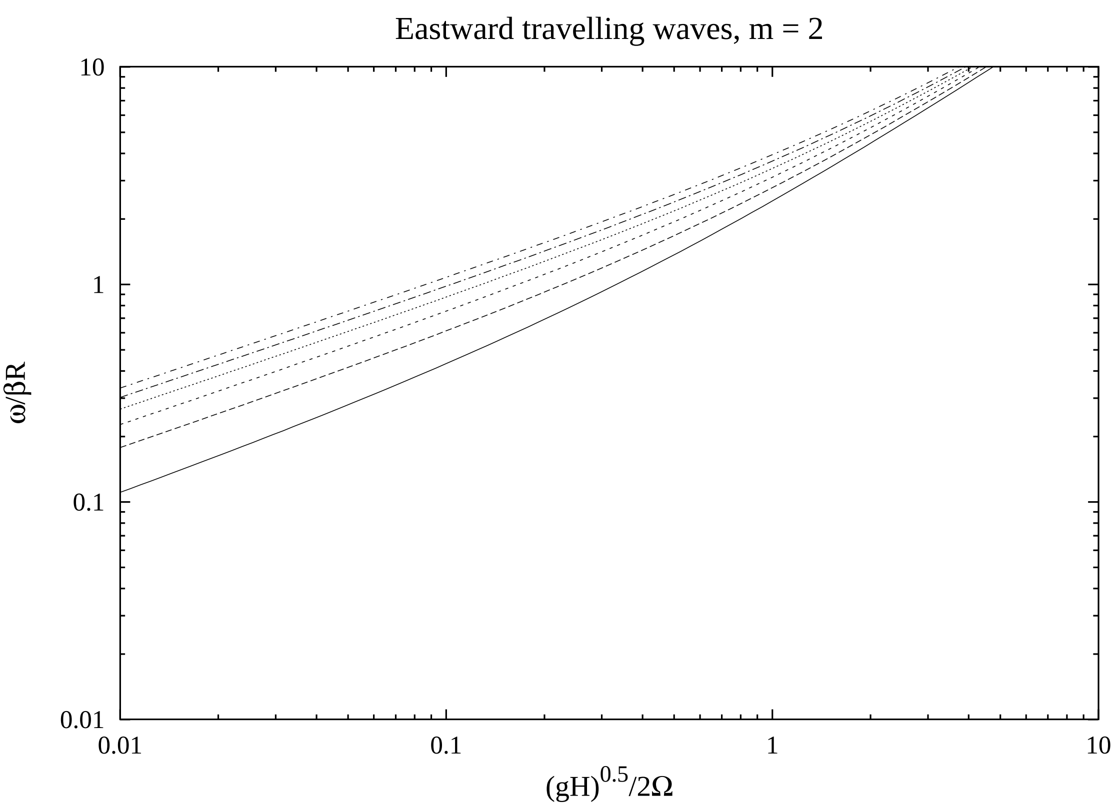
giving the dispersion relation above.

These higher modes again have three roots, two gravity waves with

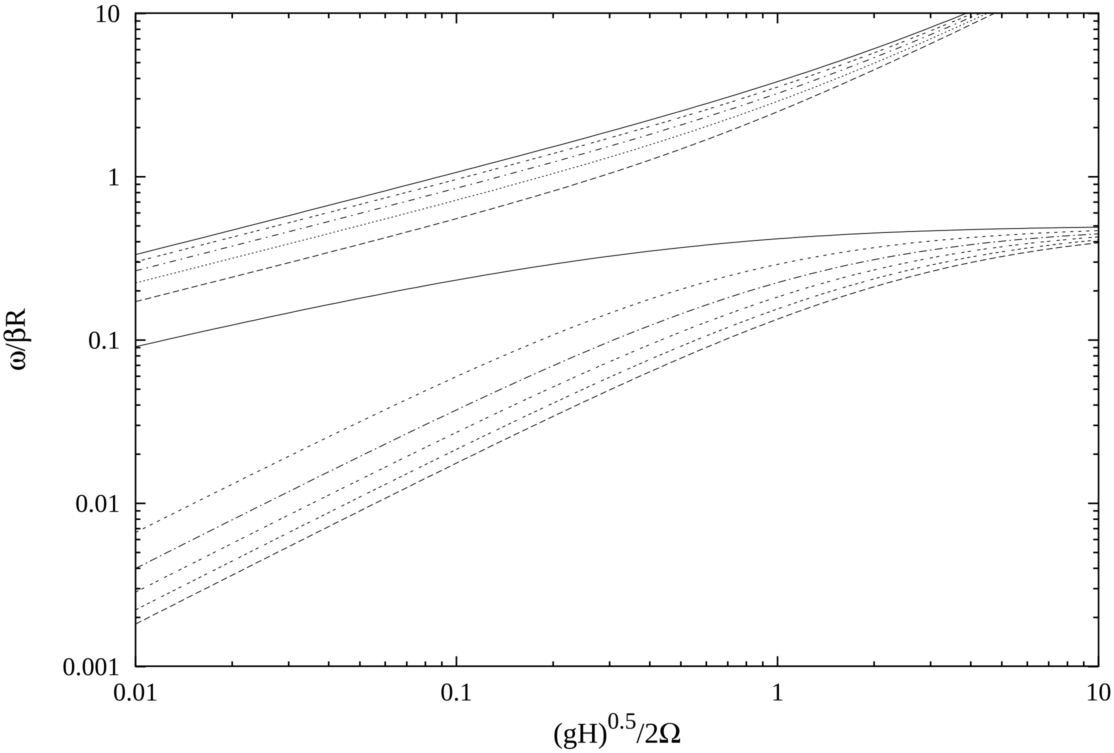
$$\omega^2 \simeq gH(k^2 + \frac{2n+1}{R^2})$$

and a Rossby mode with

$$\omega \simeq -\frac{\beta k}{k^2 + \frac{2n+1}{R^2}}$$



Westward travelling waves, m = 2



Kelvin wave

If we compare the behavior of the equatorial beta-plane to the Hough functions by plotting ω vs. $\sqrt{gH}$ in figure (2), we notice that one mode is missing; the one with ω nearly proportional to $\sqrt{gH}$. This is the kelvin wave mode, and it is connected to the $\frac{\partial}{\partial t}v = 0$ root. For finite frequency, this implies $v = 0$; equation (2) then tells us $\omega = \pm\sqrt{gH}k$ while equation (1) gives us the meridional structure

$$k\beta y P + \omega \frac{\partial P}{\partial y} = 0$$

The only well-bhaved solution will be for eastward-propagating waves with

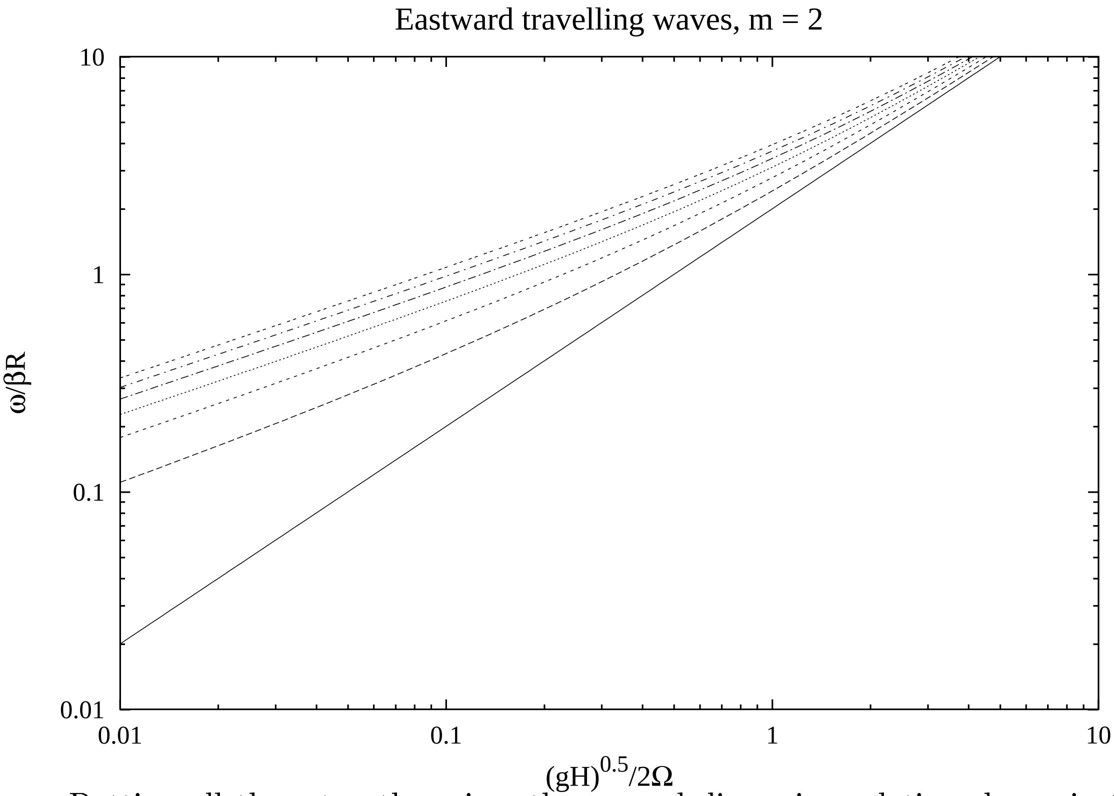
$$P = \exp(-\frac{1}{2}\frac{y^2}{R^2}) \quad , \quad u = \frac{1}{\sqrt{gH}}P$$

Dynamically, we just have geostrophic balance of the zonal flow

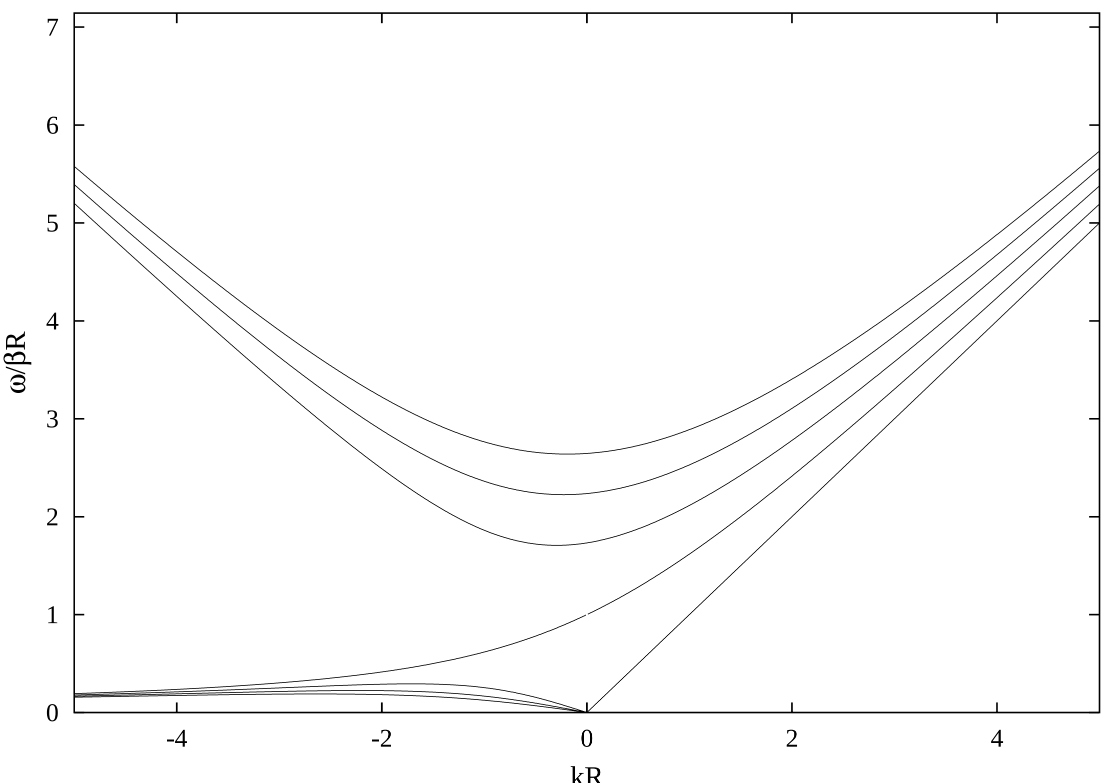
$$\beta y u = -\frac{\partial}{\partial y}P$$

and an along-equator gravity wave balance

$$\frac{\partial}{\partial t}u = -\frac{\partial}{\partial x}P \quad , \quad \frac{\partial}{\partial t}P = -gH\frac{\partial}{\partial x}u$$



Putting all these together gives the general dispersion relation shown in figures (3) and (4). Essentially, the waves on the equatorial beta plane reproduce the full spherical results fairly well in an analytically simpler regime.



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