Surface-forced waves

Suppose we specify the surface pressure

$$P = P_0 \cos(kx - \omega t) = \Re[P_0 e^{\imath(kx-\omega t)}]$$

what is the structure and the nature of the disturbances below?

We can derive an equation for the pressure in the water column using

$$\nabla^2 P = \frac{\partial}{\partial z} b$$
$$w = -\frac{1}{N^2}\frac{\partial}{\partial t} b$$
$$\frac{\partial}{\partial t} w = -\frac{\partial}{\partial z} P + b$$

Eliminating w from the last two and taking a z derivative gives

$$\left[\frac{1}{N^2}\frac{\partial^2}{\partial t^2} + 1\right]\frac{\partial b}{\partial z} = \frac{\partial^2 P}{\partial z^2}$$

or

$$\left[\frac{1}{N^2}\frac{\partial^2}{\partial t^2} + 1\right]\nabla^2 P = \frac{\partial^2 P}{\partial z^2}$$

which can also be written as

$$\frac{\partial^2}{\partial t^2}\nabla^2 P = -N^2 \nabla_h^2 P$$

as before. The pressure will have the same horizontal and temporal structure as the forcing

$$P = P(z) e^{\imath(kx-\omega t)}$$

and the vertical structure satisfies

$$\frac{\partial^2}{\partial z^2} P = -\left(\frac{N^2}{\omega^2} - 1\right) k^2 P$$

The pressure has a vertical structure like $\exp(\pm \imath m z)$ with

$$m^2 = \left(\frac{N^2}{\omega^2} - 1\right) k^2$$

(which is equivalent to our standard dispersion relation

$$\omega^2 = N^2 \frac{k^2}{k^2 + m^2}$$

but solved for m rather than ω).

Trapped disturbances

When $\omega > N$, we have imaginary vertical wavenumbers $M = \imath m$, so that the solutions are exponentially growing or decaying with depth $P(z) = P_0 \exp(\pm Mz)$. Since the forcing is at the surface, we have to choose the solutions which decay

$$P = \Re[P_0 e^{Mz} e^{\imath(kx-\omega t)}] = P_0 \cos(kx - \omega t) e^{Mz}$$

The solutions are in phase with depth and decay at a rate

$$M = k \left(1 - \frac{N^2}{\omega^2}\right)^{1/2}$$

Propagating disturbances

When $\omega < N$, the forcing is in the range of frequencies for which free waves exist, and m is real. To resolve which sign we use, we argue that the energy should be propagating downwards, away from the source, so that

$$\mathbf{c}_g \cdot \hat{\mathbf{z}} = -N \frac{mk}{(k^2 + m^2)^{3/2}} < 0$$

Thus, we must choose the positive sign for m, and the solution becomes

$$P = \Re[P_0 e^{\imath(kx+mz-\omega t)}] = P_0 \cos(kx + mz - \omega t)$$

The solutions now change phase with depth and flux energy deep into the water.