

4. Circulation

A key integral conservation property of fluids is the *circulation*. For convenience, we derive *Kelvin's circulation theorem* in an inertial coordinate system and later transform back to Earth coordinates.

In inertial coordinates, the vector form of the momentum equation may be written

$$\frac{d\mathbf{V}}{dt} = -\alpha\nabla p - g\hat{k} + \mathbf{F}, \quad (4.1)$$

where \hat{k} is the unit vector in the z direction and \mathbf{F} represents the vector frictional acceleration.

Now define a material curve that, however, is constrained to lie at all times on a surface along which some *state variable*, which we shall refer to for now as s , is a constant. (A state variable is a variable that can be expressed as a function of temperature and pressure.) The picture we have in mind is shown in Figure 3.1.

The arrows represent the projection of the velocity vector onto the s surface, and the curve is a *material curve* in the specific sense that each point on the curve moves with the vector velocity, projected onto the s surface, at that point.

Now the *circulation* is defined as

$$C \equiv \oint \mathbf{V} \cdot d\mathbf{l}, \quad (4.2)$$

where $d\mathbf{l}$ is an incremental length *along* the curve and \mathbf{V} is the vector velocity. The integral is a closed integral around the curve. Differentiation of (4.2) with respect to this gives

$$\frac{dC}{dt} = \oint \frac{d\mathbf{V}}{dt} \cdot d\mathbf{l} + \oint \mathbf{V} \cdot \frac{d\mathbf{l}}{dt}. \quad (4.3)$$

Since the curve is a material curve,

$$\frac{d\mathbf{l}}{dt} = d\mathbf{V},$$

so the integrand of the last term in (4.3) can be written as a perfect derivative and so the term itself vanishes. Thus

$$\frac{dC}{dt} = \oint \frac{d\mathbf{V}}{dt} \cdot d\mathbf{l},$$

and substituting (4.1) results in

$$\frac{dC}{dt} = \oint [-\alpha \nabla p + \mathbf{F}] \cdot d\mathbf{l}. \quad (4.4)$$

The gravity term vanishes because it can be expressed as the derivative of a potential, and so vanishes when integrated on a closed curve.

Now since α is a state variable (neglecting its dependence on water vapor or, in the ocean, salinity, for the time being), it can be written as a function of s and p :

$$\alpha = \alpha(s, p),$$

But because ∇p in (4.4) must be a gradient at *constant* s (since the material curve is chosen to lie on an s surface), the pressure term in (4.4) can be written

$$-\oint \alpha \nabla p \cdot d\mathbf{l} = \int \nabla \zeta \cdot d\mathbf{l} = 0,$$

where

$$\zeta \equiv \int_0^p \alpha(s_0, p') dp',$$

and s_0 is the particular value of s characterizing the s surface in question. Thus (4.4) becomes

$$\frac{dC}{dt} = \oint \mathbf{F} \cdot d\mathbf{l}. \quad (4.5)$$

Thus the only process that changes the circulation around a closed material curve on an s surface is friction.

Using the definition of circulation, (4.2), and Stokes's theorem, (4.5) can be written alternatively as

$$\frac{d}{dt} \int \int_A (\nabla \times \mathbf{V}) \cdot \hat{n} dA = \int \int_A (\nabla \times F) \cdot \hat{n} dA, \quad (4.6)$$

where now the integrals are over the area enclosed by the material curve, and \hat{n} is a unit vector orthogonal to the s surface.

In either of its two forms, (4.5) or (4.6), the Circulation Theorem expresses a fluid analog of angular momentum conservation. If the curve happens to be a circle, then the circulation can be seen to be the angular momentum per unit mass of an infinitesimal ring centered on that circle (multiplied by 2π).

Now in a local coordinate system fixed to the rotating earth, the absolute velocity (which appears in (4.5) and (4.6)) is related to the Earth-relative velocity, \mathbf{V}_r , by

$$\nabla \times \mathbf{V} = \nabla \times \mathbf{V}_r + 2\mathbf{\Omega}, \quad (4.7)$$

where $\mathbf{\Omega}$ is the vector angular velocity of the Earth's rotation. Substituting (4.7) into (4.6) gives

$$\frac{d}{dt} \int \int [\nabla \times \mathbf{V}_r + 2\mathbf{\Omega}] \cdot \hat{n} dA = \int \int (\nabla \times \mathbf{F}) \cdot \hat{n} dA. \quad (4.8)$$

This is the form of the Circulation Theorem that we shall most often refer to. It implies that on a surface along which some state variable is constant, the relative vorticity ($\nabla \times \mathbf{V}_r$) will *increase* if

1. The area enclosed by the material curve decreases (implying convergence).
2. The curve is displaced southward or is tilted, such that $\mathbf{\Omega} \cdot \hat{n}$ decreases.

Henceforth, we shall assume Earth-relative coordinates and drop the subscript r in (4.8).

It is worth noting that the dependence of specific volume, α , on water vapor in the atmosphere and on salinity in the ocean can be accounted for in forming the

circulation theorem, by a suitable choice of variables. In the atmosphere, a good choice of variables is the *virtual potential temperature*, defined

$$\theta_v \equiv T_v \left(\frac{p_0}{p} \right)^\kappa, \quad (4.9)$$

where T_v is given by (2.4), p_0 is a reference pressure (usually 1000 MBA), and κ is R_d/c_{pd} , where R_d is the gas constant for dry air, and c_{pd} is the heat capacity at constant pressure of dry air. It can be shown that θ_v is very nearly conserved in reversible adiabatic transformations. Moreover, as is clear from (2.3) and (4.9),

$$\alpha = \frac{R_d T_v}{p} = R_d \theta_v p^{\kappa-1} p_0^{-\kappa}, \quad (4.10)$$

and on a surface along which θ_v is constant,

$$\alpha \nabla p = c_{pd} \theta_v p_0^{-\kappa} \nabla p^\kappa,$$

so that once again,

$$\oint \alpha \nabla p \cdot d\mathbf{l} = 0,$$

when the curve lies on a surface of constant θ_v .

Similarly, in the ocean, we can write

$$\alpha = \sigma(s, T, p) G(p), \quad (4.11)$$

where σ is the *potential density* and is a function of salinity, temperature, and pressure. The potential density is the density sea water would have if brought reversibly to some reference pressure. Clearly, from (4.11), the pressure gradient

term also vanishes when integrated around a closed material curve on a surface along which σ is constant.

While the Circulation Theorem is useful for envisioning changes in fluid vorticity, it is an integral theorem and thus cannot be used as part of a closed system of equations describing the detailed evolution of fluid flow. We therefore proceed to develop a conserved scalar, called the potential vorticity, using the Circulation Theorem as a starting point.

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