

6. Invertibility

The potential vorticity, q , is in general a function of the distributions of 5 variables: the three velocity components, density, and either θ_v or σ . In *quasi-balanced* flows, it is possible to reduce this dependence to one that relies on a *single* variable, from which all of the others can be derived. The relationship between q and this single variable is usually through an elliptic, second-order differential equation. Under these circumstances, the spatial distribution of q can be *inverted*, given certain boundary conditions, to yield the distribution of velocity and mass. This property of q and of quasi-balanced flows is known as *invertibility*.

We shall explore various balance approximations in some detail later, but now let's have a quick look at how the dependence of q on 5 variables may be reduced to a dependence on 1 under some conditions.

First, let's expand the definition of potential vorticity out into its various components. For the atmosphere, (5.8) expands to

$$q_a = \alpha \left[\left(f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \frac{\partial \theta_v}{\partial z} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \frac{\partial \theta_v}{\partial y} + \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \frac{\partial \theta_v}{\partial x} \right]. \quad (6.1)$$

From mass continuity, w scales at *most* according to

$$\frac{\partial w}{\partial z} \sim 0 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right),$$

or

$$w \sim 0 \left(u_0 \frac{H}{L} \right), \quad (6.2)$$

where H and L are typical vertical and horizontal scales over which the flow varies, and u_0 is a typical horizontal velocity scale. (Note that in most geophysical flows, the flow is quasi-nondivergent, so actually $w \ll 0 (u_0 \frac{H}{L})$.) Thus, in terms that appear in (6.1), like

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x},$$

the order of the term is

$$\frac{u_0}{H} \left(1 - \frac{H^2}{L^2} \right).$$

Since, for virtually all flows we will be interested in, $H/L \ll 1$, the contribution of w to the potential vorticity is utterly negligible. So (6.1) may be accurately approximated by

$$q_a \simeq \alpha \left[\left(f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \frac{\partial \theta_v}{\partial z} + \frac{\partial u}{\partial z} \frac{\partial \theta_v}{\partial y} - \frac{\partial v}{\partial z} \frac{\partial \theta_v}{\partial x} \right]. \quad (6.3)$$

Now if we employ the hydrostatic approximation, (2.2), it follows that

$$\alpha \frac{\partial A}{\partial z} \simeq -g \frac{\partial A}{\partial p},$$

for any quantity A . Using this in (6.3) gives

$$q_a \simeq -g \left[\left(f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \frac{\partial \theta_v}{\partial p} + \frac{\partial u}{\partial p} \frac{\partial \theta_v}{\partial y} - \frac{\partial v}{\partial p} \frac{\partial \theta_v}{\partial x} \right], \quad (6.4)$$

This can be re-expressed *in θ_v coordinates* as

$$q_a \simeq -g \left(\frac{\partial p}{\partial \theta_v} \right)^{-1} \left(f + \left(\frac{\partial v}{\partial x} \right)_{\theta_v} - \left(\frac{\partial u}{\partial y} \right)_{\theta_v} \right). \quad (6.5)$$

Now suppose that the flow is, to a good approximation, hydrostatic and geostrophic. In θ_v coordinates, the hydrostatic and geostrophic relations are expressed in terms of the *Montgomery streamfunction*:

$$M \equiv c_{pd}T_v + gz. \quad (6.6)$$

These relations are:

Hydrostatic:

$$c_{pd} \left(\frac{p}{p_0} \right)^\kappa = \frac{\partial M}{\partial \theta_v}, \quad (6.7)$$

Geostrophic:

$$\begin{aligned} f u_g &= - \left(\frac{\partial M}{\partial y} \right)_{\theta_v}, \\ f v_g &= \left(\frac{\partial M}{\partial x} \right)_{\theta_v}. \end{aligned} \quad (6.8)$$

Substituting these into (6.5) gives

$$q_a \simeq -g p_0^{-1} c_{pd}^{\frac{1}{\kappa}} \kappa \left(f + \frac{1}{f} \nabla^2 M \right) \left[\left(\frac{\partial M}{\partial \theta_v} \right)^{\frac{1}{\kappa} - 1} \frac{\partial^2 M}{\partial \theta_v^2} \right]^{-1}. \quad (6.9)$$

Then q_a is a function of M alone, and this function is a nonlinear and usually elliptic one. (It is always elliptic when $\frac{1}{f} \nabla^2 M + f$ has the same sign as q_a , and

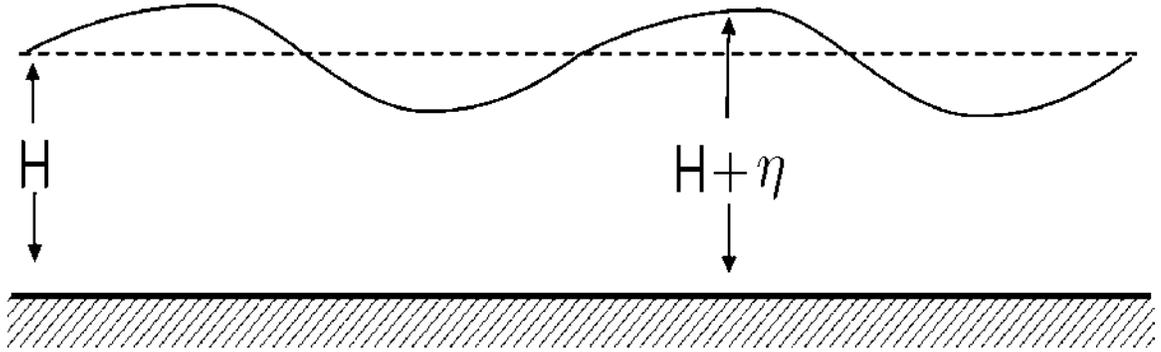


Figure 7.1

$\partial M / \partial \theta_v > 0$.) When it is elliptic, (6.9) can be inverted to find M , and therefore u_g , v_g , and p , given the distribution of q_a and certain boundary conditions.

We will be developing somewhat simpler invertibility relationships for potential vorticity. The essential elements in all of these are the definition of potential vorticity, coupled with balance approximations that link the instantaneous distribution of velocity to that of mass.

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12.803 Quasi-Balanced Circulations in Oceans and Atmospheres
Fall 2009

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