## 8. Quasigeostrophy and Pseudo-potential vorticity

The shallow water system is probably the most simple fluid system that allows for divergent flow and inertia-gravity waves. Here we develop a simple set of equations for quasi-balanced flows of a *continuously stratified fluid,* based on the approximation that the flow is nearly geostrophic. This system is called the *quasi-geostrophic system.*

We begin with the horizontal momentum equation in pressure coordinates:

$$\frac{d\mathbf{V}}{dt} + f\hat{k} \times \mathbf{V} + \nabla\varphi = \mathbf{F}, \tag{8.1}$$

where $\mathbf{F}$ is the net acceleration by frictional forces. *Geostrophic balance* is defined by the equality of the two middle terms of (8.1), so that the *geostrophic wind* is defined

$$\mathbf{V}_g \equiv \frac{1}{f}\hat{k} \times \nabla\varphi. \tag{8.2}$$

Using this definition, (8.1) may be rewritten

$$\frac{d\mathbf{V}}{dt} + f\hat{k} \times (\mathbf{V} - \mathbf{V}_g) - \mathbf{F} = 0, \tag{8.3}$$

or, equivalently,

$$\mathbf{V} = \mathbf{V}_g + \frac{1}{f}\hat{k} \times \frac{d\mathbf{V}}{dt} - \frac{1}{f}\hat{k} \times \mathbf{F}. \tag{8.4}$$

We will use (8.4) to investigate the relative magnitudes of the terms in the horizontal momentum equations. For this purpose, we shall approximate frictional acceleration as

$$\mathbf{F} \simeq -\mathbf{V}/\tau_f, \tag{8.5}$$

where $\tau_f$ is a time scale associated with frictional damping. We also define a *Lagrangian time scale,* $\boldsymbol{\tau}$, which can be thought of as a typical time scale over which a sample of fluid accelerates in a given flow. We replace the dimensional time, $t$, in (8.4) by a nondimensional time $t^*$:

$$t \to \tau t^*, \tag{8.6}$$

resulting in the scaled version of (8.4):

$$\mathbf{V} = \mathbf{V}_g + R_0 \hat{k} \times \frac{d\mathbf{V}}{dt} + R_F \hat{k} \times \mathbf{V}, \tag{8.7}$$

where $R_0$ is the *Rossby number,* defined

$$R_0 \equiv \frac{1}{f\tau}, \tag{8.8}$$

and $R_F$ is a nondimensional measure of friction:

$$R_F \equiv \frac{1}{f\tau_f}. \tag{8.9}$$

Note that because $f$ varies with latitude, both $R_0$ and $R_F$ vary with time.

An expansion of (8.7) in terms of the *geostrophic* wind alone can be made by substituting $\mathbf{V}$ as given by (8.7) into the terms involving $\mathbf{V}$ on the right side of the same equation, resulting in

$$\mathbf{V} = V_g + R_0 \hat{k} \times \frac{d\mathbf{V}_g}{dt} + R_F \hat{k} \times \mathbf{V}_g - R_0^2 \frac{d^2\mathbf{V}}{dt^2}$$
$$- R_0 R_F \frac{d\mathbf{V}}{dt} - R_F^2 \mathbf{V} - R_0 \frac{d\mathbf{V}}{dt} \frac{dR_0}{dt}.$$

By repeating the procedure, this may be written

$$\mathbf{V} = \mathbf{V}_g + R_0\hat{k} \times \frac{d\mathbf{V}_g}{dt} + R_F\hat{k} \times \mathbf{V}_g - R_0^2\frac{d^2\mathbf{V}_g}{dt^2}$$
$$- R_0R_F\frac{d\mathbf{V}_g}{dt} - R_F^2\mathbf{V}_g - R_0\frac{d\mathbf{V}_g}{dt}\frac{dR_0}{dt} \tag{8.10}$$
$$+ O(R_0^3) + O(R_F^3),$$

assuming that $\frac{dR_0}{dt}$ is no larger than $R_0$.

If $R_0 < 1$ and $R_F < 1$, we might expect that the series (8.10) converges. The order zero approximation (8.10) is just geostrophic balance:

$$\mathbf{V} \simeq \mathbf{V}_g,$$

while the order 1 approximation is called the *geostrophic momentum approximation.* Writing the order 1 approximation to (8.10) in dimensional form results in

$$\frac{d\mathbf{V}_g}{dt} + f\hat{k} \times (\mathbf{V} - \mathbf{V}_g) - \mathbf{F} \simeq 0, \tag{8.11}$$

where it must be remembered that $\mathbf{F}$ has been assumed to be *at most* order $R_0$.

The approximation (8.11) is called the geostrophic momentum approximation because it consists in replacing the inertia of the actual wind by that of the geostrophic wind. This approximation is one component of a system of approximate relations.

The second fundamental approximation to the momentum equations is to approximation advection by *geostrophic advection.* The full geostrophic momentum term may be expanded to

$$\frac{d\mathbf{V}_g}{dt} = \frac{\partial\mathbf{V}_g}{\partial t} + (\mathbf{V}_g + \mathbf{V}_a) \cdot \nabla\mathbf{V}_g + \omega\frac{\partial\mathbf{V}_g}{\partial p}, \tag{8.12}$$

where $\mathbf{V}_a$ is the ageostrophic part of the wind field, and

$$\omega \equiv \frac{dp}{dt} \tag{8.13}$$

is called the *pressure velocity* (or just "omega") and is proportional to the vertical component of velocity.

By (8.10), it is clear that

$$\frac{|\mathbf{V}_a|}{|\mathbf{V}_g|} \sim O(R_0);$$

moreover, we have already shown (by definition!) that $\left|\frac{d\mathbf{V}_g}{dt}\right|$ is $O(R_0)$ compared to $f\hat{k} \times \mathbf{V}$, so that to be consistent with the order of approximation, we need to drop the term $\mathbf{V}_a$ that appears in (8.13). In addition, the mass continuity equation in hydrostatic, pressure coordinates is

$$\nabla \cdot \mathbf{V} + \frac{\partial \omega}{\partial p} = 0, \tag{8.14}$$

which can also be written as

$$\nabla \cdot \mathbf{V}_g + \nabla \cdot \mathbf{V}_a + \frac{\partial \omega}{\partial p} = 0. \tag{8.15}$$

From the definition of geostrophic wind, (8.2),

$$\nabla \cdot \mathbf{V}_g = -\frac{\beta}{f^2}\frac{\partial \varphi}{\partial x} = -\frac{\beta}{f}v_g, \tag{8.16}$$

where $v_g$ is the meridional component of the geostrophic wind, and

$$\beta \equiv \frac{df}{dy}. \tag{8.17}$$

Comparing (8.16) to (8.15), it will be seen that if

$$\frac{\beta L_y}{f} \lesssim 0(R_0), \tag{8.18}$$

then

$$\left|\frac{\omega}{\Delta p}\right| \lesssim R_0 \left|\frac{\mathbf{V}_g}{L}\right|, \tag{8.19}$$

where $L_y$ is a typical meridional scale over which the flow varies, $L$ is an overall horizontal scale of flow variation, and $\Delta p$ is a pressure scale over which $\omega$ varies. If (8.19) is met, then we can also neglect the term involving $\omega$ in (8.12), which becomes

$$\frac{d\mathbf{V}_g}{dt} \simeq \frac{\partial \mathbf{V}_g}{\partial t} + \mathbf{V}_g \cdot \nabla \mathbf{V}_g.$$

Using this in (8.11) gives us the *quasi-geostrophic momentum equation:*

$$\frac{\partial \mathbf{V}_g}{\partial t} + \mathbf{V}_g \cdot \nabla \mathbf{V}_g + f\hat{k} \times (\mathbf{V} - \mathbf{V}_g) - \mathbf{F} = 0. \tag{8.20}$$

The accuracy of (8.20) depends both on the smallness of $R_0$ and on condition (8.18).

The final element of this series of approximations is made to the thermodynamic equation, which may be written

$$\frac{\partial \ln \theta}{\partial t} + \mathbf{V} \cdot \nabla \ln \theta + \omega \frac{\partial \ln \theta}{\partial p} = \dot{Q}, \tag{8.21}$$

where for atmospheric applications, $\theta$ is the potential temperature and

$$\dot{Q} = \frac{\dot{H}}{c_p T},$$

where $\dot{H}$ is the heating and $c_p$ is the heat capacity at constant pressure. For the ocean, $\theta$ is the potential density and $Q$ is its source, divided by the potential density itself.

It may at first seem that the approximation to (8.21) that is consistent with the approximation we made to the momentum equation is to drop the ageostrophic advection and the term involving $\omega$ in (8.21), but this is not the case because in rotating stratified flows, the vertical gradient of $\theta$ scales very differently from its horizontal gradient. To see this, let's compare the magnitude of the horizontal and vertical advection terms in (8.21). The magnitude of the horizontal advection is approximately

$$|\mathbf{V} \cdot \nabla \ln \theta| \simeq \left| v_g \frac{\partial \ln \theta}{\partial y} \right| = \left| \frac{f}{g} v_g \frac{\partial u_g}{\partial z} \right|, \tag{8.22}$$

where we have used the thermal wind equation, and $u_g$ is a typical geostrophic velocity scale. The magnitude of the vertical advection term is

$$\left| \omega \frac{\partial \ln \theta}{\partial p} \right| \sim \left| R_0 u_g \frac{N^2 h}{gL} \right|, \tag{8.23}$$

where we have used the hydrostatic relation, the scaling relation (8.19), $h$ is a typical vertical scale of variation of the flow, and $N$ is the *buoyancy (or Brünt-Väisälä) frequency,* defined

$$N^2 \equiv g \frac{\partial \ln \theta}{\partial z}. \tag{8.24}$$

Now the ratio of the magnitudes of the vertical and horizontal advection terms in the thermodynamic equation is

$$R \equiv R_0 \frac{N^2 h}{f \frac{\partial u_g}{\partial z} L}. \tag{8.25}$$

As we will see shortly, the *deformation radius* in quasi-geostrophic flows is

$$L_D = h\frac{N}{f},$$

so if $L$ scales with $L_D$ in (8.25),

$$R \simeq R_0 Ri^{1/2},$$

where $Ri$ is the Richardson number,

$$Ri \equiv \frac{N^2}{\left(\frac{\partial u_g}{\partial z}\right)^2}.$$

In both the atmosphere and the ocean, $Ri$ is an *order one* quantity, because the Richardson number is quite large. For this reason, we must retain the vertical advection term in (8.21), and for consistency, we expand $\ln\theta$ as

$$\ln\theta = \ln\overline{\theta}(p) + \ln\theta'(x, y, p, t), \tag{8.26}$$

with the scaling relation

$$\frac{\partial \ln\theta'}{\partial p} = O(R_0)\frac{\partial \ln\overline{\theta}}{\partial p}. \tag{8.27}$$

Then (8.21) is approximated by

$$\frac{\partial \ln\theta'}{\partial t} + \mathbf{V}_g \cdot \ln\theta' + \omega\frac{\partial \ln\overline{\theta}}{\partial p} = \dot{Q}. \tag{8.28}$$

(Note that $Q$ is permitted to be order 1.)

*Summary of quasi-geostrophic system:*

The quasi-geostrophic equations may be summarized:

$$D_g \mathbf{V}_g + f\hat{k} \times (\mathbf{V} - \mathbf{V}_g) = \mathbf{F}, \tag{8.29}$$

$$D_g \theta + \omega \frac{d\overline{\theta}}{dp} = \dot{Q}, \tag{8.30}$$

$$\boldsymbol{\nabla} \cdot \mathbf{V} + \frac{\partial \omega}{\partial p} = 0, \tag{8.31}$$

$$\mathbf{V}_g = \frac{1}{f} \hat{k} \times \nabla \varphi, \tag{8.32}$$

$$\frac{\partial \varphi}{\partial p} = \begin{cases} -\frac{R}{p} \left( \frac{p}{p_0} \right)^{R/c_p} \theta & \text{atmosphere,} \\ -G\sigma & \text{ocean.} \end{cases} \tag{8.33}$$

In this set, the geostrophic operator is defined

$$D_g \equiv \frac{\partial}{\partial t} + \mathbf{V}_g \cdot \boldsymbol{\nabla},$$

and $\mathbf{F}$ is assumed to be of order $R_0$. In (8.33) $G$ is a function of $p$ that depends on the equation of state for sea water, and $\theta$, $\sigma$, and $\varphi$ (except where overbarred) are deviations from the basic state values of those quantities.

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