16. Rossby waves

We have seen that the existence of potential vorticity gradients supports the propagation of a special class of waves known as *Rossby waves.* These waves are the principal means by which information is transmitted through quasi-balanced flows and it is therefore fitting to examine their properties in greater depth. We begin by looking at the classical problem of barotropic Rossby wave propagation on a sphere and continue with quasi-geostrophic Rossby waves in three dimensions.

a. Barotropic Rossby waves on a sphere

The vorticity equation for barotropic disturbances to fluid at rest on a rotating sphere is

$$\frac{d\eta}{dt} = 0, \tag{16.1}$$

where

$$\eta \equiv 2\Omega \sin\varphi + \zeta.$$

Here ζ is the relative vorticity in the z direction. Now the equation of mass continuity for two-dimensional motion on a sphere may be written

$$\frac{1}{a}\left[\frac{\partial u}{\partial \lambda} + \frac{\partial}{\partial \varphi}(v \cos\varphi)\right] = 0, \tag{16.2}$$

where u and v are the eastward and northward velocity components, λ and φ are longitude and latitude, and a is the (mean) radius of the earth. Using (16.2) we may define a velocity streamfunction ψ such that

$$u = -\frac{1}{a}\frac{\partial \psi}{\partial \varphi},$$

and (16.3)

$$v = \frac{1}{a\cos\varphi}\frac{\partial \psi}{\partial \lambda}.$$

The Eulerian expansion of (16.1) can be written

$$\frac{\partial \eta}{\partial t} + \frac{u}{a\cos\varphi}\frac{\partial \eta}{\partial \lambda} + \frac{v}{a}\frac{\partial \eta}{\partial \varphi} = 0,$$

or using (16.3),

$$\frac{\partial \eta}{\partial t} + \frac{1}{a^2\cos\varphi}\left[\frac{\partial \psi}{\partial \lambda}\frac{\partial \eta}{\partial \varphi} - \frac{\partial \psi}{\partial \varphi}\frac{\partial \eta}{\partial \lambda}\right] = 0. \tag{16.4}$$

We next linearize (16.4) about the resting state ($u = v = 0$), for which $\overline{\eta} = 2\Omega\sin\varphi$, giving

$$\frac{\partial \eta'}{\partial t} + \frac{2\Omega}{a^2}\frac{\partial \psi'}{\partial \lambda} = 0, \tag{16.5}$$

where the primes denote departures from the basic state.

In spherical coordinates,

$$\eta' = \zeta' = \hat{k} \cdot \nabla \times \mathbf{V}'$$
$$= \frac{1}{a^2 \cos^2 \varphi} \left[\frac{\partial^2 \psi'}{\partial \lambda^2} + \cos \varphi \frac{\partial}{\partial \varphi} \left(\cos \varphi \frac{\partial \psi'}{\partial \varphi} \right) \right]. \quad (16.6)$$

Let's look for modal solutions of the form

$$\psi' = \Psi(\varphi) e^{im(\lambda - \sigma t)},$$

where m is the zonal wavenumber and σ is an *angular* phase speed. Using this and (16.6) in (16.5) gives

$$\frac{d^2\Psi}{d\varphi^2} - \tan \varphi \frac{d\Psi}{d\varphi} - \left[\frac{2\Omega}{\sigma} + \frac{m^2}{\cos^2 \varphi} \right] \Psi = 0. \quad (16.7)$$

This can be transformed into canonical form by transforming the independent variable using

$$\mu \equiv \sin \varphi,$$

yielding

$$(1 - \mu^2) \frac{d^2\Psi}{d\mu^2} - 2\mu \frac{d\Psi}{d\mu} - \left[\frac{2\Omega}{\sigma} + \frac{m^2}{1 - \mu^2} \right] \Psi = 0. \quad (16.8)$$

The only solutions of (16.8) that are bounded at the poles ($\mu = \pm 1$) have the form

$$\Psi = A P_m^n, \quad (16.9)$$

Table 16.1. Meridional Structure of $P_m^n(\varphi)$ Rossby Waves on a Sphere

		m			
		0	1	2	3
	1	$\sin\varphi$	$\cos\varphi$	–	–
n	2	$\frac{1}{2}(3\sin^2\varphi - 1)$	$-3\sin\varphi\cos\varphi$	$3\cos^2\varphi$	–
	3	$\frac{3}{2}\sin\varphi(5\sin^2\varphi - 3)$	$-\frac{9}{2}(5\sin^2\varphi - 1)\cos\varphi$	$45\sin\varphi\cos^3\varphi$	$-45\cos^3\varphi$

where P_m^n is an associated Legendre function of degree n and order m, with $n > m$. The angular frequency must satisfy

$$\sigma = \frac{-2\Omega}{n(n+1)}. \qquad (16.10)$$

As in the case of barotropic Rossby waves in a fluid at rest on a β plane, spherical Rossby waves propagate westward. Their zonal phase speed is given by

$$c = a\cos\varphi\sigma = -2\Omega a\frac{\cos\varphi}{n(n+1)}. \qquad (16.11)$$

The first few associated Legendre functions are given in Table 16.1. The lowest order modes, for which $m = 0$, are zonally symmetric and have zero frequency. These are just east-west flows that do not perturb the background vorticity gradient and thus are not oscillatory. The lowest order wave mode, for which $n = m = 1$, has an angular frequency of $-\Omega$ and is therefore stationary relative to absolute space. This zonal wavenumber 1 mode has maximum amplitude on the equator and decays as $\cos\varphi$ toward the poles. Modes of greater values of n have increasingly fine meridional structure.

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