17. Quasi-geostrophic Rossby waves

Baroclinic flows can also support Rossby wave propagation. This is most easily described using quasi-geostrophic theory. We begin by looking at the behavior of small perturbations to a zonal background flow that varies only in the meridional and vertical directions. Beginning with the definition of pseudo-potential vorticity (9.11), we let φ and q_p be represented by zonally invariant background fields plus perturbations to them:

$$\begin{aligned} \varphi &= \overline{\varphi}(y,p) + \varphi'(x,y,p,t) \\ q_p &= \overline{q}_p(y,p) + q_p'(x,y,p,t) \end{aligned} \tag{17.1}$$

We next linearize the adiabatic, frictionless form of the conservation equation for q_p (9.10) using (17.1):

$$\frac{\partial q_p'}{\partial t} + \overline{u}_g \frac{\partial q_p'}{\partial x} + v_g' \frac{\partial \overline{q}_p}{\partial y} = 0, \tag{17.2}$$

where

$$\begin{aligned} \overline{u}_g &= -\frac{1}{f}\frac{\partial \overline{\varphi}}{\partial y}, \\ v_g' &= \frac{1}{f}\frac{\partial \varphi'}{\partial x}. \end{aligned} \tag{17.3}$$

We also note from the definition of pseudo-potential vorticity (9.11), that

$$\begin{aligned} \frac{\partial \overline{q}_p}{\partial y} &= \frac{1}{f_0}\nabla^2 \frac{\partial \overline{\varphi}}{\partial y} + \beta_0 + \frac{\partial}{\partial p}\frac{f_0}{\mathcal{S}}\frac{\partial}{\partial p}\frac{\partial \overline{\varphi}}{\partial y} \\ &= \beta - \frac{\partial^2 \overline{u}}{\partial y^2} - \frac{\partial}{\partial p}\frac{f_0^2}{\mathcal{S}}\frac{\partial \overline{u}}{\partial p}. \end{aligned} \tag{17.4}$$

Thus the meridional gradient of background pseudo-potential vorticity depends on β, the meridional gradient of the vorticity of the zonal wind, and a measure of the curvature of the vertical profile of the mean zonal wind.

Using the second of the geostrophic relations in (17.3) as well as the definition of pseudo-potential vorticity, (9.11), the linearized conservation relation (17.2) may be written

$$\left(\frac{\partial}{\partial t}+\overline{u}_g\frac{\partial}{\partial x}\right)\left[\frac{1}{f_0}\nabla^2\varphi'+\frac{\partial}{\partial p}\frac{f_0}{\mathcal{S}}\frac{\partial\varphi'}{\partial p}\right]+\frac{1}{f_0}\frac{\partial\overline{q}_p}{\partial y}\frac{\partial\varphi'}{\partial x}=0. \tag{17.5}$$

We will examine solutions of (17.5) in the special case that the stratification is constant, $\mathcal{S}$ = constant. We will also assume that the background zonal wind, $\overline{u}_g$, and the associated gradient of background pseudo-potential vorticity given by (17.4) are slowly varying compared to the structures of perturbations to the flow. If this is the case, we can make the W.K.B. approximation and represent *modal* solutions to (17.5) as

$$\varphi'=\Phi e^{ik(x-ct)+i\int^y l(y',p)dy'+i\int^p m(y,p')dp'}, \tag{17.6}$$

where $l(y,\rho)$ and $m(y,p)$ are slowly varying functions of latitude and pressure. Substituting (17.6) into (17.5) gives a dispersion relation:

$$c=\overline{u}_g-\frac{\partial\overline{q}_p/\partial y}{k^2+l^2+\frac{f_0^2}{\mathcal{S}}m^2}. \tag{17.7}$$

Comparing this to the strictly barotropic dispersion relation (10.8) shows the strong similarity between barotropic and baroclinic waves. The main differences are that in the baroclinic case, the meridional gradient of potential (rather than actual) vorticity serves as the refractive index for Rossby waves, and the vertical structure contributes to the dispersion properties of the waves.

The wave frequency, which remains invariant along the ray paths followed by the wave energy as long as the background flow is considered to be steady, is given by

$$\omega = kc = k\overline{u}_g - \frac{k(\partial \overline{q}_p/\partial y)}{k^2 + l^2 + \frac{f_0^2}{\mathcal{S}} m^2}. \tag{17.8}$$

Letting $k_i \equiv (k, l, m)$, the three components of the group velocity are given by

$$c_{g_i} = \frac{\partial \omega}{\partial k_i} = \left[\left(\overline{u}_g + \frac{\overline{q}_{py}}{r^4} \left(k^2 - l^2 - \frac{f_0^2}{\mathcal{S}} m^2 \right) \right), \frac{2kl\overline{q}_{py}}{r^4}, \frac{2km\frac{f_0^2}{\mathcal{S}}\overline{q}_{py}}{r^4} \right], \tag{17.9}$$

where

$$\overline{q}_{py} \equiv \frac{\partial \overline{q}_p}{\partial y},$$
$$r^2 \equiv k^2 + l^2 + \frac{f_0^2}{\mathcal{S}} m^2.$$

It is of some interest to compare these group velocities to the phase speeds, which are given by

$$\begin{aligned} c_r \equiv \frac{\omega}{k_i} &= \left[\left(\overline{u}_g - \frac{\overline{q}_{py}}{r^2} \right), -\frac{k}{l}\frac{\overline{q}_{py}}{r^2}, -\frac{k}{m}\frac{\overline{q}_{py}}{r^2} \right] \\ &= \left[\left(c_{g_x} - 2\frac{k^2\overline{q}_{py}}{r^4} \right), -\frac{1}{2}\frac{r^2}{l^2}c_{gy}, -\frac{1}{2}\frac{\mathcal{S}}{f_0^2}\frac{r^2}{m^2}c_{gp} \right] \end{aligned} \tag{17.10}$$

Thus, for quasi-geostrophic Rossby waves, the flow-relative group velocity in the meridional and vertical directions is of opposite sign from the phase speeds in those directions.

As quasi-geostrophic Rossby waves disperse in three dimensions, the associated wave numbers evolve following the vector group velocity, according to the relationship for the refraction of wave energy:

$$\frac{dk_i}{dt} = -\frac{\partial \omega}{\partial x_i}, \tag{17.11}$$

where the total derivative indicates the rate of change following the group velocity:

$$\frac{dk_i}{dt} = \frac{\partial k_i}{\partial t} + c_{g_j}\frac{\partial k_i}{\partial x_j}.$$

Using (17.8), the evolution of wavenumber (17.11) is

$$\frac{dk_i}{dt} = \left[0, k\left(-\frac{\partial \overline{u}_g}{\partial y} + \frac{\partial^2\overline{q}_p/\partial y^2}{r^2}\right), k\left(-\frac{\partial \overline{u}_g}{\partial \rho} + \frac{\partial^2\overline{q}_p/\partial y\partial p}{r^2}\right)\right]. \qquad (17.12)$$

The interaction between Rossby waves and the background flow is of great interest, because in quasi-balanced flows these waves are responsible for conveying information from one place to another. An elegant way of quantifying the interaction between quasi-geostrophic Rossby waves and the background state on which they are assumed to propagate is through the examination of *Eliassen-Palm fluxes.* The *Eliassen-Palm theory* is derived as follows.

Since quasi-geostrophic flow on an f plane is nondivergent, we may write the conservation equation for pseudo-potential vorticity, (9.10), in the form

$$\frac{\partial q_p}{\partial t} = -\frac{\partial}{\partial x}(u_g q_p) - \frac{\partial}{\partial y}(v_g q_p). \qquad (17.13)$$

Consider now the time rate of change of zonal mean pseudo-potential vorticity. First define a zonal average operator $\{\quad\}$, such that for any scalar A,

$$\{A\} \equiv \frac{1}{L}\int_0^L A\,dx, \qquad (17.14)$$

where L is the distance around a latitude circle. Applying this operator to (17.13) gives

$$\frac{\partial}{\partial t}\{q_p\} = -\frac{\partial}{\partial y}\{v_g q_p\}. \qquad (17.15)$$

Now let

$$v_g = \{v_g\} + v_g',$$

$$q_p = \{q_p\} + q_p',$$

where v_g' is the local, instantaneous departure of v_g from $\{v_g\}$, but since

$$v_g = \frac{1}{f_0}\frac{\partial \varphi}{\partial x},$$

$\{v_g\} = 0$. Thus (17.15) becomes

$$\frac{\partial}{\partial t}\{q_p\} = -\frac{\partial}{\partial y}\{v_g' q_p'\}. \tag{17.16}$$

The time rate of change of zonal mean pseudo-potential vorticity is equal to the convergence of the meridional eddy flux of pseudo-potential vorticity.

Using the definitions of q_p and v_g,

$$\begin{aligned} q_p' &= \frac{1}{f_0}\nabla^2\varphi' + \frac{\partial}{\partial p}\frac{f_0}{\mathcal{S}}\frac{\partial \varphi'}{\partial p}, \\ v_g' &= \frac{1}{f_0}\frac{\partial \varphi'}{\partial x}, \end{aligned}$$

where φ' is the departure of φ from its zonal average, we can write

$$\begin{aligned} v_g' q_p' &= \frac{1}{f_0^2}\left[\frac{\partial \varphi'}{\partial x}\frac{\partial^2 \varphi'}{\partial x^2} + \frac{\partial \varphi'}{\partial x}\frac{\partial^2 \varphi'}{\partial y^2}\right] + \frac{\partial \varphi'}{\partial x}\frac{\partial}{\partial p}\frac{1}{\mathcal{S}}\frac{\partial \varphi'}{\partial p} \\ &= \frac{1}{f_0^2}\left[\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \varphi'}{\partial x}\right)^2 + \frac{\partial}{\partial y}\left(\frac{\partial \varphi'}{\partial x}\frac{\partial \varphi'}{\partial y}\right) - \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \varphi'}{\partial y}\right)^2\right] \\ &\quad + \frac{\partial}{\partial p}\left(\frac{\partial \varphi'}{\partial x}\frac{1}{\mathcal{S}}\frac{\partial \varphi'}{\partial p}\right) - \frac{1}{2\mathcal{S}}\frac{\partial}{\partial x}\left(\frac{\partial \varphi'}{\partial p}\right)^2. \end{aligned} \tag{17.17}$$

Taking the zonal average of this gives

$$\{v_g' q_p'\} = \frac{1}{f_0^2}\frac{\partial}{\partial y}\left\{\frac{\partial \varphi'}{\partial x}\frac{\partial \varphi'}{\partial y}\right\} + \frac{\partial}{\partial p}\left\{\frac{\partial \varphi'}{\partial x}\frac{1}{\mathcal{S}}\frac{\partial \varphi'}{\partial p}\right\}. \tag{17.18}$$

Using the geostrophic relations and the hydrostatic relation (15.8), this may be written

$$
\begin{aligned}
\{v_g' q_p'\} &= -\frac{\partial}{\partial y}\{u_g' v_g'\} - \frac{\partial}{\partial p}\left\{\frac{f_0}{\mathcal{S}\pi} v_g' \theta'\right\} \\
&\equiv \nabla \cdot \mathbf{F},
\end{aligned}
\tag{17.19}
$$

where $\mathbf{F}$ is the *Eliassen-Palm flux*, given by

$$
\mathbf{F} \equiv -\{u_g' v_g'\}\hat{j} - \left\{\frac{f_0}{\mathcal{S}\pi} v_g' \theta'\right\}\hat{p},
\tag{17.20}
$$

with $\hat{j}$ and $\hat{p}$ unit vectors in y and p. Thus the northward component of the Eliassen-Palm flux is the geostrophic northward eddy flux of zonal (geostrophic) momentum, while the vertical component of the EP flux is the geostrophic northward eddy heat flux.

The utility of the Eliassen-Palm flux lies in its role as a source of *wave activity.* This is a measure of the variance of pseudo-potential vorticity, and is defined as

$$
\mathcal{A} \equiv \frac{1}{2}\frac{q_p'^2}{\overline{q}_{py}}.
\tag{17.21}
$$

We can form an equation for wave activity by multiplying (17.2), modified to take into account dissipation, by q_p' and taking the zonal average of the result:

$$
\frac{\partial}{\partial t}\frac{1}{2}\{q_p'^2\} + \frac{\partial \overline{q}_p}{\partial y}\{v' q_p'\} = \{\mathrm{D} q_p'\}.
\tag{17.22}
$$

Now letting

$$
\overline{q}_{py} \equiv \frac{\partial \overline{q}_p}{\partial y},
$$

and noting that $\overline{q}_{py}$ is not a function of time or longitude, divide (17.22) through by $\overline{q}_{py}$:

$$\frac{\partial}{\partial t}\left\{\frac{1}{2}\frac{q_p'^2}{\overline{q}_{py}}\right\}+\{v'q_p'\}=\left\{\mathrm{D}\frac{q_p'}{\overline{q}_{py}}\right\},$$

or using (17.21) and (17.19),

$$\frac{\partial \mathcal{A}}{\partial t}+\nabla\cdot\mathbf{F}=\mathcal{D}, \tag{17.23}$$

where

$$\mathcal{D}\equiv\left\{\mathrm{D}\frac{q_p'}{\overline{q}_{py}}\right\}.$$

In the absence of dissipation of pseudo-potential vorticity, the rate of change of wave activity is proportional to the divergence of the Eliassen-Palm flux. Conversely, in a steady flow, creation or dissipation of wave activity is signified by a nonzero divergence of the Eliassen-Palm flux.

In the case of plane waves, the Eliassen-Palm flux may be interpreted as the flux of wave activity along wave ray paths, traveling at the group velocity. This is shown as follows:

First, using the definitions of wave activity (17.21) and pseudo-potential vorticity (9.11) and the modal decomposition (17.6), we have

$$\begin{aligned}\mathcal{A}&=\frac{1}{2\overline{q}_{py}}\left\{\left[\frac{1}{f_0}(k^2+l^2)+\frac{f_0}{\mathcal{S}}m^2\right]^2\Phi^2e^{2ik(x-ct)+2i\int^y l\,dy'+2i\int^p m\,dp'}\right\}\\&=\frac{1}{4\overline{q}_{py}}\Phi^2\left[\frac{1}{f_0}(k^2+l^2)+\frac{f_0}{\mathcal{S}}m^2\right]^2e^{2i\int^y l\,dy'+2i\int^p m\,dp'}.\end{aligned} \tag{17.24}$$

On the other hand, using (17.6) in the definition of the Eliassen-Palm flux vector, (17.20), together with the usual geostrophic and hydrostatic relations, gives

$$\mathbf{F} = \left[\frac{1}{2f_0^2} kl\Phi^2 e^{2i\int^y l\,dy' + 2i\int^p m\,dp'}\right]\hat{j} + \left[\frac{1}{2}\frac{mk}{\mathcal{S}}\Phi^2 e^{2i\int^y l\,dy' + 2i\int^p m\,dp}\right]\hat{p}. \quad (17.25)$$

Now comparing (17.25) to (17.24), and using the group velocity relations (17.9) together with the definition of wave activity, (17.25), shows that

$$F_i = c_{g_i}\mathcal{A}. \quad (17.26)$$

Thus, for individual plane waves, the Eliassen-Palm flux is just the product of the Rossby wave group velocity and the wave activity. This does not hold for disturbances consisting of more than one plane wave, or nonmodal disturbances, as will be discussed in Chapter (?). Later on, we will find that the Eliassen-Palm flux is very useful for diagnosing the sources and sinks of Rossby waves from atmospheric observations.

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